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Ternary Quadratic Forms and Eta Quotients

Kenneth S. Williams

Abstract. Let $\eta(z)$ $(z \in \mathbb{C}, \mathrm{Im}(z) > 0)$ denote the Dedekind eta function. We use a recent product-to-sum formula in conjunction with conditions for the non-representability of integers by certain ternary quadratic forms to give explicitly ten eta quotients

$$f(z) := \eta^{a(m_1)}(m_1 z) \cdots \eta^{a(m_r)}(m_r z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}, \quad z \in \mathbb{C},\ \mathrm{Im}(z) > 0,$$

such that the Fourier coefficients $c(n)$ vanish for all positive integers n in each of infinitely many non-overlapping arithmetic progressions. For example, we show that if $f(z) = \eta^4(z)\eta^9(4z)\eta^{-2}(8z)$ we have $c(n) = 0$ for all n in each of the arithmetic progressions $\{16k + 14\}_{k \geq 0}$, $\{64k + 56\}_{k \geq 0}$, $\{256k + 224\}_{k \geq 0}$, $\{1024k + 896\}_{k \geq 0}, \ldots$.

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{H}$ denote the Poincaré upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The Dedekind eta function is defined by

$$\eta(z) := e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}), \quad z \in \mathbb{H}.$$

An eta quotient $f(z)$ is a holomorphic function of the form

$$f(z) := \eta^{a(m_1)}(m_1 z) \cdots \eta^{a(m_r)}(m_r z), \quad z \in \mathbb{H},$$

where $r \in \mathbb{N}$, $m_1, \ldots, m_r \in \mathbb{N}$ satisfy $m_1 < \cdots < m_r$, and $a(m_1), \ldots, a(m_r)$ are nonzero integers. We suppose that

$$m_1 a(m_1) + \cdots + m_r a(m_r) = 24$$

so that the eta quotient $f(z)$ has a Fourier expansion of the form

$$f(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}, \quad c(1) = 1,$$

where the Fourier coefficients $c(n)$ are integers. We adopt the notation

$$[f(z)]_n := c(n), \quad n \in \mathbb{N}.$$

Many questions concerning the vanishing or non-vanishing of the Fourier coefficients of eta quotients have been addressed; see, for example, [2], [3, p. 133], [4], [5], and [6]. In this note we are interested in determining explicit eta quotients $f(z)$ such that $[f(z)]_n = 0$ for all n in infinitely many non-overlapping arithmetic progressions. We

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do this by using the author's recent product-to-sum formula [7, Theorem 1.1, p. 80] to express the Fourier coefficients $[f(z)]_n$ of certain eta quotients $f(z)$ in the form

$$[f(z)]_n = \sum_{\substack{(x_1,x_2,x_3)\in\mathbb{Z}^3 \\ Q(x_1,x_2,x_3)=n}} P(x_1,x_2,x_3), \quad n \in \mathbb{N}, \tag{1.1}$$

where $P \in \mathbb{Q}[x_1,x_2,x_3]$ and $Q = x_1^2 + ax_2^2 + bx_3^2$ for some $a, b \in \mathbb{N}$ with $1 \le a \le b \le 4$. Classical results on the representability of $n \in \mathbb{N}$ by the ternary quadratic form Q [1, pp. 111–113] give infinitely many arithmetic progressions such that if n belongs to any one of them, then n is not represented by Q, and so by (1.1), $[f(z)]_n = 0$ for these n. The 10 eta quotients constructed in this manner are given in Theorem 1.1(i)–(x). Theorem 1.1 is proved in Section 2.

