Canad. J. Math. Vol. **57** (1), 2005 pp. 61–81

On Operators with Spectral Square but without Resolvent Points

Paul Binding and Vladimir Strauss

Abstract. Decompositions of spectral type are obtained for closed Hilbert space operators with empty resolvent set, but whose square has closure which is spectral. Krein space situations are also discussed.

1 Introduction

Operators A with empty resolvent set arise in various ways. For example, in a Hilbert space $\mathfrak{H}$ a closed symmetric operator A with a proper self-adjoint extension $\tilde{A}$ is of this type [11, p. 271]. One can then shrink the spectrum from $\mathbb{C}$ to (a subset of) $\mathbb{R}$ by passing to $\tilde{A}$, and thereby obtain a spectral decomposition of $\mathfrak{H}$.

Various examples have also been given of such operators A which are already self-adjoint, but with respect to an indefinite inner product [3, p. 113], [4, p. 148], [6]. In [2] a Sturm-Liouville problem is described, leading to such an operator (see [6] for discussion). In these cases the "standard" spectral theorem, which requires nonemptiness of the resolvent [9, 5], does not apply and it is not immediately clear how to obtain decompositions of spectral type.

Further instances appear in certain applications. For example operators of the form

$$A = \begin{pmatrix} 0 & S^{-1} \\ S & 0 \end{pmatrix} \tag{1.1}$$

in $\mathfrak{H} \oplus \mathfrak{H}$, where S is self-adjoint on $\mathfrak{H}$ with $\operatorname{Ker}(S) = \{0\}$ and at least one of the operators S and S^{-1} is unbounded, arise in connection with Maxwell's equations [8]. Another example is change of independent variable, *e.g.*, $A\colon f(t) \mapsto f(1/t)$ in $L_2(\mathbb{R})$, *cf.* [10].

It turns out that most of the above examples do have enough structure to permit some decomposition of spectral type. A key condition is

(1.2) the closure C of A^2 is similar to a bounded symmetric operator.

In many of the above examples, $C = I$. In Bognar's example A is of the form

$$\begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}$$

Received by the editors March 10, 2003; revised April 5, 2004.

The first author's research was supported by NSERC of Canada and the I. W. Killam Foundation. The second author's research was supported by project CONICIT (Venezuela) No 97000668.

AMS subject classification: 47A05, 47A15, 47B40, 47B50, 46C20.

Keywords: unbounded operators, closed operators, spectral resolution, indefinite metric.

and then $C = 0$. The two block-matrix examples above are special cases of

$$A = \begin{pmatrix} 0 & (M^*)^{-1}ZM^{-1} \\ MSM^* & 0 \end{pmatrix},$$

where M is a bounded and boundedly invertible operator, S and Z are self-adjoint operators with commuting resolvents, SZ is a bounded operator, but at least one of the operators S and Z is unbounded. In this case (1.2) holds although C is not symmetric.

The above condition (1.2) on C, and another technical condition, turn out to be enough to give a decomposition of $\mathfrak{H}$ of spectral type related to A. In Section 2 we treat the case $C = I$ and we establish the existence of two orthogonal A-invariant subspaces $\mathfrak{F}$ and $\mathfrak{G}$ in which $A|_{\mathfrak{F}}$ is self-adjoint (permitting the usual decomposition) and $\mathfrak{G}$ permits a decomposition so that

$$A|_{\mathfrak{G}} = \begin{pmatrix} 0 & S^{-1}T^{-1} \\ TS & 0 \end{pmatrix}, \tag{1.3}$$

T being isometric and S positive and contracting. In Section 3 we study the general situation and find that $A|_{\mathfrak{F}}$ is normal and $A|_{\mathfrak{G}}$ takes the form

$$\begin{pmatrix} 0 & ZT^{-1} \\ TS & 0 \end{pmatrix},$$

where S and Z commute. We also discuss integral representations for some of these constructions.

In Sections 4 and 5 we admit an (indefinite) inner product, generated by a self-adjoint operator J, which makes $\mathfrak{H}$ into a Krein space, and we assume that A is J-symmetric. For example, the operator A of (1.1) is J-self-adjoint in $\mathfrak{H} \oplus \mathfrak{H}$ if

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Roughly, the results we obtain parallel those of Sections 2 and 3, but where the decompositions respect both A and J. For example, if $C = I$ then $A|_{\mathfrak{G}}$ is as in (1.3) and

$$J|_{\mathfrak{G}} = \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix}.$$

On the other hand some extra conditions are needed in general, *e.g.*, to obtain the analogue of Section 3 we require A to be J-nonnegative for the existence of a J-self-adjoint extension.

2 A Matrix Representation for a Square Root of the Identity

Let B be a closed operator acting in a Hilbert space $\mathfrak{H}$ with a dense domain $\mathfrak{D}(B)$ and such that

$$\begin{cases} \text{(i)} & \text{the closure of the restriction of } B \text{ on } \mathfrak{D}(B^2) \text{ coincides with } B; \\ \text{(ii)} & \text{the closure } C \text{ of } B^2 \text{ is the identity operator.} \end{cases} \tag{2.1}$$

https://doi.org/10.4153/CJM-2005-003-4 Published online by Cambridge University Press

Lemma 2.1 *Let B satisfy conditions* (2.1)*. Then there are closed subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ such that $\mathfrak{D}(B) = \mathfrak{L}_+ \dot{+} \mathfrak{L}_-$. In fact $\mathfrak{L}_+$ and $\mathfrak{L}_-$ can be taken as eigenspaces of B, corresponding to eigenvalues* 1 *and* -1 *respectively.*

Proof Let us consider the operators $C_+ = \frac{1}{2}(I + B)$, $C_- = \frac{1}{2}(I - B)$. Note that the closure of $C_+|_{\mathcal{D}(B^2)}$ is just C_+. Thus for any $x \in \mathcal{D}(B)$ there is a sequence $x_n \in \mathcal{D}(B^2)$ with $x_n \to x$ and $y_n := C_+x_n \to y := C_+x$. Now $C_+y_n = C_+^2x_n = C_+x_n \to y$. So since C_+ is closed, $y = C_+y = C_+^2x$. It follows that $C_+^2 = C_+$ and $By = (2C_+ - I)y = y$. Let $\mathfrak{L}_+ = C_+\mathfrak{D}(B)$. If $y_n = C_+x_n$ with $x_n \in \mathfrak{D}(B)$ and $y_n \to y$ then $C_+y_n = C_+x_n \to y$ so $y \in \mathfrak{L}_+$ since C_+ is closed. Similarly $C_-^2 = C_-$ and $\mathfrak{L}_- = C_-\mathcal{D}(B)$ is closed too, and $Bz = -z$ if $z \in \mathfrak{L}_-$. Finally, $C_+ + C_- = I$ on $\mathcal{D}(B)$, so $\mathcal{D}(B) = \mathfrak{L}_+ \dot{+} \mathfrak{L}_-$. ■

Remark 2.2 For $x \in \mathcal{D}(B)$, $x = x_+ + x_-$, $x_+ \in \mathfrak{L}_+$, $x_- \in \mathfrak{L}_-$, the equality $Bx = x_+ - x_-$ holds. It follows that

(i) $\mathcal{D}(B) = \mathcal{D}(B^2) = \mathcal{R}(B)$ and
(ii) for every complex number ξ the relation $(B - \xi I)\mathcal{D}(B) \subset \mathcal{D}(B)$ holds.

Thus if B is unbounded, then $B - \xi I$ cannot have a (closed!) bounded inverse, and the resolvent set of B must be empty.

Remark 2.3 If B satisfies condition (i) of (2.1), but instead of condition (ii) we have $C = -I$, then $\mathfrak{D}(B) = \mathfrak{L}_+ \dot{+} \mathfrak{L}_-$, where $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are the eigenspaces of B, corresponding to eigenvalues i and $-i$, respectively.

Remark 2.4 In this paper we deal with operators in Hilbert (or Krein) spaces, but Lemma 2.1 and the above Remarks can be easily generalized to linear topological spaces where the Closed Graph Theorem holds, *e.g.*, Fréchet spaces [7, p. 57].

Remark 2.5 We can also find B^* explicitly if B satisfies (2.1). By definition, for all $f \in \mathfrak{D}(B^*)$ we have $(Bx, f) = (B(x_+ + x_-), f) = ((x_+ - x_-), f) = ((x_+ + x_-), B^*f)$, where $x_+ \in \mathfrak{L}_+$ and $x_- \in \mathfrak{L}_-$. Thus $(x_+, (f - B^*f)) = 0$ and $(x_-, (f + B^*f)) = 0$ for all $x_+ \in \mathfrak{L}_+$ and $x_- \in \mathfrak{L}_-$, *i.e.*, $f_+ := (f - B^*f) \in \mathfrak{L}_+^\perp$ and $f_- := (f + B^*f) \in \mathfrak{L}_-^\perp$. Then $2f = f_+ + f_-$ and $2B^*f = f_- - f_+$.

In what follows we shall give analogues of $\mathfrak{L}_+$ and $\mathfrak{L}_-$, and more refined decompositions of spectral type, under appropriate conditions.

Lemma 2.6 *Let B satisfy conditions* (2.1)*. Then there is a decomposition $\mathfrak{H} = \mathfrak{H}_{sa} \oplus$*

https://doi.org/10.4153/CJM-2005-003-4 Published online by Cambridge University Press

$\mathfrak{H}_{nsa}$, *invariant with respect to B, and such that*

$$(2.2)\qquad \begin{cases} \bullet\ B|_{\mathfrak{H}_{sa}} \text{ is a self-adjoint operator,} \\ \bullet\ \text{there are a decomposition } \mathfrak{H}_{nsa} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, \text{ an isometric operator} \\ \quad T\colon \mathfrak{H}_1 \mapsto \mathfrak{H}_2 \text{ with } \Re(T) = \mathfrak{H}_2 \text{ and a positive self-adjoint ope-} \\ \quad \text{rator } S\colon \mathfrak{H}_1 \mapsto \mathfrak{H}_1,\ \|S\| \le 1,\ 1 \notin \sigma_p(S), \text{ such that } \mathfrak{D}(B|_{\mathfrak{H}_{nsa}}) = \\ \quad \mathfrak{H}_1 \oplus T\Re(S) \text{ and} \\ \qquad\qquad B|_{\mathfrak{H}_{nsa}} = \begin{pmatrix} 0 & S^{-1}T^{-1} \\ TS & 0 \end{pmatrix}. \end{cases}$$

