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POISSON CONVERGENCE FOR POINT PROCESSES ON THE PLANE

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Abstract

A compensator is defined for a point process in two dimensions. It is shown that a Poisson process is characterized by a continuous deterministic compensator. Sufficient conditions are given for convergence in distribution of a sequence of two-dimensional point processes in the Skorokhod topology to a Poisson process when the corresponding sequence of compensators converges pointwise in probability to a continuous deterministic function.

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Introduction

The study of simple point processes on $\mathbb{R}_+ = [0, \infty)$ may frequently be simplified by considering their compensators (see, for example, Liptser and Shiryayev (1978)). In particular, if the compensators converge pointwise in probability to a continuous deterministic function, the point processes converge (both vaguely and in the Skorokhod topology) to a Poisson process. This technique has been extended to marked point processes by Jacod (1975) and Brown (1981).

In this article, point processes on $\mathbb{R}_+^2$ will be considered. If it is known that if there is at most one point on every vertical line, the process may be treated as a

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marked process, and Brown's (1981) approach is appropriate. However, if it is assumed only that the process is simple (i.e. there exists no more than one point at any single location in $\mathbb{R}_+^2$), techniques that are genuinely two-dimensional may be appropriate. As an example, consider the process in which all points fall in a random set of vertical lines. The lines intersect the horizontal axis at points which form a Poisson process with intensity μ (μ denotes Lebesgue measure on $\mathbb{R}^1$) and the points on each line form independent Poisson processes, each with intensity μ as well. Using Jacod's (1975) definition of the compensator, the compensator of this process would be indistinguishable from that of an ordinary Poisson process on $\mathbb{R}_+^2$.

In Section 1, a two-dimensional compensator will be defined. In Section 2 it will be shown that a point process with a continuous deterministic compensator is a (non-homogeneous) Poisson process. In Section 3, it will be proven that under certain conditions a sequence of point processes whose compensators converge pointwise in probability to a continuous deterministic function converge in distribution (in the Skorokhod topology) to a Poisson process. Since a Poisson process has at most one point on each vertical line, it seems intuitively clear that the approximating sequence should be an "asymptotic marked point process sequence" as defined by Brown (1981). It will be shown that Theorem 3.1 does imply that the conditions of Brown's (1981) Proposition 1 are satisfied. However, in practice it may sometimes be easier to prove convergence using 2-dimensional compensators as this method avoids direct verification that the processes do in fact form an asymptotic point process sequence.

The theorems in two dimensions are not as general as those in one dimension simply because it is not known how to stop general two-dimensional processes so as to ensure that they remain bounded (see Meyer (1981), page 33, for a discussion of this point). This is also a problem in the study of the properties of the compensator and it appears that completely new techniques will be required. We plan to investigate this problem elsewhere.

As in the case of one-dimensional point processes, this approach has the advantage that general σ-fields are permitted, and one is not restricted to minimal σ-fields as is the case when conditional intensities are used (cf. Kallenberg (1978)).

1. Notation and definitions

Let $\mathbb{R}_+ = [0, \infty)$. For $z = (s, t)$, $z' = (s', t') \in \mathbb{R}^2$ we shall write $z < z'$ if and only if $s \leqslant s'$ and $t \leqslant t'$. We write $z \ll z'$ if both inequalities are strict. If $(s, t) = z \ll z' = (s', t')$, then $(z, z'] = (s, s'] \times (t, t']$. With each point $(s, t) \in \mathbb{R}_+^2$ are associated the following four quadrants:

$$Q_1(s,t) = \{(u,v): u \geqslant s, v \geqslant t\}, \quad Q_2(s,t) = \{(u,v): u < s, v \geqslant t\},$$
$$Q_3(s,t) = \{(u,v): u < s, v < t\}, \quad Q_4(s,t) = \{(u,v): u \geqslant s, v < t\}.$$

We denote by D the set of functions f: $\mathbb{R}_+^2 \to \mathbb{R}$ which have limits in all four quadrants at each point $z = (s, t)$ (respectively $f(s+, t+)$, $f(s-, t)$, $f(s-, t-)$, $f(s, t-)$) and such that $f(s, t) = f(s+, t+)$. Let

$$\Delta f(z) = f(s,t) - f(s-,t) - f(s,t-) + f(s-,t-),$$

$$\Delta^1 f(z) = f(s,t) - f(s-,t) \quad \text{and} \quad \Delta^2 f(z) = f(s,t) - f(s,t-).$$

For $z = (s, t) \ll (s', t') = z'$, let $f(z, z'] = f(z') - f(s', t) - f(s, t') + f(z)$. A function $f \in D$ is said to be increasing if $f(s, t) = 0$ whenever $s = 0$ or $t = 0$ and if $f(z, z'] \geqslant 0$ whenever $z \ll z'$.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. We assume the existence of a complete right-continuous filtration $\{\mathscr{F}_z: z \in \mathbb{R}_+^2\}$ (i.e. $\mathscr{F}(z) \subseteq \mathscr{F}$; $z < z' \Rightarrow \mathscr{F}(z) \subseteq \mathscr{F}(z')$; $\mathscr{F}(z) = \bigcap_{z' \gg z} \mathscr{F}(z')$; $\mathscr{F}(0,0)$ contains all the null sets of $\mathscr{F}$) such that $\mathscr{F}(s,0) = \mathscr{F}(0,t) = \mathscr{F}(0,0)$, for all s, t. Let $\mathscr{F}^1(s) = \bigvee_t \mathscr{F}(s,t)$, $\mathscr{F}^2(t) = \bigvee_s \mathscr{F}(s,t)$, and $\mathscr{F}^*(s,t) = \mathscr{F}^1(s) \vee \mathscr{F}^2(t)$. Let $\mathscr{F}^*(z-) = \bigvee_{z' \ll z} \mathscr{F}^*(z')$. Occasionally, we shall invoke the following condition:

(F4) for all $(s, t) \in \mathbb{R}_+^2$, $\mathscr{F}^1(s)$ and $\mathscr{F}^2(t)$ are conditionally independent given $\mathscr{F}(s, t)$.