Theorem 1.1 *For all $e \in \mathbb{N}_0$ we have*

(i) $[\eta^2(z)\eta(2z)\eta^{-2}(3z)\eta(4z)\eta^5(6z)\eta^2(8z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 5 \cdot 2^{2e+1} \pmod{2^{2e+4}}$,
(ii) $[\eta^{-2}(z)\eta^8(2z)\eta^{-2}(4z)\eta^3(6z)]_n = 0$ *for* $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$,
(iii) $[\eta^3(2z)\eta^{-2}(3z)\eta^8(6z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 2 \cdot 3^{2e} \pmod{3^{2e+1}}$,
(iv) $[\eta^3(2z)\eta^{-2}(4z)\eta^3(6z)\eta^5(8z)\eta^{-2}(16z)]_n = 0$ *for* $n \equiv 5 \cdot 3^{2e+1} \pmod{2 \cdot 3^{2e+2}}$,
(v) $[\eta^{-2}(3z)\eta^6(4z)\eta^5(6z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 8 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$,
(vi) $[\eta^{-2}(3z)\eta^6(4z)\eta^5(6z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 11 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$,
(vii) $[\eta^2(z)\eta^7(2z)\eta^2(4z)]_n = 0$ *for* $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$,
(viii) $[\eta^4(z)\eta^9(4z)\eta^{-2}(8z)]_n = 0$ *for* $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$,
(ix) $[\eta^4(z)\eta^2(2z)\eta^{-2}(3z)\eta^4(4z)\eta^5(6z)\eta^{-2}(12z)]_n = 0$ *for* $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$,
(x) $[\eta^4(z)\eta^2(2z)\eta^2(4z)\eta^5(8z)\eta^{-2}(16z)]_n = 0$ *for* $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$.

2 Proof of Theorem 1.1

For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty}(1 - q^{kn}),$$

so that $E_k = E_1(q^k)$.

We begin with a lemma that enables us to eliminate some uninteresting cases that arise in the proof of Theorem 1.1 as well as assisting in the formulation of some conjectures in Section 3.

Lemma 2.1 *Let $n \in \mathbb{N}$.*

(i) *If* $n \equiv 0, 3 \pmod 4$, *then* $[qE_1^2E_2^{-1}E_4^6]_n = 0$.
(ii) *If* $n \equiv 0, 6 \pmod 8$, *then* $[qE_1^2E_2E_4^{-1}E_8^7E_{16}^{-2}]_n = 0$.
(iii) *If* $n \equiv 0, 3 \pmod 4$, *then* $[qE_1^{-2}E_2^5E_4^4]_n = 0$.
(iv) *If* $n \equiv 2, 3 \pmod 4$, *then* $[qE_3^{-2}E_4^6E_6^5E_{12}^{-2}]_n = 0$.
(v) *If* $n \equiv 3 \pmod 4$, *then* $[qE_1^4E_2^2E_4^2E_8^5E_{16}^{-2}]_n = 0$.

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Proof (i) By a classical theorem of Jacobi, we have

$$E_1^2E_2^{-1} = 1 + 2\sum_{m=1}^{\infty}(-1)^m q^{m^2}.$$

Thus for $n \in \mathbb{N}$, we have

$$[qE_1^2E_2^{-1}]_n = \begin{cases} 1 & \text{if } n = 1, \\ 2(-1)^m & \text{if } n = m^2+1, m \in \mathbb{N}, \\ 0 & \text{if } n \neq m^2+1, m \in \mathbb{N}_0. \end{cases}$$

If $n \equiv 0, 3 \pmod 4$, then we have $n \neq m^2+1$ $(m \in \mathbb{N}_0)$ so that $[qE_1^2E_2^{-1}]_n = 0$. Hence,

$$[qE_1^2E_2^{-1}E_4^6]_n = 0 \text{ if } n \equiv 0, 3 \pmod 4.$$

(ii) Replacing q by q^2 in Jacobi's identity, we obtain

$$E_2^2E_4^{-1} = 1 + 2\sum_{m=1}^{\infty}(-1)^m q^{2m^2}.$$

Multiplying the series for $E_1^2E_2^{-1}$ and $E_2^2E_4^{-1}$ together, we deduce

$$qE_1^2E_2E_4^{-1} = q + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2+1} + 2\sum_{n=1}^{\infty}(-1)^n q^{2n^2+1} + 4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{\ell,m\geq 1 \\ \ell^2+2m^2+1=n}}(-1)^{\ell+m}\Bigg)q^n.$$

As $n^2+1 \equiv 1, 2, 5 \pmod 8$, $2n^2+1 \equiv 1, 3 \pmod 8$ and $\ell^2+2m^2+1 \equiv 1, 2, 3, 4, 5, 7 \pmod 8$, we have $[qE_1^2E_2E_4^{-1}]_n = 0$ for $n \equiv 0, 6 \pmod 8$. Thus,

$$[qE_1^2E_2E_4^{-1}E_8^7E_{16}^{-2}]_n = 0 \text{ if } n \equiv 0, 6 \pmod 8.$$

(iii) By another classical identity of Jacobi, we have

$$E_1^{-2}E_2^5E_4^{-2} = 1 + 2\sum_{m=1}^{\infty} q^{m^2}.$$

Thus for $n \in \mathbb{N}$ we have

$$[qE_1^{-2}E_2^5E_4^{-2}]_n = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = m^2+1, m \in \mathbb{N}, \\ 0 & \text{if } n \neq m^2+1, m \in \mathbb{N}_0. \end{cases}$$