Proof By Lemma 2.1, $\mathfrak{D}(B) = \mathfrak{L}_+ \dot{+} \mathfrak{L}_-$, where $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are the eigenspaces of B, corresponding to eigenvalues 1 and -1 respectively. Put $\mathfrak{H}_{sa} = (\mathfrak{L}_-^{\perp} \cap \mathfrak{L}_+) \oplus (\mathfrak{L}_+^{\perp} \cap \mathfrak{L}_-)$, $\mathfrak{H}_{nsa} = \mathfrak{H} \ominus \mathfrak{H}_{sa}$. It is easy to check that the subspace $\mathfrak{H}_{sa}$ is as required, so without loss of generality we can suppose

$$(2.3)\qquad \mathfrak{H} = \mathfrak{H}_{nsa}.$$

Let P_+ be the ortho-projector onto $\mathfrak{L}_+$, let

$$(2.4)\qquad P\colon\ \mathfrak{L}_- \mapsto \mathfrak{L}_+$$

be the restriction of P_+ to $\mathfrak{L}_-$ and let

$$(2.5)\qquad P^* = UK,$$

be the polar decomposition of P^* [11, §VI.2.7]. By (2.3), $P\mathfrak{L}_-$ is dense in $\mathfrak{L}_+$, so $\operatorname{Ker}(P^*) = \{0\}$. Since $K = (PP^*)^{1/2}$, K is a one-to-one symmetric operator on $\mathfrak{L}_+$; also U is an isometric operator from $\mathfrak{L}_+$ onto $\mathfrak{L}_-$. Now suppose $0 \neq x \in \mathfrak{L}_+$. Then $(Ux, x) = (PUx, x) = (Kx, x)$, so $\|x+Ux\|^2 = (x, x)+(Ux, x)+(x, Ux)+(Ux, Ux) = 2((I+K)x, x) > 2\|x\|^2$. Thus the corresponding lineal $\mathfrak{H}_1 := \{x + Ux\}_{x\in\mathfrak{L}_+}$ is closed [11, p. 231], and Lemma 2.1 shows that

$$(2.6)\qquad \mathfrak{H}_1 \subset \mathfrak{D}(B).$$

Similarly we find $\|B(x + Ux)\|^2 = \|x - Ux\|^2 = 2((I - K)x, x) < \|x + Ux\|^2$. In this case the lineal $\{x - Ux\}_{x\in\mathfrak{L}_+}$ need not be closed and we denote its closure by $\mathfrak{H}_2$. Thus $B|_{\mathfrak{H}_1}: \mathfrak{H}_1 \mapsto \mathfrak{H}_2$ is a strict contraction which is one-to-one (since $Ux = x \in \mathfrak{L}_+$ implies $x = 0$) and we take TS as its polar decomposition. The result follows. ■

Remark 2.7 For a given scalar product the subspaces $\mathfrak{H}_{sa}$, $\mathfrak{H}_1$, $\mathfrak{H}_2$ and the operators T, S from (2.2) are uniquely determined.

Proof First, the subspace $\mathfrak{H}_{sa}$ is defined by the formula $\mathfrak{H}_{sa} = \operatorname{Ker}(B - B^*)$. Second, we claim that the subspace $\mathfrak{H}_1$ must be representable in the form $\mathfrak{H}_1 = \{x+Wx\}_{x\in\mathfrak{L}_+}$, where $W : \mathfrak{L}_+ \mapsto \mathfrak{L}_-$ is a bounded and boundedly invertible operator. Indeed, (2.6) shows that all elements $y \in \mathfrak{H}_1$ take the form $y = x_+ + x_- \in \mathfrak{D}(B)$, where $x_+ \in \mathfrak{L}_+$,

$x_- \in \mathfrak{L}_-$. Since $By = x_+ - x_- \in \mathfrak{H}_2 \cap \mathfrak{D}(B) \Leftrightarrow y = x_+ + x_- \in \mathfrak{H}_1$, x_+ must run through all $\mathfrak{L}_+$ and x_- must run through all $\mathfrak{L}_-$. Moreover, $\mathfrak{H}_1 \cap \mathfrak{H}_2 = \{0\}$, so $x_+ = 0 \Leftrightarrow x_- = 0 \Leftrightarrow y = 0$. Thus, the relation $x_- = Wx_+ \Leftrightarrow y = x_+ + x_-$ generates a linear operator and similarly W is invertible. It is easy also to check that W is closed, so the second part is finished. Third, $\mathfrak{H}_1 \perp \mathfrak{H}_2$, therefore (with P as above) $(x + Wx, x - Wx) = (x, x) + (PWx, x) - (x, PWx) - (Wx, Wx) = 0$. Since $(x, x) - (Wx, Wx)$ is real and $(PWx, x) - (x, PWx)$ is imaginary, the operator PW is self-adjoint and W is an isometry. Thus, $P^* = W(PW)$. Finally, by (2.2), $(Sy, Sy) \le (y, y)$ for every $y = x+Wx \in \mathfrak{H}_1$ and $(y, y) = 2((I+PW)x, x)$, $(Sy, Sy) = 2((I - PW)x, x)$, so $PW > 0$. Thus, W and PW are the elements of the (uniquely determined) polar decomposition for P^*. The rest is straightforward. ■

Now let us consider another way to reconstruct the domain $\mathfrak{D}(B)$ of B satisfying conditions (2.1) by using the range of a suitable bounded operator. There is a general way to do this for any closed operator A in a Hilbert space $\mathfrak{H}$. Indeed, the operator $M := (I + (A^*A)^{1/2})^{-1}$ has the required property: $M\mathfrak{H} = \mathfrak{D}(A)$. In our particular case there is a more direct solution.

Lemma 2.8 *Let:*

- *B satisfy conditions* (2.1),
- *$\mathfrak{D}(B) = \mathfrak{L}_+ \dot{+} \mathfrak{L}_-$ as in Lemma* 2.1,
- *P_+ and P_- be ortho-projectors onto the subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ respectively and $\Theta = P_+ + P_-$.*

Then $\Theta^{1/2}\mathfrak{H} = \mathfrak{D}(B)$.

Proof Without loss of generality suppose that (2.3) holds. Let P, U and K be as in (2.4) and (2.5). Then $\Theta(I + U)|_{\mathfrak{L}_+} = (I + U)(I + K)|_{\mathfrak{L}_+}$ and $\Theta(I - U)|_{\mathfrak{L}_+} = (I - U)(I - K)|_{\mathfrak{L}_+}$. In the same way

$$\Theta^k(I + U)|_{\mathfrak{L}_+} = (I + U)(I + K)^k|_{\mathfrak{L}_+},\qquad (2.7)$$
$$\Theta^k(I - U)|_{\mathfrak{L}_+} = (I - U)(I - K)^k|_{\mathfrak{L}_+}$$

for all natural k. Since $\Theta^{1/2}$ is a limit of polynomials in Θ [12, §9.4], we obtain

$$\Theta^{1/2}(I + U)|_{\mathfrak{L}_+} = (I + U)(I + K)^{1/2}|_{\mathfrak{L}_+},\qquad (2.8)$$
$$\Theta^{1/2}(I - U)|_{\mathfrak{L}_+} = (I - U)(I - K)^{1/2}|_{\mathfrak{L}_+}$$

from (2.7). Next, we have

$$\mathfrak{D}(B) = (I + U)\mathfrak{L}_+ \oplus (I - U)\mathfrak{L}_+.\qquad (2.9)$$

Let $K = \int_0^1 \lambda dG_\lambda$, where G_λ is the spectral function of K. Put

$$M_1 \colon \mathfrak{L}_+ \mapsto \mathfrak{H}_1, \quad M_1 \colon = (I + U)\int_0^1 \frac{1}{\sqrt{2(1 + \lambda)}} dG_\lambda x\qquad (2.10)$$

https://doi.org/10.4153/CJM-2005-003-4 Published online by Cambridge University Press

and

$$M_2\colon \mathfrak{L}_+ \mapsto \mathfrak{H}_2, \quad M_2\colon = (I-U)\int_0^1 \frac{1}{\sqrt{2(1-\lambda)}} dG_\lambda x, \tag{2.11}$$

where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are from (2.2). Let us show that M_1 and M_2 are one-to-one maps. Indeed, for $y \in \mathfrak{L}_+$ we have $\|(I+U)y\|^2 = 2((I+K)y, y)$ and $\|(I-U)y\|^2 = 2((I-K)y, y)$. Then replacing y respectively by $\int_0^1 \frac{1}{\sqrt{2(1+\lambda)}}\, dG_\lambda x$ and by $\int_0^1 \frac{1}{\sqrt{2(1-\lambda)}}\, dG_\lambda x$, from (2.10) and (2.11) we have $\|M_1 x\| = \|x\|$ and $\|M_2 x\| = \|x\|$. Thus, $M_1\mathfrak{L}_+ = \mathfrak{H}_1$ and $M_2\mathfrak{L}_+ = \mathfrak{H}_2$.

Now, from (2.10) and Lemma 2.8 we have $\Theta^{1/2}\mathfrak{H}_1 = \Theta^{1/2}M_2\mathfrak{L}_+ =$

$$(I+U)(I+K)^{1/2}\int_0^1 \frac{1}{\sqrt{2(1+\lambda)}}\, dG_\lambda \mathfrak{L}_+ = \frac{1}{\sqrt{2}}(I+U)\mathfrak{L}_+, \tag{2.12}$$

so $\Theta^{1/2}\mathfrak{H}_1 = (I+U)\mathfrak{L}_+$. In the same way one can show that $\Theta^{1/2}\mathfrak{H}_2 = (I-U)\mathfrak{L}_+$. The rest is clear from (2.9). ■

Remark 2.9 Let B satisfy (2.1) and (2.3) and let $\Theta^{1/2} = \int_0^{\sqrt{2}} \mu\, dH_\mu$. Then $(I-H_1)\mathfrak{H} = \mathfrak{H}_1$.

Proof Indeed, from (2.10) and (2.12) we have that $\Theta^{1/2}\mathfrak{H}_1 = \mathfrak{H}_1$ and $\|\Theta^{1/2}y\|^2 = \|\frac{1}{\sqrt{2}}(I+U)x\|^2 = ((I+K)x, x) > \|x\|^2 = \|y\|^2$ for $0 \neq x \in \mathfrak{L}_+$, and $(I+U)\int_0^1 \frac{1}{\sqrt{2(1+\lambda)}} dG_\lambda x = y \in \mathfrak{H}_1$. In the same way one can prove that $\Theta^{1/2}\mathfrak{H}_2 = \mathfrak{H}_2$ and $\|\Theta^{1/2}y\|^2 = ((I-K)x, x) < \|x\|^2 = \|y\|^2$ for $(I-U)\int_0^1 \frac{1}{\sqrt{2(1-\lambda)}} dG_\lambda x = y \in \mathfrak{H}_2$, $0 \neq x \in \mathfrak{L}_+$. The rest is clear. ■

We conclude this section with the case where $C = I$ in (2.1)(ii) is replaced by $C = -I$.

Lemma 2.10 *Let B satisfy condition* (2.1)(i) *and let the closure C of $-B^2$ be the identity operator. Then there is a decomposition $\mathfrak{H} = \mathfrak{H}_n \oplus \mathfrak{H}_{nn}$, invariant with respect to B, and such that*

(2.13)
- *$B|_{\mathfrak{H}_n}$ is a normal operator,*
- *there are a decomposition $\mathfrak{H}_{nn} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, an isometric operator $T\colon \mathfrak{H}_1 \mapsto \mathfrak{H}_2$ with $\mathfrak{R}(T) = \mathfrak{H}_2$ and a negative self-adjoint operator $S\colon \mathfrak{H}_1 \mapsto \mathfrak{H}_1$, $\|S\| \le 1$, $1 \notin \sigma_p(S)$, such that $\mathfrak{D}(B|_{\mathfrak{H}_{nn}}) = \mathfrak{H}_1 \oplus T\mathfrak{R}(S)$ and*

$$B|_{\mathfrak{H}_{nn}} = \begin{pmatrix} 0 & ZT^{-1} \\ TS & 0 \end{pmatrix},$$

where $ZS = -I$.