A simple point process N on $\mathbb{R}_+^2$ is defined to be a random element of D, which is an increasing step function taking its values in $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ such that $\Delta N(z) = 0$ or 1 for each $z \in \mathbb{R}$. By definition, $N(0, t) = N(s, 0) = 0$, for all s, t. The intensity $m(\cdot)$ of $N(\cdot)$ is defined by $E(N(s,t)) = m(s,t)$. *It will always be assumed that* $m(z) < \infty$ *for all* $z \in \mathbb{R}_+^2$ (we impose this condition because of the problem of stopping a process on $\mathbb{R}_+^2$). It is a trivial consequence of uniform integrability that if the intensity is finite everywhere then N is right continuous in L_1 (right continuity refers to limits in the quadrant Q_1).

In what follows, let X be a random process taking its values in D, such that $X(\cdot)$ is adapted to the filtration $\{\mathscr{F}(z)\}$, and such that X is integrable (ie. $E(|X(z)|) < \infty$, for all $z \in \mathbb{R}_+^2$). We call X a *weak* (*sub*) *martingale* if for every pair $z, z' \in \mathbb{R}_+^2$, $z \ll z'$,

$$E(X(z,z'] | \mathscr{F}(z)) = (\geqslant) 0. \tag{1.1}$$

Suppose that $X(s, t) = 0$ if $s = 0$ or $t = 0$. Then X is a *strong* (*sub*) *martingale* if for $z \ll z'$

$$E(X(z,z'] | \mathscr{F}^*(z)) = (\geqslant) 0. \tag{1.2}$$

Trivially, an adapted point process is a strong submartingale, and we obtain the following.

PROPOSITION 1.1. *Let N be a simple point process* (*with finite intensity*) *adapted to the filtration $\{\mathscr{F}(z)\}$. Then there exist processes M and A adapted to the filtration $\{\mathscr{F}^*(z)\}$ such that M is a weak martingale, A is an increasing process, and*

$$N = M + A. \tag{1.3}$$

Furthermore, if (F4) *holds, M and A are adapted to $\{\mathscr{F}(z)\}$, and M is a strong martingale.*

PROOF. This proposition follows easily from Theorem 3.5 of Brennan (1979) and Proposition 1 of Dozzi (1981).

In addition, according to Dozzi (1981), there exists a unique (up to indistinguishability) increasing process A in the decomposition (1.3) such that for all $z' \in \mathbb{R}_+^2$

$$E\left[\int_{(0,z']} H(z)\,dA(z)\right] = E\left[\int_{(0,z']} E(H(z)|\mathscr{F}^*(z-))\,dA(z)\right],$$

where $H = \chi(B \times F)$, $B \in \mathscr{B}$, $F \in \mathscr{F}$ ($\mathscr{B}$ denotes the Borel sets of $\mathbb{R}_+^2$ and χ is the indicator function). Following Brown (1978), and Kabanov, Liptser, and Shiryayev (1980), we call this increasing process A the *compensator* of N. If we consider Dozzi's (1981) construction of the compensator, it is clear that the compensator of the example in the introduction will not be deterministic. However, the compensator of a Poisson process will simply be its intensity.

We now turn to the definitions of stopping times. Following Ivanoff (1983), the random variable S is a 1-stopping time relative to $\{\mathscr{F}(z)\}$ if $\{S \leqslant s\}$ is $\mathscr{F}^1(s)$-measurable for all $s \in \mathbb{R}_+$. Likewise, T is a 2-stopping time relative to $\{\mathscr{F}(z)\}$ if $\{T \leqslant t\}$ is $\mathscr{F}^2(t)$-measurable for all $t \in \mathbb{R}_+$.

In the notation of Wong and Zakai (1976), $\tau(z,\omega)$, $z \in \mathbb{R}_+^2$, $\omega \in \Omega$, is a stopping time relative to $\{\mathscr{F}(z)\}$ if $\tau(\cdot,\cdot)$ is measurable and adapted, and for almost all ω, $z < z' \Rightarrow \tau(z,\omega) \geqslant \tau(z',\omega)$, and τ takes only the values 0 or 1. Let $D_\tau(\omega) = \overline{\{z\colon \tau(z,\omega) = 1\}}$. D_τ is the stopping domain associated with τ. For any adapted random process X of bounded variation, define X_τ (X stopped at τ) pointwise by

$$X_\tau(z) = X(\tau \wedge z) = \int_{D_\tau \cap (0,z]} dX(z'). \tag{1.4}$$

We shall denote convergence in probability and convergence in distribution by "$\to_p$" and "$\to_{\mathscr{D}}$" respectively.

2. Lemmas

In this section we shall state several basic lemmas which will be required subsequently.

We consider a sequence $\{(X_n, \mathscr{F}_n, P_n), n = 1,2,3,\ldots\}$ such that for each n, X_n is a random element of D, which vanishes on the axes and is adapted to the complete right-continuous filtration $\{\mathscr{F}_n(z) \subseteq \mathscr{F}_n\}$. To show tightness of the

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sequence, we need only to consider the restriction of the processes to sets of the form $[0, K]^2$. We say that condition $[A]$ is satisfied if

$[A]$ for every $K > 0$, $\varepsilon > 0$, $\eta > 0$ there exist $\delta > 0$ and $n_0 < \infty$ such that for every 1-stopping time S_n and 2-stopping time T_n on $(X_n(z), \mathscr{F}_n(z), P_n; z < (K, K))$

$\sup_{n \geqslant n_0} \sup_{\theta \in [0,\delta]} P_n(\sup_{0 \leqslant t \leqslant K} |X_n(S_n + \theta, t) - X_n(S_n, t)| > \eta) < \varepsilon$, and

$\sup_{n \geqslant n_0} \sup_{\theta \in [0,\delta]} P_n(\sup_{0 \leqslant s \leqslant K} |X_n(s, T_n + \theta) - X_n(s, T_n)| > \eta) < \varepsilon$. (We assume that $S_n \leqslant K$, $T_n \leqslant K$ and interpret $S_n + \theta$ and $T_n + \theta$ as $(S_n + \theta) \wedge K$ and $(T_n + \theta) \wedge K$, respectively.)

LEMMA 2.1. *Let $\{(X_n, \mathscr{F}_n, P_n)\}$ be a sequence which is defined as above and which satisfies $[A]$. If $\{X_n(z)\}$ is tight for each $z \in \mathbb{R}^2_+$, then $\{X_n\}$ is tight in D.*

PROOF. This lemma is an easy consequence of Corollary 4.2 of Ivanoff (1980) and Theorem 3.1 of Ivanoff (1983).