If $n \equiv 0, 3 \pmod 4$ then $n \neq m^2+1$ $(m \in \mathbb{N}_0)$ so $[qE_1^{-2}E_2^5E_4^{-2}]_n = 0$. Thus,

$$[qE_1^{-2}E_2^5E_4^4]_n = [qE_1^{-2}E_2^5E_4^{-2} \cdot E_4^6]_n = 0 \text{ if } n \equiv 0, 3 \pmod 4.$$

(iv) Mapping q to q^3 in the identity of Jacobi given in the proof of (iii), we have

$$E_3^{-2}E_6^5E_{12}^{-2} = 1 + 2\sum_{m=1}^{\infty} q^{3m^2}.$$

Hence for $n \in \mathbb{N}$, we have

$$[qE_3^{-2}E_6^5E_{12}^{-2}]_n = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 3m^2+1, m \in \mathbb{N}, \\ 0 & \text{if } n \neq 3m^2+1, m \in \mathbb{N}_0. \end{cases}$$

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If $n \equiv 2, 3 \pmod 4$, then $n \neq 3m^2 + 1$ ($m \in \mathbb{N}_0$) so $[qE_3^{-2}E_6^5E_{12}^{-2}]_n = 0$. Thus

$$[qE_3^{-2}E_4^6E_6^5E_{12}^{-2}]_n = 0 \text{ if } n \equiv 2, 3 \pmod 4.$$

(v) Ramanujan defined the theta function ϕ by

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$

and proved that

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2), \quad \phi(q)\phi(-q) = \phi^2(-q^2).$$

Hence,

$$\begin{aligned}\sum_{j=0}^{3}&(-1)^j\phi(i^jq)\phi^3(-i^jq)\\ &= \phi(q)\phi(-q)\big(\phi^2(q) + \phi^2(-q)\big) - \phi(iq)\phi(-iq)\big(\phi^2(iq) + \phi^2(-iq)\big)\\ &= 2\phi^2(-q^2)\phi^2(q^2) - 2\phi^2(q^2)\phi^2(-q^2) = 0.\end{aligned}$$

Thus,

$$[\phi(q)\phi^3(-q)]_n = 0 \text{ for } n \equiv 2 \pmod 4.$$

Now $\phi(q) = E_1^{-2}E_2^5E_4^{-2}$ and $\phi(-q) = E_1^2E_2^{-1}$ so $\phi(q)\phi^3(-q) = E_1^4E_2^2E_4^{-2}$. Hence,

$$[E_1^4E_2^2E_4^{-2}]_n = 0 \text{ for } n \equiv 2 \pmod 4.$$

Then

$$[E_1^4E_2^2E_4^2E_8^5E_{16}^{-2}]_n = [E_1^4E_2^2E_4^{-2} \cdot E_4^4E_8^5E_{16}^{-2}]_n = [E_1^4E_2^2E_4^{-2}]_n = 0 \text{ for } n \equiv 2 \pmod 4,$$

and finally

$$[qE_1^4E_2^2E_4^2E_8^5E_{16}^{-2}]_n = 0 \text{ for } n \equiv 3 \pmod 4.$$

This completes the proof of the lemma. ∎

Taking $q = e^{2\pi iz}$ ($z \in \mathbb{H}$) so that $|q| < 1$, we have

$$\eta(kz) = e^{\pi ikz/12}E_k = q^{k/24}E_k.$$

We now state the product-to-sum formula proved by the author in [7, Theorem 1.1, p. 80], which we will use in the proof of Theorem 1.1.

Product-to-Sum Formula *Let $k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$. Let $r, s, t, u \in \mathbb{N}_0$ be such that $r + s + t + u = k$. Let $v, w, x, y \in \mathbb{N}_0$ be such that $v + w + x + y = \ell$. Set $m = k + 2\ell$ so that $m \in \mathbb{N}$ and $m \geq 2$. Let*

$$\begin{aligned}P(x_1, \dots, x_m) = \frac{1}{2^\ell}&\prod_{g=r+1}^{r+v}(x_g^2 - 2x_{g+s+\ell+y}^2)\prod_{g=r+v+1}^{r+v+w}(x_g^2 - 3x_{g+s+t+\ell+y}^2)\\ &\times\prod_{g=r+v+w+1}^{r+v+w+x}(x_g^2 - 4x_{g+s+t+\ell+y+u}^2)\prod_{g=r+v+w+x+1}^{r+\ell}(x_g^4 - 3x_g^2x_{g+y}^2)\end{aligned}$$