Proof The operator iB satisfies (2.1), so by Lemma 2.6 we have the representation

$$iB|_{\mathfrak{H}_{nsa}} = \begin{pmatrix} 0 & S^{-1}\tilde{T}^{-1} \\ \tilde{T}S & 0 \end{pmatrix}, \text{ whence } B|_{\mathfrak{H}_{nsa}} = \begin{pmatrix} 0 & -S^{-1}(-i\tilde{T})^{-1} \\ -i\tilde{T}S & 0 \end{pmatrix}.$$

Note that $\tilde{T}$ is an isometry, so $T = -i\tilde{T}$ is also an isometry. ■

3 Operators with Squares Similar to Self-Adjoint Operators

Let B be a closed operator acting in a separable Hilbert space $\mathfrak{H}$ with a dense domain $\mathfrak{D}(B)$ and such that

$$\text{(3.1)} \qquad \begin{cases} \text{(i)} & \text{the naturally defined operator } B^2 \text{ is bounded;} \\ \text{(ii)} & \text{the closure of the restriction of } B \text{ on } \mathfrak{D}(B^2) \text{ coincides with } B; \end{cases}$$

Proposition 3.1 *If B satisfies* (3.1) *then all natural powers B^k are well defined on* $\mathfrak{D}(B)$.

Proof Our assertion means that $B\mathfrak{D}(B) \subseteq \mathfrak{D}(B)$. Let $x \in \mathfrak{D}(B)$, $y = Bx$. Thanks to condition (ii) from (3.1) there is a sequence $\{x_k\}_1^\infty$, $x_k \in \mathfrak{D}(B^2)$, such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} y_k = y$ for $y_k = Bx_k$. Then thanks to boundedness of B^2, $\lim_{k\to\infty} B^2x_k := z$ exists. Thus for $z_k := B^2x_k = By_k$ we have $\lim_{k\to\infty} y_k = y$ and $\lim_{k\to\infty} By_k = z$. Since B is closed, $y \in \mathfrak{D}(B)$. ■

In consequence we have the following (*cf.* Remark 2.2)

Corollary 3.2 *If B satisfies* (3.1) *then either B is bounded or else its resolvent set is empty.*

Proof By Proposition 3.1, $B\mathfrak{D}(B) \subseteq \mathfrak{D}(B)$, so if $\mathfrak{D}(B) \neq \mathfrak{H}$, then $(B - \xi I)\mathfrak{D}(B) \neq \mathfrak{H}$ for all $\xi \in \mathbb{C}$. ■

As before, let C be the closure of B^2, so by (3.1(i)) C is defined everywhere on $\mathfrak{H}$. Let $\mathrm{s} - \mathrm{Alg}(C)$ be the strong operator closure of $\{\mathcal{P}(C)\}$, where $\mathcal{P}(\xi)$ runs over the set of all polynomials.

Proposition 3.3 *Let $D \in \mathrm{s} - \mathrm{Alg}(C)$. Under condition* (3.1), *D commutes with B. If in addition D^{-1} exists and $D^{-1} \in \mathrm{s} - \mathrm{Alg}(C)$, then $D\mathfrak{D}(B) = \mathfrak{D}(B)$.*

Proof Let $x \in \mathfrak{D}(B)$ and $y = Bx$. Under our conditions there is a sequence $\{\mathcal{P}_n(\xi)\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \mathcal{P}_n(C)x = Dx$ and $\lim_{n\to\infty} \mathcal{P}_n(C)y = Dy$. Then since B is closed, $B\lim_{n\to\infty} \mathcal{P}_n(C)x = \lim_{n\to\infty} B\mathcal{P}_n(C)x = \lim_{n\to\infty} \mathcal{P}_n(C)Bx = \lim_{n\to\infty} \mathcal{P}_n(C)y = Dy$. So $Dx \in \mathfrak{D}(B)$ and $BDx = DBx$. Similar arguments hold for D^{-1}. ■

https://doi.org/10.4153/CJM-2005-003-4 Published online by Cambridge University Press

We now impose in addition the following condition on C from Section 1:

(3.2) The closure C of B^2 is similar to a bounded self-adjoint operator.

Equivalently, C is a scalar spectral operator with real spectrum. Let E_λ be the spectral function of C (continuous from the left in the strong topology).

Theorem 3.4 *Let B satisfy conditions* (3.1) *and* (3.2)*. Then $\mathfrak{H}$ and $\mathfrak{D}(B)$ admit the representations*

$$\mathfrak{H} = \mathfrak{H}_{im} \dot{+} \mathfrak{H}_{ni} \dot{+} \mathfrak{H}_{re},$$
$$\mathfrak{D}(B) = \Big(\mathfrak{H}_{im} \cap \mathfrak{D}(B)\Big) \dot{+} \Big(\mathfrak{H}_{ni} \cap \mathfrak{D}(B)\Big) \dot{+} \Big(\mathfrak{H}_{re} \cap \mathfrak{D}(B)\Big), \tag{3.3}$$

where the subspaces $\mathfrak{H}_{im}$, $\mathfrak{H}_{ni}$ and $\mathfrak{H}_{re}$ are invariant with respect to B and the operators $B_{im} = B|_{\mathfrak{H}_{im}}$, $B_{ni} = B|_{\mathfrak{H}_{ni}}$ and $B_{re} = B|_{\mathfrak{H}_{re}}$ have the following properties:

- $\overline{B_{im}^2}$ *has non-positive spectrum,* $B_{ni}^2 = 0$, $\overline{B_{re}^2}$ *has non-negative spectrum and* $\operatorname{Ker}(B_{im}) = \operatorname{Ker}(B_{re}) = \{0\}$;
- $\mathfrak{H}_{im} = \overline{\mathfrak{L}_{im}^+ \dot{+} \mathfrak{L}_{im}^-}$*, where the subspaces $\mathfrak{L}_{im}^+$ and $\mathfrak{L}_{im}^-$ are invariant with respect to B, $\mathfrak{L}_{im}^+ \subseteq \mathfrak{D}(B)$, $\mathfrak{L}_{im}^- \subseteq \mathfrak{D}(B)$, the spectrum of $B|_{\mathfrak{L}_{im}^+}$ belongs to the upper imaginary half-line, the spectrum of $B|_{\mathfrak{L}_{im}^-}$ belongs to the lower imaginary half-line, the operators $B|_{\mathfrak{L}_{im}^+}$ and $B|_{\mathfrak{L}_{im}^-}$ are bounded scalar spectral operators and $B_{im} = \overline{B|_{\mathfrak{L}_{im}^+} \dot{+} B|_{\mathfrak{L}_{im}^-}}$;*
- $\mathfrak{H}_{re} = \overline{\mathfrak{L}_{re}^+ \dot{+} \mathfrak{L}_{re}^-}$*, where the subspaces $\mathfrak{L}_{re}^+$ and $\mathfrak{L}_{re}^-$ are invariant with respect to B, $\mathfrak{L}_{re}^+ \subseteq \mathfrak{D}(B)$, $\mathfrak{L}_{re}^- \subseteq \mathfrak{D}(B)$, the spectrum of $B|_{\mathfrak{L}_{re}^+}$ belongs to the non-negative real half-line, the spectrum of $B|_{\mathfrak{L}_{re}^-}$ belongs to the non-positive real half-line, the operators $B|_{\mathfrak{L}_{re}^+}$ and $B|_{\mathfrak{L}_{re}^-}$ are bounded scalar spectral operators and $B_{re} = \overline{B|_{\mathfrak{L}_{re}^+} \dot{+} B|_{\mathfrak{L}_{re}^-}}$.*

Proof In what follows a concrete form of the Hilbert scalar product on $\mathfrak{H}$ is not really essential, we need to fix only the norm topology. Thanks to this remark we can change the Hilbert scalar product for a new one such that C is a self-adjoint operator and, thus, its spectral function is orthogonal. Throughout the proof we shall suppose that $(\cdot,\cdot)$ has the above-mentioned property.

Put $\mathfrak{H}_{im} = E_0\mathfrak{H}$, $\mathfrak{H}_{ni} = (E_{+0} - E_0)\mathfrak{H}$ and $\mathfrak{H}_{re} = (E_{+\infty} - E_{+0})\mathfrak{H}$. Invariance of these subspaces follows from Proposition 3.3.

The assertion of the theorem is evident for $\mathfrak{H}_{ni}$, so let us check it for $\mathfrak{H}_{re}$. Fix an arbitrary $\mu > 0$. Then the subspace $\mathfrak{H}_\mu := (I - E_\mu)\mathfrak{H}$ is invariant with respect to B and $\mathfrak{H}_\mu \subseteq \mathfrak{H}_{re}$. Let $B_\mu := B|_{\mathfrak{H}_\mu}$ and $A_\mu := B\int_\mu^{+\infty} \lambda^{-1/2} dE_\lambda|_{\mathfrak{H}_\mu}$. Then A_μ has the same domain as B_μ. Since A_μ^2 is the identity on $\mathfrak{H}_\mu$, Lemma 2.1 shows that $\mathfrak{D}(A_\mu) = \mathfrak{L}_\mu^+ \dot{+} \mathfrak{L}_\mu^-$, $A_\mu|_{\mathfrak{L}_\mu^+} = I|_{\mathfrak{L}_\mu^+}$, $A_\mu|_{\mathfrak{L}_\mu^-} = -I|_{\mathfrak{L}_\mu^-}$. So the subspaces $\mathfrak{L}_\mu^+$ and $\mathfrak{L}_\mu^-$ are invariant with respect to B_μ, $B_\mu|_{\mathfrak{L}_\mu^+} = \int_\mu^{+\infty} \lambda^{1/2} dE_\lambda|_{\mathfrak{L}_\mu^+}$ and $B_\mu|_{\mathfrak{L}_\mu^-} = -\int_\mu^{+\infty} \lambda^{1/2} dE_\lambda|_{\mathfrak{L}_\mu^-}$. Put $\mathfrak{L}_{re}^+ = \overline{\bigcup_{\mu>0} \mathfrak{L}_\mu^+}$, $\mathfrak{L}_{re}^- = \overline{\bigcup_{\mu>0} \mathfrak{L}_\mu^-}$. It is easy to show that the subspaces $\mathfrak{L}_{re}^+$ and $\mathfrak{L}_{re}^-$ are those sought. Finally note that for every $x \in \mathfrak{D}(B_{re})$ the equality $\lim_{\mu\to+0} B_{re}(I - E_\mu)x = B_{re}x$ holds. Since $(I - E_\mu)x \in \mathfrak{L}_{re}^+ \dot{+} \mathfrak{L}_{re}^-$, all properties of $\mathfrak{L}_{re}^+$ and $\mathfrak{L}_{re}^-$ have been proved.

https://doi.org/10.4153/CJM-2005-003-4 Published online by Cambridge University Press

The analysis for the operator B_{im} is the same as that for B_{re} because the spectrum of operator $(iB_{im})^2$ is non-negative. ■

For future reference let us mention the following theorem [14, Proposition 1, subsection 2, §41].