The following lemma is probably well-known, but we have been unable to find a proof in the literature.

LEMMA 2.2. *Let $(M, \mathscr{F}, P)$ be the weak martingale in the decomposition* (1.3). *For any stopping time τ (relative to $\{\mathscr{F}^*(z)\}$) whose associated stopping domain is contained in some fixed bounded subset of $\mathbb{R}^2_+$, M_τ is also a weak martingale with respect to the filtration $\{\mathscr{F}^*(z)\}$.*

PROOF. See Appendix A.

COROLLARY 2.3. *Let $(M, \mathscr{F}, P)$ be as in Lemma 2.2. Let S be any uniformly bounded 1-stopping time and T any uniformly bounded 2-stopping time with respect to $\{\mathscr{F}(z)\}$. Then $M^1(s, t) = M(s \wedge S, t)$ and $M^2(s, t) = M(s, T \wedge t)$ are each weak martingales with respect to the filtration $\{\mathscr{F}^*(z)\}$.*

PROOF. See Appendix A.

The final lemma in the section shows that in two dimensions, a Poisson process is characterized by a continuous deterministic compensator.

LEMMA 2.4. *Suppose that the simple point process $(N, \mathscr{F}, P)$ has the decomposition $N = M + A$ where M is a strong martingale adapted to $\{\mathscr{F}(z)\}$ and A is a continuous deterministic increasing process. Then N is a Poisson process with intensity A.*

Proof. For any rectangle $R = (z, z']$ denote $N(z, z']$ and $A(z, z']$ by $N(R)$ and $A(R)$, respectively. It is sufficient to show that for any disjoint set of bounded rectangles $\{R_1, \ldots, R_n\}$, $N(R_1), \ldots, N(R_n)$ are independent Poisson random variables with parameters $A(R_1), \ldots, A(R_n)$, respectively.

Without loss of generality, assume that there exists $K < \infty$ such that $(0, K]^2$ may be divided into a grid with vertical lines at $0 = s_0 < s_1 < \cdots < s_p = K$, and horizontal line at points $0 = t_0 < t_1 < \cdots < t_q = K$ such that, for $i = 1, \ldots, n$, $R_i = \big((s_{k-1}, t_{j-1}), (s_k, t_j)\big]$ for some pair (k, j), $1 \leqslant k \leqslant p$, $1 \leqslant j \leqslant q$. We shall show that $\{N\big((s_{k-1}, t_{j-1}), (s_k, t_j)\big], 1 \leqslant k \leqslant p, 1 \leqslant j \leqslant q)\}$ are independent Poisson random variables with parameters $\{A\big((s_{k-1}, t_{j-1}), (s_k, t_j)\big]\}$.

The problem can be reduced to one dimension. Let $0 < u \leqslant pK$, and if $(j-1)K < u \leqslant jK$, $1 \leqslant j \leqslant p$, define the one-dimensional process N' by

$$N'(u) = N((0,0), (s_{j-1}, K)] + N\big((s_{j-1}, 0), (s_j, u - (j-1)K)\big] \text{ and}$$
$$\mathscr{F}'(u) = \mathscr{F}(s_{j-1}, K) \vee \mathscr{F}\big(s_j, u - (j-1)K\big).$$

Essentially, what has been done is the following: $(0, K]^2$ has been divided up into p vertical strips. Beginning with the first strip, N' counts up the number of points in the strip, with $N'(u)$ equal to the number of points in the strip up to and including the horizontal line $t = u$, $0 \leqslant u \leqslant K$. For $K < u \leqslant 2K$, $N'(u)$ includes all the points in the first strip plus those in the second up to and including the line $t = u - K$. We continue in this way to count the points in all the strips sequentially. Therefore, $N\big((s_i, t_j), (s_{i+1}, t_{j+1})\big] = N'\big(iK + t_j, iK + t_{j+1}\big]$, and so it is sufficient to show that N' is a one-dimensional Poisson process with the appropriate intensity.

Define one-dimensional processes A' and M' from A and M in an analogous manner. It is clear that $N' = A' + M'$, that A' is continuous and deterministic, and that N' and M' are adapted to $\{\mathscr{F}'(u)\}$. Also, the σ-fields $\{\mathscr{F}'(u)\}$ form a complete right-continuous filtration.

We shall show that M' is a martingale with respect to $\{\mathscr{F}'(u)\}$. Let $0 < v < u \leqslant pK$. Then, for some integers i, j, $1 \leqslant i \leqslant j \leqslant p$, $(i-1)K < v \leqslant K$, $(j-1)K < u \leqslant jK$. If $i = j$,

$$M'(u) - M'(v) = M\big((s_{i-1}, v - (i-1)K), (s_i, u - (i-1)K)\big],$$
$$\mathscr{F}'(v) \subseteq \mathscr{F}^*(s_{i-1}, v - (i-1)K),$$

and

$$\begin{aligned} E(M'(u) - M'(v) | \mathscr{F}'(v)) &= E\big(E\big(M\big((s_{i-1}, v - (i-1)K), (s_i, u - (i-1)K)\big] \\ &\qquad \cdot | \mathscr{F}^*(s_{i-1}, v - (i-1)K)\big) | \mathscr{F}^*(v)\big) \\ &= 0, \end{aligned}$$

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since M is a strong martingale with respect to $\{\mathscr{F}(z)\}$. If $i < j$,

$$M'(u) - M'(v) = M\big((s_{i-1}, v-(i-1)K),(s_i, K)\big] + M\big((s_i,0),(s_{j-1}, K)\big] + M\big((s_{j-1},0),(s_j, u-(j-1)K)\big],$$

where the second term on the right hand side of the equation is interpreted as 0 if $i = j-1$. Also, $\mathscr{F}'(v) \subseteq \mathscr{F}^*(s_{i-1}, v-(i-1)K)$ and $\mathscr{F}'(v) \subseteq \mathscr{F}^*(s_i, 0) \subseteq \mathscr{F}^*(s_{j-1}, 0)$. This implies that, since M is a strong martingale, $E(M'(u) - M'(v)|\mathscr{F}'(v)) = 0$. Thus, M' is a one-dimensional martingale with respect to $\{\mathscr{F}'(u)\}$.

