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A COMPACTIFICATION DUE TO FELL

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We give an alternative construction of a Hausdorff compactification due to Fell [2]. We say that a space is compact if it has the Heine–Borel property, locally compact if each point has a fundamental system of compact neighbourhoods. The interesting spaces from the point of view of this paper, are the *non-Hausdorff* ones since for locally compact Hausdorff spaces Fell's compactification is the usual one-point compactification. The motivation for the compactification comes from the theory of continuous fields of C^*-algebras: the primitive spectrum of a C^*-algebra A is a locally compact T_0 space X and Fell [3] realizes A as an algebra of fields of operators over the compactification of X. This note is based on a discussion of the author with Professor Fell.

A net is said to be *universal* if, for every set A, it is eventually inside or eventually outside A. The *limit set* of a net n in a topological space X is defined to be the set of y in X such that n converges to y; we denote it by $\lim n$. The universal nets are particular examples of the so-called *primitive nets*, those nets for which every cluster point is also a limit. The notion of primitive net is topological whereas that of universal net is set-theoretic.

Let X be a locally compact space. Denote the family of closed sets of X by $\mathscr{C}(X)$. For each compact C in X and every finite set $\mathscr{F}$ of opens of X, let $U(C;\mathscr{F})$ be the set of Y in $\mathscr{C}(X)$ such that $Y\cap C=\varnothing$ and $Y\cap A\neq\varnothing$ for each $A\in\mathscr{F}$. Fell has shown that the $U(C;\mathscr{F})$ form a base for a topology in $\mathscr{C}(X)$ such that it is compact Hausdorff whenever X is locally compact. Denote by λ_X the mapping of a point of X to its closure; it is not necessarily continuous. If X is T_0 then λ_X is injective. The closure of $\lambda_X(X)$ in $\mathscr{C}(X)$ is denoted by $\mathscr{H}(X)$. The points of $\mathscr{H}(X)$ were characterized in [3] as the sets of X which are limit sets of primitive nets. If X is Hausdorff, λ_X is continuous and $\mathscr{H}(X)$ is the one-point compactification of X.

The following construction is motivated by [1, Example 10.21]. Let X_0 denote the discrete space with the same underlying set as X. Denote by $X_0^\vee$ its Stone–Cech compactification which, since X_0 is discrete, can be described as the (compact Hausdorff) space of universal nets in X_0. The topology of $X_0^\vee$ is determined by the base of open sets consisting of all $\{n\in X_0^\vee: n \text{ is eventually in } W\}$, where W runs over all subsets of X. We define X^c to be $X_0^\vee/R$ where R is the equivalence relation of identifying universal nets in X_0 which have the same limit set when considered as primitive nets of X. The above identification gives rise to a bijective mapping, which we call *canonical*, of X^c onto $\mathscr{H}(X)$.

THEOREM. *Let X be a locally compact space. The canonical mapping of X^c onto $\mathscr{H}(X)$ is a homeomorphism.*

Proof. It is sufficient to show that the canonical mapping is continuous. Let $S \subset X$ be the limit set of a net n. Without loss of generality we may assume that a neighbourhood of S in $\mathscr{H}(X)$ is of one of the following forms:

$$\mathscr{U} = \{T \in \mathscr{H}(X) : T \cap U \neq \varnothing, \text{ where } U \text{ is a compact neighbourhood of } x \in S\},$$
$$\mathscr{V} = \{T \in \mathscr{H}(X) : C \cap T = \varnothing \text{ where } C \subset X \text{ is compact, } C \cap S = \varnothing\}.$$

Given a neighbourhood of S in $\mathscr{H}(X)$ we wish to find a neighbourhood of n, i.e. a subset W of X with n eventually in W, such that its limit set is in the given neighbourhood of S. The proof for a neighbourhood $\mathscr{U}$ is easy. We give a proof for a neighbourhood $\mathscr{V}$. The set $X-S$ is open. Choose points $x_1, \ldots, x_j \in C$ with compact neighbourhoods D_{x_i} such that $D_{x_i} \subset X-S$ and $Z = (D_{x_1})^0 \cup \cdots \cup (D_{x_j})^0 \supset C$. Define W to be $X-(D_{x_1} \cup \cdots \cup D_{x_j})$; so $W \supset S$. The net n is eventually in W. Indeed, if not, it is eventually in D_{x_i} for some i, so n has a limit in D_{x_i}; this contradicts the fact that $\lim n \cap D_{x_i} = \varnothing$. We must show that if m is any universal net eventually in W, the limit set of m does not intersect C. Let m be eventually in W. Then m is eventually outside the open set Z. But $C \subset Z$, so $\lim m$ is disjoint from C.

Bibliography

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3. J. M. G. Fell, *The structure of operator algebra fields*, Acta Math. **106** (1961), 233–280.

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