Theorem 3.5 *Let $\mathfrak{E}$ be a Hilbert space. Every bounded operator A in*

$$\mathcal{H} = L^2_\mu([0,T],\mathfrak{E}),$$

which commutes with all bounded multiplication operators, has the form $A\colon f(t) \mapsto A_t f(t)$, $f\colon [0,T] \mapsto \mathfrak{E}$, where $A_t\colon \mathfrak{E} \mapsto \mathfrak{E}$ is a μ-measurable essentially bounded operator valued function.

Proposition 3.6 *Let $\mathfrak{M}$ and $\mathfrak{N}$ be subspaces of a Hilbert space $\mathfrak{H}$ and let $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ be the corresponding ortho-projectors. Then*

(a) *the linear span of $\mathfrak{M}$ and $\mathfrak{N}$ is dense in $\mathfrak{H}$ if and only if* $\operatorname{Ker}(P_{\mathfrak{M}} + P_{\mathfrak{N}}) = \{0\}$;
(b) $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ *if and only if* $\operatorname{Ker}(P_{\mathfrak{M}}P_{\mathfrak{N}} - I) = \{0\}$.

Proof (a) Since $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ are non-negative, we have the following chain $(P_{\mathfrak{M}} + P_{\mathfrak{N}})x = 0 \Leftrightarrow P_{\mathfrak{M}}x = 0$ and $P_{\mathfrak{N}}x = 0 \Leftrightarrow x \perp \mathfrak{M} \cup \mathfrak{N}$.

(b) Note that $P_{\mathfrak{M}}P_{\mathfrak{N}}x = x$ if and only if $\|P_{\mathfrak{M}}P_{\mathfrak{N}}x\| = \|P_{\mathfrak{N}}x\| = \|x\|$. Thus $x \in \mathfrak{M}$ and similarly $x \in \mathfrak{N}$. ■

Theorem 3.7 *Let B satisfy conditions* (3.1) *and* (3.2), *the operator C have non-negative spectrum and* $\operatorname{Ker}(C) = \{0\}$. *Then one can define on $\mathfrak{H}$ a new scalar product $(\,\cdot\,,\,\cdot\,)'$ topologically equivalent to the initial scalar product and such that $\mathfrak{H}$ is represented as a direct integral*

$$\mathfrak{H} = \oplus' \int_0^{\omega+0} \mathfrak{H}_\lambda \, d\sigma(\lambda), \tag{3.4}$$

and if

$$x = \oplus' \int_0^{\omega+0} x_\lambda \, d\sigma(\lambda)$$

is a corresponding element of $\mathfrak{H}$, then

$$Bx = \oplus' \int_0^{\omega+0} \sqrt{\lambda} B_\lambda x_\lambda d\sigma(\lambda), \tag{3.5}$$

where $\omega = \|C\|$, and the operator $B_\lambda\colon \mathfrak{H}_\lambda \mapsto \mathfrak{H}_\lambda$ satisfies condition (2.1).

Proof As at the beginning of the proof of Theorem 3.4 we suppose that C is a self-adjoint operator with respect to $(\cdot,\cdot)$ and, thus, its spectral function is orthogonal.

Note that in the present case one cannot apply Theorem 3.5 directly because B is, generally speaking, an unbounded operator with empty resolvent set. So, first let us apply Theorem 3.4. In our case $\mathfrak{H}=\mathfrak{H}_{re}$, so for short we put $\mathfrak{L}^+ := \mathfrak{L}^+_{re}$ and $\mathfrak{L}^- := \mathfrak{L}^-_{re}$ and we denote by P^+ and P^- the ortho-projectors on the subspaces $\mathfrak{L}^+$ and $\mathfrak{L}^-$ respectively. Since the subspaces $\mathfrak{L}^+$ and $\mathfrak{L}^-$ are invariant with respect to E_λ, the operators P^+ and P^- commute with E_λ. Applying Theorem 3.5 to each subspace of constant multiplicity for C [7, p. 916], we have $\mathfrak{H}=\oplus\int \mathfrak{H}_\lambda\, d\sigma(\lambda)$, $\mathfrak{L}^+ = \oplus\int \mathfrak{L}^+_\lambda\, d\sigma(\lambda)$, $\mathfrak{L}^- = \oplus\int \mathfrak{L}^-_\lambda\, d\sigma(\lambda)$, $P^+ = \oplus\int P^+_\lambda\, d\sigma(\lambda)$ and $P^- = \oplus\int P^-_\lambda\, d\sigma(\lambda)$. Next, applying Proposition 3.6, we have that almost everywhere $\mathfrak{H}_\lambda = \overline{\mathfrak{L}^+_\lambda \dot{+} \mathfrak{L}^-_\lambda}$. Let us put $\mathfrak{D}(B_\lambda)=\mathfrak{L}^+_\lambda\dot{+}\mathfrak{L}^-_\lambda$, $B_\lambda|_{\mathfrak{L}^+_\lambda}=I$, $B_\lambda|_{\mathfrak{L}^-_\lambda}=-I$. The rest is clear. ■

Since from (3.5) $\|Bx\|^2 = \oplus'\int_0^{\omega+0} \lambda\|B_\lambda x_\lambda\|^2\, d\sigma(\lambda)$, we obtain

Corollary 3.8 *If the operator B satisfies the conditions of Theorem 3.7, then it is bounded if and only if*

$$\operatorname*{ess\,sup}_{\lambda>0}\{\sqrt{\lambda}\|B_\lambda\|\}<\infty.$$

We are now ready to generalize Lemma 2.10 to the present assumptions.

Theorem 3.9 *Let B satisfy conditions* (3.1) *and* (3.2). *Then one can define on $\mathfrak{H}$ a new scalar product $(\cdot,\cdot)'$ topologically equivalent to the initial scalar product and such that there is a decomposition $\mathfrak{H}=\mathfrak{H}_n\oplus'\mathfrak{H}_{nn}$, invariant with respect to B and such that*

$$\text{(3.6)}\qquad \begin{cases} \bullet\ B|_{\mathfrak{H}_n} \text{ is a bounded normal operator,}\\ \bullet\ \text{there are a decomposition } \mathfrak{H}_{nn}=\mathfrak{H}_1\oplus'\mathfrak{H}_2, \text{ an isometric operator}\\ \quad T\colon \mathfrak{H}_1\mapsto\mathfrak{H}_2, \text{ a bounded self-adjoint operator } S\colon \mathfrak{H}_1\mapsto\mathfrak{H}_1 \text{ and}\\ \quad \text{a positive (maybe unbounded) self-adjoint operator } Z\colon \mathfrak{H}_1\mapsto\mathfrak{H}_1,\\ \quad ZS=SZ \text{ and } (|S|x,x)<(Zx,x) \text{ for all } x\neq 0 \text{ on } \mathfrak{D}(Z),\ \|ZS\|<\\ \quad \infty, \text{ such that}\\ \qquad\qquad B|_{\mathfrak{H}_{nn}}=\begin{pmatrix} 0 & ZT^{-1}\\ TS & 0\end{pmatrix}. \end{cases}$$

Proof According to Theorem 3.4 we have the representation $\mathfrak{H}=\mathfrak{H}_{im}\dot{+}\mathfrak{H}_{ni}\dot{+}\mathfrak{H}_{re}$. Note also that we can find on $\mathfrak{H}$ a new scalar product $(\cdot,\cdot)'$ topologically equivalent to initial one and such that the operator C is self-adjoint. In this case the mentioned above representation takes a form $\mathfrak{H}=\mathfrak{H}_{im}\oplus'\mathfrak{H}_{ni}\oplus'\mathfrak{H}_{re}$. For simplicity we shall omit the symbol "$'$". We shall construct the required representation of B for each of subspaces $\mathfrak{H}_{im}$, $\mathfrak{H}_{ni}$ and $\mathfrak{H}_{re}$ separately.

First, let us consider $\mathfrak{H}_{re}$. Put $\mathfrak{H}^{sa}_{re}=\operatorname{Ker}(B_{re}-B^*_{re})$, $\mathfrak{H}^{nsa}_{re}=\mathfrak{H}_{re}\ominus\mathfrak{H}^{sa}_{re}$.

Let $\mathfrak{H}^{nsa}_{re}=\overline{\mathfrak{L}^+_{re}\dot{+}\mathfrak{L}^-_{re}}$ be the decomposition that corresponds to the third item of Theorem 3.4. Let P_+ and P_- be the ortho-projectors onto the subspaces $\mathfrak{L}^+_{re}$ and $\mathfrak{L}^-_{re}$, respectively, and write the operator (*cf.* Lemma 2.8) $\Theta := (P_+ + P_-)^{1/2}$ in the form $\Theta=\int_0^{\sqrt{2}}\mu\, dH_\mu$. Note that $\Theta C = C\Theta$. Next, arguing as for Theorem 3.7

and using the spectral function $E_\lambda|_{\mathfrak{H}_{re}^{nsa}}$ one can represent $\mathfrak{H}_{re}^{nsa}$ as a direct integral $\mathfrak{H}_{re}^{nsa} = \oplus\int_0^{\|C\|+0} \mathfrak{H}_\lambda \, d\sigma(\lambda)$. Note that the H_μ commute with C and by Theorem 3.5 we have the corresponding operator representation $H_\mu = \oplus\int (H_\mu)_\lambda \, d\sigma(\lambda)$.

Now (*cf.* Remark 2.9) let $\mathfrak{H}_1^{re} = (I - H_1)\mathfrak{H}_{re}^{nsa}$, $\mathfrak{H}_2^{re} = H_1\mathfrak{H}_{re}^{nsa}$ and $(\mathfrak{H}_1)_\lambda = (I - H_1)_\lambda\mathfrak{H}_\lambda$, $(\mathfrak{H}_2)_\lambda = (H_1)_\lambda\mathfrak{H}_\lambda$. Then $\mathfrak{H}_1^{re} = \oplus\int_0^{\|C\|+0} (\mathfrak{H}_1)_\lambda \, d\sigma(\lambda)$. By Lemma 2.6 we have the following representation

$$B_\lambda = \begin{pmatrix} 0 & S_\lambda^{-1}T_\lambda^{-1} \\ T_\lambda S_\lambda & 0 \end{pmatrix}$$

with respect to the decomposition $\mathfrak{H}_\lambda = (\mathfrak{H}_1)_\lambda \oplus (\mathfrak{H}_2)_\lambda$. Now, taking into account the representation (3.5), for the decomposition $\mathfrak{H}_{re}^{nsa} = \mathfrak{H}_1^{re} \oplus \mathfrak{H}_2^{re}$ we can write

$$B|_{\mathfrak{H}_{re}^{nsa}} = \begin{pmatrix} 0 & Z_{re}T_{re}^{-1} \\ T_{re}S_{re} & 0 \end{pmatrix},$$

where

$$T_{re} = \oplus\int_0^{\|C\|+0} T_\lambda \, d\sigma(\lambda),$$

$$S_{re} = \oplus\int_0^{\|C\|+0} \sqrt{\lambda} S_\lambda \, d\sigma(\lambda),$$

$$Z_{re} = \oplus\int_0^{\|C\|+0} \sqrt{\lambda} S_\lambda^{-1} \, d\sigma(\lambda).$$

Note that $Z_{re} > S_{re} > 0$. The construction for $B_{re} := B|_{\mathfrak{H}_{re}\cap\mathfrak{D}(B)}$ is complete.