From Appendix B, Lemma B.1, it follows that N' is a simple one-dimensional point process. It is well-known that N' is therefore a non-homogeneous one-dimensional Poisson process with intensity A'. This in turn implies that $N(R_1), \ldots, N(R_n)$ are independent Poisson random variables with parameters $A(R_1), \ldots, A(R_n)$, respectively.

3. The convergence theorem

In this section, it will be assumed that all filtrations have the structure described in Section 1. We obtain the following (partial) analogue to Theorem 1 (ii) of Kabanov, Liptser and Shiryayev (1980) and Theorem 1 of Brown (1978).

THEOREM 3.1. *Let $\{N_n, \mathscr{F}_n, P_n\}$ denote a sequence of simple point processes on $\mathbb{R}^2_+$. Assume that all intensity measures are finite on bounded sets, and that the sequence of compensators $\{A_n\}$ satisfies $A_n(z) \to_p A(z)$ for each $z \in \mathbb{R}^2$, where $A(\cdot)$ is a continuous deterministic function. If $\{N_n(z)\}$ and $\{A_n(z)\}$ are both uniformly integrable for each $z \in \mathbb{R}^2_+$, then $N_n \to_{\mathscr{D}} N$ in D, where N is a Poisson process with intensity A.*

PROOF. The proof follows generally along the lines of that of Theorem 1 of Brown (1981). Lemma 2.1 is used to show that $\{N_n\}$ is tight in D. Then necessarily all limits are simple point processes on $\mathbb{R}^2_+$. If $N_{n(k)} \to_{\mathscr{D}} N$ for some subsequence $\{n(k)\}$, then $M_{n(k)} = N_{n(k)} - A_{n(k)} \to_{\mathscr{D}} N - A = M$. It is then proven that M is a strong martingale with respect to the minimal σ-field generated by N. An application of Lemma 2.4 completes the proof.

We begin by verifying the conditions of Lemma 2.1. The uniform integrability assumption trivially proves tightness of $N_n(z)$ for each $z \in \mathbb{R}^2_+$. For $K < \infty$ arbitrary but fixed, let S_n and T_n be 1- and 2-stopping times with respect to $\{\mathscr{F}_n(z)\colon z \in [0, K]^2\}$, and assume $S_n \leqslant K$ and $T_n \leqslant K$, $n = 1, 2, \ldots$. We will prove condition [A] for the 1-stopping times only, as the proof for 2-stopping times is identical.

Since N_n is an increasing process,

$$\sup_{0\leqslant t\leqslant K} |N_n(S_n+\theta,t)-N_n(S_n,t)| = N_n(S_n+\theta,K)-N_n(S_n,K).$$

It is sufficient to show that, for any sequence $\{\theta_n\}$, $\theta_n\downarrow 0$, $N_n(S_n+\theta_n,K)-N_n(S_n,K)\to_p 0$. To simplify notation, let $N_n(s)=N_n(s,K)$ and $A_n(s)=A_n(s,K)$. Since $A_n(\cdot)$ is nondecreasing, by Lemma 1 of McLeish (1978), $\sup_{0\leqslant s\leqslant K}|A_n(s)-A(s)|\to_p 0$. Since A is continuous it follows that $|A_n(S_n+\theta_n)-A_n(S_n)|\to_p 0$. By uniform integrability, $E(A_n(S_n+\theta_n)-A_n(S_n))\to 0$. By Corollary 2.3, and since M_n disappears on the axes,

$$E(M_n(S_n,K)) = E\big(E\big(M_n^1((0,0),(K,K)]|\mathscr{F}_n{}^*(0,0)\big)\big)=0$$

(and similarly $E(M_n(S_n+\theta_n,K))=0$). Therefore,

$$E\big(N_n(S_n+\theta_n)-N_n(S_n)\big)=E\big(A_n(S_n+\theta_n)-A_n(S_n)\big),$$

and so

$$N_n(S_n+\theta_n,K)-N_n(S_n,K)\to_p 0.$$

This verifies that [A] holds.

Thus $\{N_n\}$ is tight in D. Suppose $N_{n(k)}\to_{\mathscr{D}} N$ for some subsequence $\{n(k)\}$. Since $A_{n(k)}\to_p A$ uniformly on compact sets, $M_{n(k)}=N_{n(k)}-A_{n(k)}\to_{\mathscr{D}} N-A=M$.

It remains to show that M is a strong martingale with respect to the minimal filtration generated by N. We remark that $\{\mathscr{F}(z)\}$ may be assumed to be right continuous by using an argument similar to that of Lemma 18.4 of Liptser and Shiryayev (1978) and by assuming that the underlying probability space consists of all integer-valued increasing functions in D.

Consider the collection of sets

$$\mathscr{G}(z)=\Big\{p_{u_1,\ldots,u_k}{}^{-1}(H),\ H\in\mathbf{Z}_+^k,\ u_1,\ldots,u_k\not\gg z,\ u_1,\ldots,u_k\in T_p,\ k\geqslant 1\Big\}$$

where

$$T_p=\{z\colon P(N\text{ is continuous at }z)=1\},\text{ and}$$

$$p_{u_1,\ldots,u_k}(N)=\big(N(u_1),\ldots,N(u_k)\big).$$

It is easily seen that $\mathscr{G}(z)$ forms a π-system which generates $\mathscr{F}^*(z)$. By Theorem 34.1 of Billingsley (1979), M is a strong martingale with respect to $\{\mathscr{F}(z)\}$ if it can be shown that if $z\ll z'$, for every $G\in\mathscr{G}(z)$

$$\int_G M(z,z']\,dP=0. \tag{3.1}$$

Consider $G=\{p_{u_1,\ldots,u_k}{}^{-1}(H)\}$ and let $\chi(G)$ be the indicator of G. Let $\chi_n(G)$ be the corresponding random variable for the process N_n. By convergence of the finite-dimensional distributions, $\chi_n(G) \to_{\mathscr{D}} \chi(G)$. By observing that for any $z \in T_p$, $k \in \mathbf{Z}^+$, $P_n(N_n(z)\chi_n(G) = k) = P_n(N_n(z) = k, \chi_n(G) = 1)$, we have that $N_n(z)\chi_n(G) \to_{\mathscr{D}} N(z)\chi(G)$. Similarly, for $z \ll z'$, z, $z' \in T_p$, $N_n(z,z']\chi_n(G) \to_{\mathscr{D}} N(z,z']\chi(G)$, and since $|A_n(z,z'] - A(z,z']| \to_p 0$,

$$(3.2) \qquad (N_n - A_n)(z,z']\chi_n(G) \overset{\mathscr{D}}{\to} (N - A)(z,z']\chi(G).$$

By uniform integrability, (3.2) implies (3.1) for z, $z' \in T_p$. The general result follows from the right continuity of N and A, and the fact that T_p is dense.