For B_{im} put $\mathfrak{H}_{im}^n = \operatorname{Ker}(B_{im} + B_{im}^*)$, $\mathfrak{H}_{im}^{nn} = \mathfrak{H}_{im} \ominus \mathfrak{H}_{im}^n$. The rest of the construction for B_{im}, taking in account the representation (2.13), is similar to the corresponding steps for B_{re}. As a result we obtain the decomposition $\mathfrak{H}_{im}^{nn} = \mathfrak{H}_1^{im} \oplus \mathfrak{H}_2^{im}$ and the corresponding matrix representation

$$B|_{\mathfrak{H}_{im}^{nn}} = \begin{pmatrix} 0 & Z_{im}T_{im}^{-1} \\ T_{im}S_{im} & 0 \end{pmatrix}$$

with $Z_{im} > -S_{im} > 0$.

Now let us consider $B_{ni} := B|_{\mathfrak{H}_{ni}\cap\mathfrak{D}(B)}$. Note that $\mathfrak{R}(B_{ni}) \subseteq \operatorname{Ker}(B_{ni})$ and put $\mathfrak{H}_{ni}^n = \{\mathfrak{R}(B_{ni})\}^\perp \cap \operatorname{Ker}(B_{ni})$, $\mathfrak{H}_{ni}^{nn} = \mathfrak{H}_{ni} \ominus \mathfrak{H}_{ni}^n$, $\mathfrak{H}_1^{ni} = \operatorname{Ker}(B_{ni}) \cap \mathfrak{H}_{ni}^{nn}$, $\mathfrak{H}_2^{ni} = \mathfrak{H}_{ni}^{nn} \ominus \mathfrak{H}_1^{ni}$. Then the polar representation of the operator $(B|_{\mathfrak{H}_{ni}^{nn}})^*$ has with respect to the decomposition $\mathfrak{H}_{ni}^{nn} = \mathfrak{H}_1^{ni} \oplus \mathfrak{H}_2^{ni}$ the following matrix form

$$(B|_{\mathfrak{H}_{ni}^{nn}})^* = \begin{pmatrix} 0 & 0 \\ T_{ni}Z_{ni} & 0 \end{pmatrix},$$

where T is an invertible isometry and $Z_{ni} > 0$, so

$$B|_{\mathfrak{H}_{ni}^{nn}} = \begin{pmatrix} 0 & Z_{ni}T_{ni}^{-1} \\ 0 & 0 \end{pmatrix}. \tag{3.7}$$

Now put $\mathfrak{H}_n = \mathfrak{H}_{re}^{sa} \oplus \mathfrak{H}_{im}^{n} \oplus \mathfrak{H}_{ni}^{n}$, $\mathfrak{H}_{nn} = \mathfrak{H}_{re}^{nsa} \oplus \mathfrak{H}_{im}^{nn} \oplus \mathfrak{H}_{ni}^{nn}$, $\mathfrak{H}_1 = \mathfrak{H}_1^{re} \oplus \mathfrak{H}_1^{im} \oplus \mathfrak{H}_1^{ni}$, $\mathfrak{H}_2 = \mathfrak{H}_2^{re} \oplus \mathfrak{H}_2^{im} \oplus \mathfrak{H}_1^{ni}$. The rest is clear. ■

Corollary 3.10 *If an operator B satisfies the conditions* (3.1) *and* (3.2) *then one can define on $\mathfrak{H}_{nn}$ a Krein space structure such that $B|_{\mathfrak{H}_{nn}}$ is a J-self-adjoint operator.*

Proof Define J by the formula

$$J|_{\mathfrak{H}_{nn}} = \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & T^{*} \\ T & 0 \end{pmatrix}. \tag{3.8}$$

■

4 A Canonical Form for a J-Positive Square Root of the Identity

In Corollary 3.10 a Krein space structure is defined corresponding to the operator B. In this section and in the next one we shall consider a different situation where a Krein space $\mathfrak{H}$ is already given with a fixed indefinite inner product generating various topologically equivalent Hilbert scalar products.

First let us consider an operator B with a domain $\mathcal{D}(B) \subset \mathfrak{H}$, such that

$$\begin{cases} \text{(a)} & B \text{ is a closed } J\text{-symmetric operator,} \\ \text{(b)} & \overline{\mathcal{D}(B)} = \mathfrak{H}, \\ \text{(c)} & \overline{B|_{\mathcal{D}(B^2)}} = B, \\ \text{(d)} & \text{the closure } C \text{ of } B^2 \text{ is the identity operator.} \end{cases} \tag{4.1}$$

Our goal is to study the properties of B. Since a J-symmetric operator has eigenvectors that are J-orthogonal when they correspond to different real eigenvalues, we have the following result.

Proposition 4.1 *If the operator B satisfies conditions* (4.1) *then its subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ from Lemma* 2.1 *are J-orthogonal.*

Similarly to Remark 2.5 we have the following result.

Proposition 4.2 *Let B satisfy the conditions* (4.1) *and let $B^{[*]}$ be the operator J-adjoint to B. Then $B^{[*]}$ is described by the conditions*

$$\begin{cases} \bullet\ \mathcal{D}(B^{[*]}) = \mathfrak{L}_+^{[\perp]} \dot{+} \mathfrak{L}_-^{[\perp]}, \\ \bullet\ B^{[*]}|_{\mathfrak{L}_+^{[\perp]}} = -I|_{\mathfrak{L}_+^{[\perp]}},\ B^{[*]}|_{\mathfrak{L}_-^{[\perp]}} = I|_{\mathfrak{L}_-^{[\perp]}}. \end{cases} \tag{4.2}$$

Corollary 4.3 *The operator B from Proposition* 4.2 *is J-self-adjoint if and only if $\mathfrak{L}_- = \mathfrak{L}_+^{[\perp]}$.*

Corollary 4.4 *The operator B from Proposition* 4.2 *is either J-self-adjoint, or else has at least one J-self-adjoint extension $\tilde{B}$ which satisfies conditions* (4.1) *and is described in the following way*

https://doi.org/10.4153/CJM-2005-003-4 Published online by Cambridge University Press

- $\tilde{\mathfrak{L}}_- = \mathfrak{L}_+^{[\perp]}$,
- $\mathcal{D}(\tilde{B}) = \mathfrak{L}_+ [\dot{+}] \tilde{\mathfrak{L}}_-$,
- $\tilde{B}|_{\mathfrak{L}_+} = I|_{\mathfrak{L}_+}$, $\tilde{B}|_{\tilde{\mathfrak{L}}_-} = -I|_{\tilde{\mathfrak{L}}_-}$.

Next let us consider an operator B with a domain $\mathcal{D}(B) \subset \mathfrak{H}$, satisfying (4.1) and such that

$$B \text{ is } J\text{-positive.} \tag{4.3}$$

Our aim is to find a canonical form for B.

Note that for $0 \neq x \in \mathfrak{L}_+$ and $0 \neq y \in \mathfrak{L}_-$ we have $[x, x] = [Bx, x] > 0$ and $[y, y] = -[By, y] < 0$, so we have the following analogue of Lemma 2.1 and the ensuing remarks.

Lemma 4.5 *A closed densely defined operator B satisfies conditions* (4.1) *and* (4.3) *if and only if there are two subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ such that*

$$\begin{cases} \bullet\ \mathfrak{L}_+ \textit{ and } \mathfrak{L}_- \textit{ are, repectively, positive and negative subspaces,} \\ \quad \mathfrak{L}_+[\perp]\mathfrak{L}_-, \\ \bullet\ \mathcal{D}(B) = \mathfrak{L}_+[\dot{+}]\mathfrak{L}_-, \\ \bullet\ B|_{\mathfrak{L}_+} = I|_{\mathfrak{L}_+},\ B|_{\mathfrak{L}_-} = -I|_{\mathfrak{L}_-}. \end{cases} \tag{4.4}$$

Moreover, under these conditions $\mathcal{D}(B) = \mathcal{D}(B^2) = \mathcal{R}(B)$.

Remark 4.6 The spaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ from (4.4) can be non-maximal definite subspaces.

On the other hand, as in Corollary 4.3 we have

Corollary 4.7 *An operator B satisfying conditions* (4.1) *and* (4.3) *is J-self-adjoint if and only if $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are maximal definite subspaces.*

See [3] for analogous situations. Moreover, as for Corollary 4.4 we have

Corollary 4.8 *An operator B satisfying conditions* (4.1) *and* (4.3) *is either J-self-adjoint, or has at least one J-self-adjoint extension which also satisfies conditions* (4.1) *and* (4.3).

Remark 4.9 The existence of a J-self-adjoint J-positive extension for a J-positive but J-non-self-adjoint operator is a simple corollary of the corresponding result for Hilbert space positive operators but this does not automatically imply condition (4.1)(c). As an example, let S be a positive non-self-adjoint operator with dense range and with two different self-adjoint positive extensions $\widehat{S}$ and $\widetilde{S}$. Then the operator (1.1) satisfies (4.1) but its J-positive J-self-adjoint extension

$$\begin{pmatrix} 0 & \widehat{S}^{-1} \\ \widetilde{S} & 0 \end{pmatrix}$$

does not.

https://doi.org/10.4153/CJM-2005-003-4 Published online by Cambridge University Press

During the rest of this section we shall suppose that the operator B acting in the Krein space $\mathfrak{H}$ satisfies conditions (4.1) and (4.3) and is J-self-adjoint, so the subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are maximal definite subspaces. We denote a canonical decomposition of the Krein space by $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$.

Note that for the maximal positive subspace $\mathfrak{L}_+$ there exists a unique operator $K\colon \mathfrak{H}_+ \mapsto \mathfrak{H}_-$, such that $\|Kx\| < \|x\|$ for every $x \neq 0$, $\mathfrak{L}_+ = \{x + Kx\}_{x\in\mathfrak{H}_+}$. K is usually called an angular operator (for $\mathfrak{L}_+$). If the positive subspace $\mathfrak{L}_+$ has angular operator K, then the negative subspace $\mathfrak{L}_+^{[\perp]}$ has angular operator $K^*\colon \mathfrak{H}_- \mapsto \mathfrak{H}_+$.

Theorem 4.10 *The space $\mathfrak{H}$ can be represented in the form $\mathfrak{H} = \mathfrak{H}_b[+]\mathfrak{H}_u = \mathfrak{H}_b \oplus \mathfrak{H}_u$, invariant with respect to B, and such that*

$$
\left\{
\begin{array}{l}
\bullet\ \|B|_{\mathfrak{H}_b}\| = 1, \\
\bullet\ \textit{there are a decomposition } \mathfrak{H}_u = \mathfrak{H}_1 \oplus \mathfrak{H}_2, \textit{ an isometric operator} \\
\quad T\colon \mathfrak{H}_1 \mapsto \mathfrak{H}_2 \textit{ and a positive self-adjoint operator } S\colon \mathfrak{H}_1 \mapsto \mathfrak{H}_1, \\
\quad \|S\| \le 1, \textit{ such that} \\
\qquad B|_{\mathfrak{H}_u} = \begin{pmatrix} 0 & S^{-1}T^{-1} \\ TS & 0 \end{pmatrix}, \quad J|_{\mathfrak{H}_u} = \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix}.
\end{array}
\right.
\tag{4.5}
$$

Proof Put $\mathfrak{H}_b = \operatorname{Ker} K \oplus \operatorname{Ker} K^*$ and $\mathfrak{H}_u = \mathfrak{H} \ominus \mathfrak{H}_b$. It is easy to see that $\mathfrak{H}_b$ is invariant with respect to B. So without loss of generality we can suppose that

$$
\operatorname{Ker} K = \operatorname{Ker} K^* = \{0\}. \tag{4.6}
$$

Under condition (4.6) the operators K and K^* have the representations

$$
K = UK_+ \quad \text{and} \quad K^* = K_+U^{-1} \tag{4.7}
$$

respectively, where $K_+ = (K^*K)^{1/2}$ is a positive self-adjoint operator acting on the space $\mathfrak{H}_+$ and U is an isometric operator mapping the space $\mathfrak{H}_+$ onto the space $\mathfrak{H}_-$ [11, §VI.2.7]. Then $K = \int_0^1 \lambda\, dUQ_\lambda$, where Q_λ is the spectral function of the operator K_+.

Let $x \in \bigcup_{0<\epsilon<1} Q_\epsilon \mathfrak{H}_+$ and $y = \int_0^1 \phi(\lambda)\, dQ_\lambda x$, where $\phi(\lambda) = 1/(1-\lambda^2)$. Then $Ky = \int_0^1 \chi(\lambda)U\, dQ_\lambda x$ and $K^*Ky = \int_0^1 \psi(\lambda)\, dQ_\lambda x$, where $\chi(\lambda) = \lambda\phi(\lambda)$ and $\psi(\lambda) = \lambda^2\phi(\lambda)$. Note that $x = y + Ky - Ky - K^*Ky$ and an analogous representation is true for corresponding elements from $\mathfrak{H}_-$. Moreover $y + Ky = \int_0^1 \phi(\lambda)\, dQ_\lambda x + \int_0^1 \chi(\lambda)\, dUQ_\lambda x \in \mathfrak{L}_+$ and $K^*Ky + Ky = \int_0^1 \psi(\lambda)\, dQ_\lambda x + \int_0^1 \chi(\lambda)\, dUQ_\lambda x \in \mathfrak{L}_-$.

Hence we have

$$
\begin{aligned}
B(x+Ux) = \int_0^1 \phi(\lambda)\,dQ_\lambda x &+ \int_0^1 \chi(\lambda)\,dUQ_\lambda x + \int_0^1 \psi(\lambda)\,dQ_\lambda x \\
&+ \int_0^1 \chi(\lambda)\,dUQ_\lambda x - \int_0^1 \phi(\lambda)\,dUQ_\lambda x - \int_0^1 \chi(\lambda)\,dQ_\lambda x \\
&- \int_0^1 \psi(\lambda)\,dUQ_\lambda x - \int_0^1 \chi(\lambda)\,dQ_\lambda x \\
= \int_0^1 \frac{1-\lambda}{1+\lambda}\,d(I-U)Q_\lambda x.
\end{aligned}
$$

Similarly one sees that $B(x-Ux) = \int_0^1 \frac{1+\lambda}{1-\lambda}\,d(I+U)Q_\lambda x$, so we can put $\mathfrak{H}_1 = \{x+Ux\}_{x\in\mathfrak{H}_+}$, $\mathfrak{H}_2 = \{x-Ux\}_{x\in\mathfrak{H}_+}$, $T(x+Ux) = (x-Ux)$ and $S(x+Ux) = \int_0^1 \frac{1-\lambda}{1+\lambda}\,d(I+U)Q_\lambda x$. ■

5 J-Non-Negative Operators with Spectral Squares

We now weaken the conditions of the previous section, specifically (4.1)(d) and (4.3).

Proposition 5.1 *Let an operator B be J-non-negative and satisfy conditions* (3.1) *and* (3.2)*. Then C has non-negative spectrum.*

Proof Let us suppose the contrary. Let E_λ be the spectral function of C. Then there is an interval $\Delta \subset (-\infty, 0)$ such that $0 \neq E(\Delta)$. Since B is J-symmetric, C is J-symmetric too, but the last operator is bounded and defined on all $\mathfrak{H}$, so it is J-self-adjoint. Thus $E(\Delta)$ is J-self-adjoint too. Note that $D := \int_\Delta 1/\sqrt{|\lambda|}\,dE_\lambda \in \mathrm{s}-\mathrm{Alg}(C)$ so $BD = DB$. Put $\check{\mathfrak{H}} := E(\Delta)\mathfrak{H}$, $\check{B} := BD|_{\check{\mathfrak{H}}}$. Then $\check{B}$ is a J-positive operator in $\check{\mathfrak{H}}$ and $\overline{\check{B}^2} = -I$. Then (see Remark 2.3) $\mathcal{D}(\check{B}) = \check{\mathfrak{L}}_+ \dot{+} \check{\mathfrak{L}}_-$, $\check{B}|_{\check{\mathfrak{L}}_+} = iI$, $\check{B}|_{\check{\mathfrak{L}}_-} = -iI$. The last, however, is impossible for a J-non-negative operator even if we take into account only its linear algebraic properties. ■

Note that Proposition 5.1 can also be proved for arbitrary J-non-negative operators B with non-empty resolvent set. First, if an operator B is J-non-negative and its resolvent set $\rho(B)$ is non-empty then B is J-self-adjoint. Indeed, let us suppose the contrary, *i.e.*, B is J-non-self-adjoint and there is λ such that $(B-\lambda I)\mathfrak{D}(B) = \mathfrak{H}$. Without loss of generality we can suppose that $\lambda \notin \mathbb{R}$. Now let $\tilde{B}$ be a J-self-adjoint J-non-negative extension of B and let $x_0 \in \mathfrak{D}(\tilde{B})$ but $x_0 \notin \mathfrak{D}(B)$. Put $y = \tilde{B}x_0$. Thanks to the hypothesis $\lambda \in \rho(B)$, there is $x_1 \in \mathfrak{D}(B)$ such that $y = (B-\lambda I)x_1$. Then $x_0 - x_1$ is an eigenvector of $\tilde{B}$ corresponding to λ, but a J-non-negative operator cannot have a nonreal eigenvalue. So B is J-self-adjoint and the rest of Proposition 5.1 follows from Langer's integral representation for J-non-negative J-self-adjoint operators [9]. In Proposition 5.1, however, the resolvent set can be empty.

Corollary 4.8 depends on choosing a maximal pair of subspaces whose existence is well-known. The analogous result in the present context is rather more delicate and uses a maximal pair result that we defer to the Appendix.

Theorem 5.2 *Let an operator B be J-non-negative and satisfy conditions* (3.1) *and* (3.2)*. Then B has a J-self-adjoint extension $\tilde{B}$ which satisfies the same conditions.*

Proof First, let us consider the case when the kernel of B is trivial. Then from Theorem 3.4 we have $\mathfrak{H} = \mathfrak{H}_{re}$. Put $\mathfrak{L}_+ = \mathfrak{L}^+_{re}$, $\mathfrak{L}_- = \mathfrak{L}^-_{re}$. Without loss of generality (if necessary changing the Hilbert scalar product as indicated earlier) we can suppose that C is self-adjoint. Then J commutes with C and with its spectral function E_λ.

Let us fix $\lambda > 0$ and consider the compression $B_{\langle\lambda\rangle} := (I - E_\lambda)B|_{\mathfrak{H}_\lambda}$ where $\mathfrak{H}_\lambda = (I - E_\lambda)\mathfrak{H}$. We next show how to extend $B_{\langle\lambda\rangle}$ to a J-self-adjoint operator. Let $D_{\langle\lambda\rangle} := FB_{\langle\lambda\rangle}$ where

$$F = \int_\lambda^{\|C\|+0} \frac{1}{\sqrt{\mu}}\, dE_\mu.$$

Using Proposition 3.3 and Lemma 4.5 on $B_{\langle\lambda\rangle}$, one can check that $D^2_{\langle\lambda\rangle}x = x$ for all $x \in \mathfrak{D}(B_{\langle\lambda\rangle})$ and thus we can apply Lemma 4.5 to $D_{\langle\lambda\rangle}$ to give an orthogonal definite pair $\{\mathfrak{L}_+, \mathfrak{L}_-\}$. In the Appendix it is shown that there is a maximal extension $\{\tilde{\mathfrak{L}}_+, \tilde{\mathfrak{L}}_-\}$ (the so-called extension with nullified complement), with the following property. If E is a projector which is simultaneously orthogonal and J-orthogonal and the subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are invariant with respect to E, then the subspaces $\tilde{\mathfrak{L}}_+$ and $\tilde{\mathfrak{L}}_-$ are also invariant with respect to E. If we take the extension of $D_{\langle\lambda\rangle}$ to $\tilde{\mathfrak{L}}_+ + \tilde{\mathfrak{L}}_-$ given by $\tilde{D}_{\langle\lambda\rangle}x_\pm = \pm x_\pm$ for $x_\pm \in \tilde{\mathfrak{L}}_\pm$, then the operator

$$\tilde{B}_{\langle\lambda\rangle} = F^{-1}\tilde{D}_{\langle\lambda\rangle} = \tilde{D}_{\langle\lambda\rangle}F^{-1}|_{\mathfrak{H}_\lambda}$$

is a J-self-adjoint extension of $B_{\langle\lambda\rangle}$.

Now put $\mathfrak{D}(\hat{B}) = \bigcup_{\lambda>0} \mathfrak{D}(\tilde{B}_{\langle\lambda\rangle})$ and $\hat{B}x = \tilde{B}_{\langle\lambda\rangle}x$ for $x \in \mathfrak{D}(\tilde{B}_{\langle\lambda\rangle})$. Finally let us write $\tilde{B}$ for the closure of $\hat{B}$. Let $y \in \mathfrak{D}(\tilde{B}^{[*]})$, where the operator $\tilde{B}^{[*]}$ is J-adjoint to $\tilde{B}$, and $\tilde{B}^{[*]}y := y^{[*]}$. Then for all $x \in \mathfrak{D}(\tilde{B})$ we have $[\tilde{B}(I-E_\lambda)x, y] = [(I-E_\lambda)x, y^{[*]}] = [x, (I - E_\lambda)y^{[*]}]$. By construction the operator $\tilde{B}(I - E_\lambda)$ is J-self-adjoint, so $\tilde{B}(I - E_\lambda)y = (I - E_\lambda)y^{[*]}$. But $y = \lim_{\lambda\to+0}(I - E_\lambda)y$ and $y^{[*]} = \lim_{\lambda\to+0}(I - E_\lambda)y^{[*]}$, so $\tilde{B}y = y^{[*]}$ and hence $\tilde{B}$ is J-self-adjoint.