This completes the proof of Theorem 3.1.

COMMENT. As mentioned in the introduction, if we consider the sequence of point processes on a bounded set (say $[0, K]^2$), the hypotheses of Proposition 1 of Brown (1981) are implied by the conditions of Theorem 3.1. This is shown in Appendix C.

In the one-dimensional case, the uniform integrability condition may be eliminated by stopping the process in such a way that the compensators are uniformly bounded above. As mentioned in the introduction, it is not known how to do this in general for processes with a two-dimensional time parameter. However, there are some processes which can be stopped in an appropriate way and for which the uniform integrability assumption of Theorem 3.1 may be weakened. We state a few lemmas and then give two examples of such processes.

LEMMA 3.2. *Let f be any nonnegative process bounded almost everywhere adapted to the filtration $\{\mathscr{F}(z)\}$ and let τ be any stopping time with respect to the filtration $\{\mathscr{F}^*(z)\}$ whose associated stopping domain is contained in some fixed bounded subset of $\mathbb{R}^2_+$. For any pair (s, t), $s, t < \infty$,*

$$(3.3) \qquad E\int_{[0,s]} f(u-,t)\, dA_\tau(u,t) = E\int_{[0,s]} f(u-,t)\, dN_\tau(u,t),$$

and

$$(3.4) \qquad E\int_{[0,t]} f(s,v-)\, dA_\tau(s,v) = E\int_{[0,t]} f(s,v-)\, dN_\tau(s,v).$$

(Note: all integrals are defined pointwise.)

PROOF. By symmetry, the proofs of (3.3) and (3.4) are equivalent, so only (3.3) will be considered.

As in the proof of Theorem 18.6 of Liptser and Shiryayev (1978), it suffices to consider functions of the form $f(u-,t) = f(u) = \chi_{\{a<u\leqslant b\}}\xi$, where ξ is a bounded $\mathscr{F}(a,t)$-measurable random variable. Now

$$\begin{aligned} E\int_{[0,s]} f(u)\, dN_\tau(u,t) &= E(\xi(N_\tau(b,t) - N_\tau(a,t))) \\ &= E(\xi(A_\tau(b,t) - A_\tau(a,t))) + E(\xi(M_\tau(b,t) - M_\tau(a,t))). \end{aligned}$$

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Since ξ is $\mathscr{F}^*(a,0)$-measurable, and M_τ is a weak $\{\mathscr{F}^*(z)\}$ martingale (Lemma 2.2) which is 0 on the axes,

$$E(\xi(M_\tau(b,t) - M_\tau(a,t))) = E(\xi E(M_\tau((a,0),(b,t)]|\mathscr{F}^*(a,0))) = 0.$$

Now (3.3) follows immediately.

LEMMA 3.3. *Let N be a point process with continuous compensator A. Let τ be a stopping time with stopping domain $D_\tau \subseteq [0,K]^2$, for some $K < \infty$ such that $A_\tau(s,t) < H$ a.s. for some constant H, $1 \leqslant H < \infty$. Then for $(s,t) \in \mathbb{R}^2_+$,*

$$E(M_\tau^2(s,t)) \leqslant 11HE(N(s,t)). \tag{3.5}$$

PROOF. Fix $(s,t) > 0$, and let $M^*(v) = M_\tau(s,v)$, and define $A^*(v)$ and $N^*(v)$ analogously, for $0 \leqslant v \leqslant t$. For any function $f(v)$, let $\Delta f(v) = f(v) - f(v-)$. From Lemma 18.7 of Liptser and Shiryayev (1978), it follows that

$$\begin{aligned}
M_\tau(s,t)^2 &= M^*(t)^2 \\
&= 2\int_0^t M^*(v-)\,dM^*(v) + \sum_{v \leqslant t} (\Delta M^*(v))^2 \\
&= 2\left(\int_0^t N^*(v-)\,dN^*(v) - \int_0^t N^*(v-)\,dA^*(v)\right. \\
&\qquad \left. - \int_0^t A^*(v)\,dN^*(v) + \int_0^t A^*(v)\,dA^*(v)\right) + \sum_{v \leqslant t} (\Delta N^*(v))^2 \\
&\leqslant 2\left(\int_0^t N(s,v-)\,dN^*(v) + \int_0^t N(s,v-)\,dA^*(v)\right) \\
&\qquad + 2H(N_\tau(s,t) + A_\tau(s,t)) + \sum_{v \leqslant t} (\Delta N^*(v))^2.
\end{aligned}$$

Consider $\Sigma_{v \leqslant t}(\Delta N^*(v))^2$. Let $0 = t_0$, and for $i > 0$ let

$$t_i = \begin{cases} \inf(u \leqslant t\colon N^*(u) - N^*(t_{i-1}) > 0), \\ t \quad \text{if no such } u \text{ exists.} \end{cases}$$

Clearly $t_n = t$ for some $n \leqslant N_\tau(K,K) \leqslant N(K,K)$, and

$$\sum_{v \leqslant t} (\Delta N^*(v))^2 = \sum_{i=1}^n (N_\tau(s,t_i) - N_\tau(s,t_{i-1}))^2.$$

Let $R_i(u) = N_\tau(u,t_i) - N_\tau(u,t_{i-1})$, $i = 1,\ldots,n$. Clearly, $R_i(u)$ is a simple one-dimensional point process. Applying Lemma 18.7 of Liptser and Shiryayev (1978) again,

$$\begin{aligned}
(R_i(s))^2 &= 2\int_0^s R_i(u-)\,dR_i(u) + \sum_{u \leqslant s} (\Delta R_i(u))^2 \\
&\leqslant 2\int_0^s N_\tau(u-,t)\,d(N_\tau(u,t_i) - N_\tau(u,t_{i-1})) + (N_\tau(s,t_i) - N_\tau(s,t_{i-1})),
\end{aligned}$$

since $\Delta R_i(u) = 0$ or 1. Therefore,

$$\begin{aligned}\sum_{v \leqslant t} (\Delta N^*(v))^2 &= \sum_{i=1}^{n} (R_i(s))^2 \\ &\leqslant 2\sum_{i=1}^{n} \int_0^s N(u-,t)\, d(N_\tau(u,t_i) - N_\tau(u,t_{i-1})) \\ &\quad + \sum_{i=1}^{n} (N_\tau(s,t_i) - N_\tau(s,t_{i-1})) \\ &= 2\int_0^s N(u-,t)\, dN_\tau(u,t) + N_\tau(s,t).\end{aligned}$$