Next we establish (3.1) and (3.2). If $x \in \mathfrak{D}(\tilde{B})$ and $y = \tilde{B}x$ then there is a sequence $x_n \in \mathfrak{D}(\hat{B})$ so that $x_n \to x$ and $y_n := \hat{B}x_n \to y$. We say that an operator A has property P if $\mathfrak{D}(A^2) = \mathfrak{D}(A)$, so by Lemma 4.5, $\tilde{D}_{\langle\lambda\rangle}$ has property P. Since F and F^{-1} are bounded, $\tilde{B}_{\langle\lambda\rangle}$, and hence $\hat{B}$, also have property P. Using Proposition 3.3 we then conclude that

$$\hat{B}^2u = F^{-2}u = Cu \tag{5.1}$$

for all $u \in \mathfrak{D}(\hat{B})$, so $\hat{B}^2x_n$ converges to a limit z, say, and $\hat{B}y_n \to z$. Thus $y \in \mathfrak{D}(\tilde{B})$ and $z = \tilde{B}y = \tilde{B}^2x$. It follows that $\tilde{B}$ has property P, and (3.1)(ii) is proven. From (5.1) we have $\hat{B}^2x_n = Cx_n \to Cx$ and the remaining contentions follow easily.

Finally we consider the general case. Under the hypothesis of Theorem 5.2, $\mathfrak{H} = \mathfrak{H}_{ni}[+]\mathfrak{H}_{re}$ and both subspaces are B-invariant. So we need only to show that $B|_{\mathfrak{H}_{ni}}$ has a J-self-adjoint extension of the desired type. But the last operator has the representation, *cf.* (3.7),

$$B|_{\mathfrak{H}_{ni}} = 0 \oplus \begin{pmatrix} 0 & Z_{ni}T_{ni}^{-1} \\ 0 & 0 \end{pmatrix}$$

with positive Z_{ni}. Since Z_{ni} has a positive self-adjoint extension the rest is clear. ■

We can now give a matrix representation for operators satisfying (3.1) and (3.2) (or at least for the extensions guaranteed by Theorem 5.2).

Theorem 5.3 *Let an operator B be J-non-negative J-self-adjoint and satisfy conditions* (3.1) *and* (3.2)*. Then there exist a suitable Hilbert scalar product and a decomposition $\mathfrak{H} = \mathfrak{H}_{sa} \oplus \mathfrak{H}_{nsa}$ with $\mathfrak{H}_{sa}[\perp]\mathfrak{H}_{nsa}$, such that $B|_{\mathfrak{H}_{sa}}$ is self-adjoint, and $B|_{\mathfrak{H}_{nsa}}$ and $J|_{\mathfrak{H}_{nsa}}$ simultaneously have matrix representations* (3.6) *and* (3.8).

Proof Note that the difference $B-B^*$ is J-self-adjoint, so $\operatorname{Ker}(B-B^*)$ and $(B-B^*)\mathfrak{H}$ are simultaneously orthogonal and J-orthogonal. Taking into account Theorem 4.10 and (if necessary) changing the Hilbert scalar product, the reasoning can be obtained as for Theorem 3.9. ■

We conclude this section with a partial converse to Proposition 5.1.

Theorem 5.4 *Let an operator B be J-symmetric and satisfy conditions* (3.1)*, and* (3.2) *and let C have nonnegative spectrum and trivial kernel. Then B has a J-self-adjoint extension that also satisfies* (3.1) *and* (3.2).

Proof The first steps, including the construction of the operator $D_{\langle\lambda\rangle}$, are the same as for Theorem 5.2. Now let us take the J-self-adjoint extension of $D_{\langle\lambda\rangle}$ corresponding to Corollary 4.4. The rest follows the proof of Theorem 5.2. ■

6 A Canonical Form for a J-Symmetric Root of Minus Identity

In the previous two sections we studied J-symmetric operators B of certain types (*e.g.*, with nonnegative C) that accept J-self-adjoint extensions of desired types. Now we turn to a different case, where J-self-adjoint extensions are not guaranteed.

Let us consider an operator B with a domain $\mathcal{D}(B) \subset \mathfrak{H}$, such that

$$\text{(6.1)} \qquad \begin{cases} \bullet\ B \text{ is a closed } J\text{-symmetric operator,} \\ \bullet\ \overline{\mathcal{D}(B)} = \mathfrak{H}, \\ \bullet\ \overline{B|_{\mathcal{D}(B^2)}} = B, \\ \bullet\ \text{the closure of } B^2 \text{ is minus the identity operator.} \end{cases}$$

Our aim is to find a canonical form for B.

Let us redefine $C_+ := \frac{1}{2}(I - iB)$, $C_- := \frac{1}{2}(I + iB)$. Then using the same reasoning as in Lemma 2.1 we have $C_+^2 = C_+$, $C_-^2 = C_-$, the lineals $\mathfrak{L}_0^+ := C_+\mathcal{D}(B)$,

$\mathfrak{L}_0^- := C_-\mathcal{D}(B)$ are closed and $B|_{\mathfrak{L}_0^\pm} = \pm iI$. Hence for $0 \neq x \in \mathfrak{L}_0^+$ and $0 \neq y \in \mathfrak{L}_0^-$ we have $[x, x] = 0$ and $[y, y] = 0$, and in this case Lemma 2.1 takes the following form.

Theorem 6.1 *An operator B satisfies conditions* (6.1) *if and only if there exist two subspaces $\mathfrak{L}_0^+$ and $\mathfrak{L}_0^-$ such that*

$$(6.2) \qquad \begin{cases} \bullet\ \mathfrak{L}_0^+ \text{ and } \mathfrak{L}_0^- \text{ are neutral subspaces, } \mathfrak{L}_0^+ \cap \mathfrak{L}_0^- = \{0\}, \\ \bullet\ \mathcal{D}(B) = \mathfrak{L}_0^+ \dot{+} \mathfrak{L}_0^-, \\ \bullet\ B|_{\mathfrak{L}_0^+} = iI|_{\mathfrak{L}_0^+},\ B|_{\mathfrak{L}_0^-} = -iI|_{\mathfrak{L}_0^-}. \end{cases}$$

Remark 6.2 If an operator B satisfies conditions (6.1) then it is J-self-adjoint if and only if simultaneously $(\mathfrak{L}_0^+)^{[\perp]} = \mathfrak{L}_0^+$ and $(\mathfrak{L}_0^-)^{[\perp]} = \mathfrak{L}_0^-$.

We leave open the extension problem for a dense pair of neutral subspaces, but we shall consider a particular case of this problem.

Lemma 6.3 *Let $\mathfrak{H} = \overline{\mathfrak{M} \dot{+} \mathfrak{N}}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are neutral subspaces and $\mathfrak{M}^{[\perp]} = \mathfrak{M}$. Then $\mathfrak{N}$ has a representation $\mathfrak{N} = \{x + i\Gamma x\}_{x \in \mathcal{D}(\Gamma)}$, where $\Gamma\colon J\mathfrak{M} \mapsto \mathfrak{M}$ is a linear operator with dense domain and $J\Gamma\colon J\mathfrak{M} \mapsto J\mathfrak{M}$ is a symmetric operator. Moreover $\mathfrak{N}^{[\perp]} = \mathfrak{N}$ if and only if $J\Gamma$ is a self-adjoint operator.*

Proof Thanks to the condition $\mathfrak{M}^{[\perp]} = \mathfrak{M}$ we have $\mathfrak{H} = \mathfrak{M} \oplus J\mathfrak{M}$. Let P be the ortho-projector onto $\mathfrak{M}$. Then for all $x \in \mathfrak{N}$ we have $x = Px + (I - P)x$. Note that the relation $(I - P)x \mapsto Px \Leftrightarrow x \in \mathfrak{N}$ generates a linear operator because $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ and it is closed. Put $\mathcal{D}(\Gamma) := \{y = (I - P)x\}_{x \in \mathfrak{N}}$, $\Gamma y := -iPx$, giving $\mathfrak{N} = \{y + i\Gamma y\}_{y \in \mathcal{D}(\Gamma)}$. It is clear that $\mathfrak{M} \dot{+} \mathfrak{N}$ is dense in $\mathfrak{H}$ if and only if $\mathcal{D}(\Gamma)$ is dense in $J\mathfrak{H}$.

Since $\mathfrak{N}$ is neutral, we have $0 = [(x+i\Gamma x), (x+i\Gamma x)] = -i[x, \Gamma x] + i[\Gamma x, x]$. Thus, $J\Gamma$ is symmetric. Now let $y+z[\perp]\mathfrak{N}$, $y \in \mathfrak{M}$, $z \in J\mathfrak{M}$. Then $0 = [(x+i\Gamma x), (y+z)] = i[\Gamma x, z] + [x, y]$. Thus $z \in \mathcal{D}(\Gamma^*)$ and $y = i\Gamma^* z$. ■

In conclusion, we remark that it is easy to construct a J-symmetric operator B with the properties (6.1) but without a J-self-adjoint extension. Indeed, we can take Γ such that $J\Gamma$ is symmetric with no self-adjoint extension and then use Lemma 6.3.

A Appendix

A.1 Preliminaries

As is well known any J-orthogonal pair $\{\mathfrak{L}_+, \mathfrak{L}_-\}$ of definite subspaces can be extended to a maximal pair. Traditional proofs of this result are non-constructive and use the existence of a maximal element in a partially ordered set. Here we provide a constructive proof which may be useful in other applications as well.

A.2 Main Construction

Let $\mathfrak{H}$ be a Krein space, the representation $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ be its canonical decomposition and J be the corresponding canonical symmetry. Next, let $\mathfrak{L}_+$ and $\mathfrak{L}_-$ be positive and negative subspaces respectively, with $\mathfrak{L}_+[\perp]\mathfrak{L}_-$. Then $\mathfrak{L}_+ = \{x + Kx\}_{x\in\mathfrak{D}(K)\subseteq\mathfrak{H}_+}$, $\mathfrak{L}_- = \{Qy + y\}_{y\in\mathfrak{D}(Q)\subseteq\mathfrak{H}_-}$, where $K\colon \mathfrak{H}_+ \mapsto \mathfrak{H}_-$ and $Q\colon \mathfrak{H}_- \mapsto \mathfrak{H}_+$ are so-called angular operators of the subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$, respectively. Note that $\|Kx\| < \|x\|$ and $\|Qy\| < \|y\|$ for all $x \neq 0$, $y \neq 0$, *i.e.,* K and Q are strong contractions. The subspace $\mathfrak{L}_+$ is not maximal non-negative iff $\mathfrak{D}(K) \neq \mathfrak{H}_+$ and similarly for $\mathfrak{L}_-$.

Let us define the subspaces: $\mathfrak{H}_+^{(1)} = \mathfrak{D}(K)$, $\mathfrak{H}_+^{(2)} = \mathfrak{H}_+ \ominus \mathfrak{H}_+^{(1)}$, $\mathfrak{H}_-^{(1)} = \mathfrak{D}(Q)$ and $\mathfrak{H}_-^{(2)} = \mathfrak{H}_- \ominus \mathfrak{H}_-^{(1)}$. We have then $Kx = K_{1,1}x \oplus K_{2,1}x$, where $K_{1,1}x \in \mathfrak{H}_-^{(1)}$, $K_{2,1}x \in \mathfrak{H}_-^{(2)}$, $x \in \mathfrak{H}_+^{(1)}$, and $Qy = Q_{1,1}y \oplus Q_{2,1}y$, where $Q_{1,1}y \in \mathfrak{H}_+^{(1)}$, $Q_{2,1}y \in \mathfrak{H}_+^{(2)}$, $y \in \mathfrak{H}_-^{(1)}$. We need to define $\tilde{K}$ and $\tilde{Q}$ in the form $\tilde{K}x = K_{1,2}x \oplus K_{2,2}x$, where $K_{1,2}x \in \mathfrak{H}_-^{(1)}$, $K_{2,2}x \in \mathfrak{H}_-^{(2)}$, for $x \in \mathfrak{H}_+^{(2)}$, and $\tilde{Q}y = Q_{1,2}y \oplus Q_{2,2}y$, where $Q_{1,2}y \in \mathfrak{H}_+^{(1)}$, $Q_{2,2}y \in \mathfrak{H}_+^{(2)}$, for $y \in \mathfrak{H}_-^{(2)}$. The operators $K_{1,2}, K_{2,2}, Q_{1,2}$ and $Q_{2,2}$ must be such that the extensions $\tilde{K}$ and $\tilde{Q}$ are contractions and the corresponding extensions of the spaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are J-orthogonal. The last condition implies

$$Q_{j,k} = K_{k,j}^*, \quad j, k = 1, 2, \tag{A.1}$$

so the operators $K_{1,2}$ and $Q_{1,2}$ are already defined. Note also that since K and Q are strong contractions, we have

$$\|K_{1,1}x\|^2 + \|K_{2,1}x\|^2 < \|x\|^2, \quad \|Q_{1,1}y\|^2 + \|Q_{2,1}y\|^2 < \|y\|^2, \tag{A.2}$$

for all $0 \neq x \in \mathfrak{H}_+^{(1)}$ and $0 \neq y \in \mathfrak{H}_-^{(1)}$.

Let $u \in \mathfrak{H}_+^{(1)}$ and $v \in \mathfrak{H}_+^{(2)}$, $\|u\| + \|v\| > 0$. Then (A.1) and (A.2) imply

$$\begin{aligned}
|(y, K_{1,1}u) + (y, K_{1,2}v)| &= |(Q_{1,1}y, u) + (Q_{2,1}y, v)| \\
&\leq \{\|Q_{1,1}y\|^2 + \|Q_{2,1}y\|^2\}^{1/2}\{\|u\|^2 + \|v\|^2\}^{1/2} \\
&< \|y\|\{\|u\|^2 + \|v\|^2\}^{1/2}.
\end{aligned}$$

The last chain implies

$$\|K_{1,1}u + K_{1,2}v\|^2 < \|u\|^2 + \|v\|^2. \tag{A.3}$$

In what follows we use a scheme from [1] and [13] to prove existence of a norm preserving extension of a self-adjoint contraction. Taking into account (A.3), let us introduce a new norm on $\mathfrak{H}_+$ by:

$$\|u \oplus v\|_0^2 = \|u\|^2 + \|v\|^2 - \|K_{1,1}u + K_{1,2}v\|^2. \tag{A.4}$$

With respect to this norm on the domain, the operator $K_{2,1}$ is a contraction too. In fact, from (A.1) for $0 \neq u$ we have $\|K_{2,1}u\|^2 < \|u\|^2 - \|K_{1,1}u\|^2 = \|u\|_0^2$. Our aim is to extend $K_{2,1}$ to a contraction acting from the whole space $\mathfrak{H}_+$ with norm $\|\cdot\|_0$

into $\mathfrak{H}_-^{(2)}$. Let $\tilde{\mathfrak{H}}$ and $\tilde{\mathfrak{H}}^{(1)}$ be respectively the completion of the lineals $\mathfrak{H}_+$ and $\mathfrak{H}_+^{(1)}$ with respect to the norm $\|\cdot\|_0$. Next, let $\tilde{K}_{2,1}$ be the closure of $K_{2,1}$ in $\tilde{\mathfrak{H}}$ and let P be the ortho-projector (with respect to the norm $\|\cdot\|_0$) from $\tilde{\mathfrak{H}}$ onto $\tilde{\mathfrak{H}}^{(1)}$. Then the operator $\tilde{K}_2 = \tilde{K}_{2,1}P$ is a well defined contraction, acting from $\tilde{\mathfrak{H}}$ into $\mathfrak{H}_2^-$. Now let us put $K_{2,1} \oplus K_{2,2} = \tilde{K}_2|_{\mathfrak{H}_+}$. Then for all $x \in \mathfrak{H}_+^{(1)}$ and $z \in \mathfrak{H}_+^{(2)}$ we have $\|\tilde{K}(x \oplus z)\|^2 = \|K_{1,1}x+K_{1,2}z\|^2+\|K_{2,1}x+K_{2,2}z\|^2 \le \|K_{1,1}x+K_{1,2}z\|^2+\|x\|^2+\|z\|^2-\|K_{1,1}x+K_{1,2}z\|^2 = \|x\|^2+\|z\|^2$.

Let us call the extension $\tilde{\mathfrak{L}}_+ = \{x \oplus \tilde{K}x\}_{x\in\mathfrak{H}_+}$, $\tilde{\mathfrak{L}}_- = \tilde{\mathfrak{L}}_+^{[\perp]}$, with

$$\tilde{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

as above, the *extension with nullified complement.*

A.3 Theorem of Invariance

Theorem A.1 *Let $\mathfrak{L}_+, \mathfrak{L}_-$ be a pair of J-orthogonal subspaces respectively J-positive and J-negative, and let E be a projector simultaneously orthogonal and J-orthogonal and such that $E\mathfrak{L}_+ \subset \mathfrak{L}_+$ and $E\mathfrak{L}_- \subset \mathfrak{L}_-$. Then for the extension $\{\tilde{\mathfrak{L}}_+, \tilde{\mathfrak{L}}_-\}$ with nullified complement the relations $E\tilde{\mathfrak{L}}_+ \subset \tilde{\mathfrak{L}}_+$ and $E\tilde{\mathfrak{L}}_- \subset \tilde{\mathfrak{L}}_-$ hold and the pair $\{E\tilde{\mathfrak{L}}_+, E\tilde{\mathfrak{L}}_-\}|_{E\mathfrak{H}}$ is an extension of $\{E\mathfrak{L}_+, E\mathfrak{L}_-\}|_{E\mathfrak{H}}$ with nullified complement.*

Proof Let us keep the previous notation. Since E is self-adjoint and J-self-adjoint, $E\mathfrak{H}_+ \subset \mathfrak{H}_+$ and $E\mathfrak{H}_- \subset \mathfrak{H}_-$. On the other hand, under our assumptions $E(x \oplus Kx)$ is of the form $y \oplus Ky$, where $x, y \in \mathfrak{D}(K) = \mathfrak{H}_+^{(1)}$. Thus $E\mathfrak{H}_+^{(1)} \subset \mathfrak{H}_+^{(1)}$, $E\mathfrak{H}_+^{(2)} \subset \mathfrak{H}_+^{(2)}$ and $EK = KE|_{\mathfrak{H}_+^{(1)}}$. Then similar reasoning gives us the relations $E\mathfrak{H}_-^{(1)} \subset \mathfrak{H}_-^{(1)}$, $E\mathfrak{H}_-^{(2)} \subset \mathfrak{H}_-^{(2)}$ and $EQ = QE|_{\mathfrak{H}_-^{(1)}}$. This implies that $EK_{1,1} = K_{1,1}E|_{\mathfrak{H}_+^{(1)}}$, $EK_{2,1} = K_{2,1}E|_{\mathfrak{H}_+^{(1)}}$, $EQ_{1,1} = Q_{1,1}E|_{\mathfrak{H}_-^{(1)}}$ and $EQ_{2,1} = Q_{2,1}E|_{\mathfrak{H}_-^{(1)}}$. Now it is easy to see that $E|_{\mathfrak{H}_+}$ is also orthogonal with respect to the norm $\|\cdot\|_0$ from (A.4). The rest is straightforward. ■

Acknowledgement This paper was written while the second author was visiting the Department of Mathematics and Statistics, University of Calgary. He would like to thank the department for its hospitality.

References

[1] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space.* vol. 2, Pitman, Boston, 1981.

[2] F. V. Atkinson and A. B. Mingarelli, *Asymptotics of the number of zeros and of the eigenvalues of general weighted Sturm-Liouville problems.* J. Reine Angew. Math. **375/376** (1987), 380–393.

[3] T. Ya. Azizov and I. S. Iokhvidov, *Linear operators in spaces with an indefinite metric.* John Wiley and Sons, Chichester, 1989.

[4] J. Bognár, *Indefinite inner product spaces.* ergeb. math. Grenzgeb., 78, Springer-Verlag, New York, 1974.

[5] ———, *A proof of the spectral theorem for J-positive operators.* Acta Sci. Math. (Szeged) **45**(1983), 75–80.

[6] B. Ćurgus and H. Langer, *A Kreĭn space approach to symmetric ordinary differential operators with an indefinite weight function.* J. Differential Equations **79**(1989), 31–61.
[7] N. Dunford and J. T. Schwartz, *Linear Operators. I.* General Theory, Interscience, London, 1958.
[8] G. Freiling and V. Yurko, *On constructing differential equations with singularities from incomplete spectral information.* Inverse Problems **14**(1998), 1131–1150.
[9] H.Langer, *Spectral functions of definitizable operators in Krein space.* Lect. Notes in Math. 948, Springer, Berlin, 1982, pp. 1–46.
[10] N. Karapetiants and S. Samko, *Equations with involutive operators.* Birkhäuser, Boston, 2001.
[11] T. Kato, *Perturbation theory for linear operators.* Springer-Verlag, New York, 1966.
[12] E. Kreyszig, *Introductory functional analysis with applications.* John Wiley and Sons, New York, 1978.
[13] A. V. Kuzhel and S. A. Kuzhel, *Regular extensions of Hermitian operators.* VSP, Utrecht, 1998.
[14] M. A. Naimark, *Normed algebras.* Wolters-Nordhoff, 1972.

Department of Mathematics and Statistics
University of Calgary
Calgary, AB
e-mail: binding@ucalgary.ca

Department of Pure and Applied Mathematics
Simón Bolívar University
Caracas, Venezuela
e-mail: str@usb.ve