Using the fact that $A_\tau(s,t) \leqslant H$, and applying Lemma 3.2, we obtain

$$\begin{aligned}E\left(M_\tau(s,t)^2\right) &\leqslant 4E\left(\int_0^t N(s,v-)\, dA_\tau(s,v)\right) + 4HE(N(s,t)) \\ &\quad +2E\left(\int_0^s N(u-,t)\, dA_\tau(u,t)\right) + E(N(s,t)) \\ &\leqslant 11HE(N(s,t)).\end{aligned}$$

We now consider processes which may be stopped. Given a sequence of point processes $\{N_n, \mathscr{F}_n, P_n\}$ with compensators $\{A_n\}$, we say that the sequence of compensators satisfies condition $[S]$ if

$[S]$ for every set $[0, K]^2$, $K < \infty$, a constant $C_K < \infty$ may be chosen such that there exists a sequence of stopping times $\{\tau_n\}$ adapted to $\{\mathscr{F}_n^*\}$ for which $D_{\tau_n} \subseteq [0, K]^2$, $A_{n,\tau_n}(z) \leqslant C_K$ for all z and n, and $P_n\{\tau_n(z) \neq 1$ for some $z < (K, K)\} \to 0$ as $n \to \infty$.

THEOREM 3.4. *Let $\{N_n, \mathscr{F}_n, P_n\}$ be a sequence of point processes with continuous compensators $\{A_n\}$ satisfying $[S]$. If $A_n(z) \to_p A(z)$ for each $z \in \mathbb{R}_+^2$ where A is a continuous deterministic function, then $N_n \to_{\mathscr{D}} N$ in D where N is a Poisson process with compensator A.*

PROOF. Fix $K > 0$ and note that $\{A_{n,\tau_n}\}$ is uniformly integrable since the compensators are uniformly bounded. By Lemma 3.3, the sequence $\{M_{n,\tau_n}\}$ is uniformly integrable. The proof then proceeds as in Theorem 3.1 using the equivalent stopped sequences $\{N_{n,\tau_n}\}$, $\{A_{n,\tau_n}\}$ and $\{M_{n,\tau_n}\}$ to give convergence on $[0, K]^2$. Since K is arbitrary, this is sufficient.

We now give two examples of a sequence of point processes whose compensators satisfy $[S]$ if $A_n(z) \to_p A(z)$, all $z \in \mathbb{R}_+^2$.

(a) Suppose that the filtrations $\{\mathscr{F}_n\}$ each satisfy (F4) and the compensators are all continuous. Then $\{A_n\}$ satisfies $[S]$, since A_n is $\{\mathscr{F}_n\}$-adapted and for

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arbitrary K the stopping time

$$\tau_n(s,t) = \begin{cases} 1 & \text{if } (s,t) < (K,K) \text{ and } A_n(K,t) \leqslant A(K,K)+1, \\ 0 & \text{otherwise.} \end{cases}$$

is $\mathscr{F}_n^*$-adapted. Also, $D_{\tau_n} \subseteq [0, K]^2$, $P_n\{\tau_n(z) \neq 1 \text{ for some } z < (K,K)\} \to 0$ and $A_{n,\tau_n}(s,t) \leqslant A_{n,\tau_n}(K,K) \leqslant A(K,K)+1$.

(b) Suppose that each of the compensators A_n is absolutely continuous (a.s.) with $\mathscr{F}_n^*$-adapted density a_n. If a is the density of A, assume that for each $K < \infty$ there is a constant $\varepsilon < \infty$ such that $\lim_{n\to\infty} P_n(\sup_{z<(K,K)} |a_n(z) - a(z)| > \varepsilon) = 0$. It may be seen that $\{A_n\}$ satisfies $[S]$ by using the following argument: let $\phi_n(z) = \sup_{z'<z} |a_n(z) - a(z)|$. For $K < \infty$, define τ_n as follows:

$$\tau_n(z) = \begin{cases} 1 & \text{if } z < (K,K) \text{ and } \phi_n(z) \leqslant \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2 of Yeh (1981) implies that τ_n is a stopping time adapted to $\{\mathscr{F}_n^*\}$. By definition, $D_{\tau_n} \subseteq [0, K]^2$, $P_n\{\tau_n(z) \neq 1 \text{ for some } z < (K,K)\} \to 0$, and

$$\begin{aligned} A_{n,\tau_n}(s,t) &= \int_{\{[0,s]\times[0,t]\}\cap D_{\tau_n}} a_n(u,v)\,du\,dv \\ &\leqslant \int_{[0,K]^2} (a(u,v)+\varepsilon)\,du\,dv \\ &= A(K,K) + \varepsilon K^2. \end{aligned}$$

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Appendix A

PROOF OF LEMMA 2.2. We begin with a discrete-time result. Suppose $z \in I = \{(i,j); i = 0,1,\ldots,m, j = 0,1,\ldots,n\}$, $m, n \leqslant \infty$. Let $\{\mathscr{F}(z)\}$ be an increasing set of complete sub-σ-fields of $\mathscr{F}$ and let X be an adapted process which vanishes on the axes. We will use Walsh's ((1979) pages 179–180) definition of a stopping domain D in the discrete case. Note that his definition extends D to be continuous rather than discrete. We define a discrete stopping time τ to be the characteristic function of a stopping domain. In analogy to (1.4) define the stopped process X_τ by

$$\text{(A.1)} \quad X_\tau(z) = X(\tau \wedge z) = \sum_{(i,j) \ll z} \tau(i+1, j+1) X((i,j),(i+1,j+1)].$$

The following lemma is easily verified.

LEMMA A.1. *Let $(M, \mathscr{F}, P)$ be a weak martingale with discrete time parameter $z \in I$. For any discrete stopping time τ, M_τ is also a weak martingale with respect to the filtration $\{\mathscr{F}(z), z \in I\}$.*

We return now to the proof of Lemma 2.2. Note first that if D_τ is a bounded stopping domain, it is contained entirely in $[0, K]^2$ for some $K < \infty$. Since both N and A are increasing processes, $N_\tau(z) \leqslant N(K, K)$ and $A_\tau(z) \leqslant A(K, K)$ for every $z \in \mathbb{R}_+^2$. Thus, $\{N_\tau(z)\}$, $\{A_\tau(z)\}$ and in particular $\{M_\tau(z)\}$ are each uniformly integrable, for all $z \in \mathbb{R}_+^2$, and for any set of stopping times $\{\tau\}$ whose stopping domains are uniformly bounded.

Let D_τ be the stopping domain associated with the continuous stopping time τ in the statement of Lemma 2.2. Let

$$I_n = \{(i,j): i = 0, 2^{-n}, 2 \cdot 2^{-n}, 3 \cdot 2^{-n}, \ldots, j = 0, 2^{-n}, 2 \cdot 2^{-n}, \ldots\}.$$

Define a discrete stopping domain D_n in the following way: $D \subseteq D_n$, and for $(i,j) \in I_n$, $(i,j) \in D_n$ if and only if $(i-1, j-1) \in D$. The domains $\{D_n, D\}$ are uniformly bounded. Let τ_n be the (discrete) stopping time associated with D_n. By Lemma A.1, if $z, z' \in I_n$, $z \ll z'$, $E(M_{\tau_n}(z,z'] | \mathscr{F}^*(z)) = 0$. For any two points $z, z' \in \mathbb{R}_+^2$, $z \ll z'$, choose $z_n, z_n' \in I_n$, $z_n \ll z_n'$ such that $z \ll z_n$, $z' \ll z_n'$, $z_m > z_n$ and $z_m' > z_n'$ if $m < n$, and $z_n \to z$, $z_n' \to z'$. By the assumption of right continuity, $M_{\tau_n}(z_n, z_n'] \to M_\tau(z, z']$ a.s. If $A \in \mathscr{F}^*(z)$, then $A \in \mathscr{F}^*(z_n)$ for all n, and

$$\int_A E\left(M_{\tau_n}(z_n, z_n'] | \mathscr{F}^*(z_n)\right) dP = \int_A M_{\tau_n}(z_n, z_n'] \, dP = 0.$$

Therefore, by almost sure convergence and uniform integrability, $\int_A M_\tau(z,z'] dP = 0$.

PROOF OF COROLLARY 2.3. It must be shown that for any pair $z \gg z'$, $E(M^i(z,z'] | \mathscr{F}^*(z)) = 0$, $i = 1, 2$. Consider $i = 1$ and choose $K < \infty$ such that $S \leqslant K$ a.s. and $z' \ll (K, K)$. Define a stopping time τ as follows:

$$\tau(u) = \begin{cases} 1 & \text{if } u < (S, K), \\ 0 & \text{otherwise.} \end{cases}$$

If $(s,t) \in (0, K]^2$, $M_\tau(s,t) = M^1(s,t)$, and since z, $z' \in [0, K]^2$, Lemma 2.2 gives the result for $i = 1$. The case $i = 2$ follows by symmetry.

Appendix B

LEMMA B.1. *Let N be a simple point process whose compensator A is continuous and deterministic. Then with probability one, any finite horizontal or vertical line segment contains at most one point.*

PROOF. By symmetry, it is sufficient to consider horizontal line segments. For $K < \infty$ arbitrary, let $\mathscr{A}$ be the event that there exists a horizontal line in $[0, K]^2$ with more than one point. For $n \in \mathbf{Z}^+$, define the following sets:

$$S_n(i, j) = \left(\left(\frac{(i-1)K}{2^n}, \frac{(j-1)K}{2^n}\right), \left(\frac{iK}{2^n}, \frac{jK}{2^n}\right)\right] \qquad 1 \leqslant i, j \leqslant 2^n,$$

$$T_n(k, j) = \bigcup_{i=k}^{2^n} S_n(i, j) \qquad 1 \leqslant k, j \leqslant 2^n.$$

Let $\mathscr{A}_n$ be the event that at least one of $T_n(1,1), \ldots, T_n(1, 2^n)$ contains more than one point. Thus $\mathscr{A} \subset \mathscr{A}_n$ for all n. Let $\mathscr{B}_n(k, j)$ be the event that $N(T_n(k, j)) > 0$, $\mathscr{C}_n(k, j)$ the event that $N(S_n(k, j)) > 0$, and $\mathscr{D}_n$ the event that $N(S_n(i, j)) > 1$ for some pair (i, j). Since

$$\text{(B.1)} \qquad \mathscr{A}_n \subseteq \left[\bigcup_{k=1}^{2^n-1} \bigcup_{j=1}^{2^n} (\mathscr{C}_n(k, j) \cap \mathscr{B}_n(k+1, j))\right] \cup \mathscr{D}_n,$$

$$P(\mathscr{A}_n) \leqslant \sum_{k=1}^{2^n-1} \sum_{j=1}^{2^n} P(\mathscr{B}_n(k+1, j) | \mathscr{C}_n(k, j)) P(\mathscr{C}_n(k, j)) + P(\mathscr{D}_n).$$

Since N is simple $P(\mathscr{D}_n) \to 0$ as $n \to \infty$. Now, note that $\mathscr{C}_n(k, j) \in \mathscr{F}^*(k, j-1)$ and A is deterministic and continuous. Thus, for any $\varepsilon > 0$ if n is sufficiently large,

$$P(\mathscr{B}_n(k+1, j) | \mathscr{C}_n(k, j)) \leqslant E(N(T_n(k+1, j)) | \mathscr{C}_n(k, j))$$
$$= E(A(T_n(k+1, j)) | \mathscr{C}_n(k, j)) \leqslant A\left(K, \frac{jK}{2^n}\right) - A\left(K, \frac{(j-1)K}{2^n}\right) \leqslant \varepsilon.$$

Finally, for n sufficiently large,

$$\sum_k \sum_j P(\mathscr{B}_n(k+1, j) | \mathscr{C}_n(k, j)) P(\mathscr{C}_n(k, j)) \leqslant \varepsilon \sum_k \sum_j P(\mathscr{C}_n(k, j))$$
$$\leqslant \varepsilon \sum_k \sum_j E(N(S_n(k, j)) = \varepsilon EN(K, K).$$

From (B.1), $P(\mathscr{A}) \leqslant \lim_n P(\mathscr{A}_n) = 0$.

Appendix C

In this appendix, it will be shown that under the hypotheses of Theorem 3.1

(i) if $Q_n(s, t) = \{\text{number of } u \leqslant s \text{ such that } \Delta^1 N(u, t) = \Delta^1 N(u, K) = 1\}$ and $R_n(s, t) = N_n(s, t) - Q_n(s, t)$, then as $n \to \infty$, $E(R_n(K, K)) \to 0$.

(ii) If $A_n^1(\cdot, t)$ is the (1-dimensional) compensator of $N_n(\cdot, t)$ with respect to $\{\mathscr{F}_n^1(s)\}$, then $A_n^1(s, t) \to_p A(s, t)$, for each $(s, t) \in (0, K]^2$.

Thus, the conditions of Proposition 1 of Brown (1981) are satisfied.

Proof of (i). Divide $(0, K]^2$ into vertical strips as follows: if $A(K, K) = a$, then define $0 = s_0 < s_1 < \cdots < s_r = K$ iteratively via $s_i = \{\inf s\colon A(s, K) - A(s_{i-1}, K) = a/r\}$, $i = 1, \dots, r-1$. Let L_i be the strip $\left((s_{i-1}, 0), (s_i, K)\right]$. By Theorem 3.1, $P(N_n(L_i) > 1) \to 1 - e^{-a/r}(1 + a/r)$, as $n \to \infty$. Therefore,

$$\begin{aligned} P(R_n(K, K) > 0) = P\left(\bigcup_{i=1}^{r} (R_n(L_i) > 0) \right) &\leqslant \sum_{i=1}^{r} P(N_n(L_i) > 1) \\ &\to \sum_{i=1}^{r} \left(1 - e^{-a/r}(1 + a/r)\right) \qquad \text{as } n \to \infty \\ &\leqslant \sum_{i=1}^{r} \left(1 - (1 - a/r)(1 + a/r)\right) = a^2/r. \end{aligned}$$

Since r is arbitrary, $P(R_n(K, K) = 0) \to 1$ and by uniform integrability, $E(R_n(K, K)) \to 0$.

Proof of (ii). It should first be pointed out that the fact that A_n^1 is well-defined as an increasing 2-dimensional process is well-known (see, for example, Merzbach and Zakai (1980)). Let $M_n(s, t) = N_n(s, t) - A_n(s, t)$, and $M_n^1(s, t) = N_n(s, t) - A_n^1(s, t)$. Then $M_n^1(\cdot, t)$ is a (1-dimensional) martingale with respect to $\{\mathscr{F}_n^1(s)\}$, and for $u \leqslant s$,

(C.1)

$$\begin{aligned} &E\left[\left(A_n^1(s, t) - A_n(s, t)\right) - \left(A_n^1(u, t) - A_n(u, t)\right) | \mathscr{F}^1(u)\right] \\ &\quad = -E\left[M_n^1(s, t) - M_n^1(u, t) | \mathscr{F}^1(u)\right] + E\left[M_n((u, 0), (s, t)] | \mathscr{F}^*(u, 0)\right] = 0. \end{aligned}$$

Suppose it can be shown that for each fixed $t \in (0, K]$, $\{A_n^1(\cdot, t)\}$ is tight in $C[0, K]$. If some subsequence $\{A_{n(k)}^1(\cdot, t)\}$ converges in distribution to $B \in C[0, K]$, then $(A_{n(k)}^1 - A_{n(k)})(\cdot, t) \to_{\mathscr{D}} B - A(\cdot, t)$. Using (C.1) it is straightforward to show that $B - A(\cdot, t)$ is a martingale with respect to the minimal filtration generated by B. Since both B and A are continuous, $B \equiv A(\cdot, t)$, proving (ii).

Thus, it suffices to show that $\{A_n^1(\cdot, t)\}$ is tight in $C[0, K]$. Using the one-dimensional stopping time condition for tightness due to Aldous (1978), it is easy to show that $\{A_n^1(\cdot, t)\}$ is tight in $D[0, K]$ since $\{N_n(\cdot, t)\}$ is. It remains to show that any limit must be continuous with probability one. Fix $\varepsilon > 0$. If

$$S_n = \begin{cases} \inf\left(s \colon \Delta^1 A_n^1(s, t) > \varepsilon\right) \\ \infty \quad \text{if no such } s \text{ exists} \end{cases},$$

then it is enough to show that $P(S_n \leqslant K) \to 0$ as $n \to \infty$, since ε is arbitrary. We have

$$\text{(C.2)} \qquad \begin{aligned} P(S_n \leqslant K) &\leqslant \varepsilon^{-1} E\left[\chi(S_n \leqslant K) \Delta^1 A_n^1(S_n, t)\right] \\ &= \varepsilon^{-1} E\left[\chi(S_n \leqslant K) E\left(\Delta^1 N_n(S_n, t) | \mathscr{F}_n^1(S_n -)\right)\right]. \end{aligned}$$

Using approximation arguments similar to those in Appendix A and the fact that S_n is predictable, it may be shown that for any 1-stopping time σ with respect to $\mathscr{F}_n$, with $\sigma < S_n$, $E(\Delta^1 M_n(S_n, t) | \mathscr{F}_n^1(\sigma)) = 0$, and hence that $E(\Delta^1 N_n(S_n, t) | \mathscr{F}_n^1(S_n -)) = E(\Delta^1 A_n(S_n, t) | \mathscr{F}_n^1(S_n -))$. Substituting in (C.2) we obtain, for any $\delta > 0$

$$\begin{aligned} P(S_n \leqslant K) &\leqslant \varepsilon^{-1} E\left[\chi(S_n \leqslant K) \Delta^1 A_n(S_n, t)\right] \\ &\leqslant \varepsilon^{-1} E\left[\chi(S_n \leqslant K) \chi\left(\Delta^1 A_n(S_n, t) \geqslant \delta\right)\right] + \delta/\varepsilon \to \delta/\varepsilon \qquad \text{as } n \to \infty. \end{aligned}$$

Since δ is arbitrary, $P(S_n \leqslant K) \to 0$ for all ε, proving (ii).

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