and

$$\begin{aligned}Q(x_1, \dots, x_m) = x_1^2 &+ \cdots + x_{r+\ell+y}^2 + 2x_{r+\ell+y+1}^2 + \cdots + 2x_{r+s+\ell+v+y}^2\\ &+ 3x_{r+s+\ell+v+y+1}^2 + \cdots + 3x_{r+s+t+\ell+v+w+y}^2 + 4x_{r+s+t+\ell+v+w+y+1}^2 + \cdots + 4x_m^2.\end{aligned}$$

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Let

$$a_1 = -2r + 2v + 4y, \qquad a_6 = 5t + 3w,$$
$$a_2 = 5r - 2s + v + 3w + 2y, \qquad a_8 = -2s + 5u + 2v,$$
$$a_3 = -2t, \qquad a_{12} = -2t,$$
$$a_4 = -2r + 5s - 2u + v + 6x + 4y, \qquad a_{16} = -2u.$$

Then for $n \in \mathbb{N}$ with $n \geq \ell$, we have

$$\Big[q^{\ell} E_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_6^{a_6} E_8^{a_8} E_{12}^{a_{12}} E_{16}^{a_{16}} \Big]_n = \sum_{\substack{(x_1,\dots,x_m)\in\mathbb{Z}^m \\ Q(x_1,\dots,x_m)=n}} P(x_1,\dots,x_m)$$

and

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 8a_8 + 12a_{12} + 16a_{16} = 24\ell.$$

Proof of Theorem 1.1 We use the special case of the product-to-sum formula when $k = \ell = 1$, so that $m = 3$ and Q is a positive diagonal ternary quadratic form all of whose coefficients are 1, 2, 3, or 4 with at least one of the coefficients equal to 1. Let $r, s, t, u, v, w, x, y \in \mathbb{N}_0$ satisfy

$$r + s + t + u = 1, \quad v + w + x + y = 1. \tag{2.1}$$

Define $A \in \mathbb{N}$ and $B, C, D \in \mathbb{N}_0$ by

$$A := r + y + 1, \quad B := s + v, \quad C := t + w, \quad D := u + x, \tag{2.2}$$

so that

$$A + B + C + D = 3. \tag{2.3}$$

Then

$$P(x_1,x_2,x_3) = \frac{1}{2} \prod_{g=r+1}^{r+v} (x_g^2 - 2x_{g+s+y+1}^2) \prod_{g=r+v+1}^{r+v+w} (x_g^2 - 3x_{g+s+t+y+1}^2)$$
$$\times \prod_{g=r+v+w+1}^{r+v+w+x} (x_g^2 - 4x_{g+s+t+y+u+1}^2) \prod_{g=r+v+w+x+1}^{r+1} (x_g^4 - 3x_g^2 x_{g+y}^2)$$

and

$$Q(x_1,x_2,x_3) = \sum_{i=1}^{A} x_i^2 + 2\sum_{i=A+1}^{A+B} x_i^2 + 3\sum_{i=A+B+1}^{A+B+C} x_i^2 + 4\sum_{i=A+B+C+1}^{A+B+C+D} x_i^2. \tag{2.4}$$

Define the integers $a_1, a_2, a_3, a_4, a_6, a_8, a_{12}$ and a_{16} as in the product-to-sum formula. Then, as $\ell = 1$, by the product-to-sum formula we have

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 8a_8 + 12a_{12} + 16a_{16} = 24$$

and

$$\begin{aligned} &\Big[\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_8}(8z)\eta^{a_{12}}(12z)\eta^{a_{16}}(16z) \Big]_n \\ &= \Big[qE_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_6^{a_6} E_8^{a_8} E_{12}^{a_{12}} E_{16}^{a_{16}} \Big]_n \\ &= \sum_{\substack{(x_1,x_2,x_3)\in\mathbb{Z}^3 \\ Q(x_1,x_2,x_3)=n}} P(x_1,x_2,x_3). \end{aligned} \tag{2.5}$$

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We next examine the 16 possible values of the vector $(r, s, t, u, v, w, x, y) \in \mathbb{N}_0^8$ subject to the restrictions in (2.1). The ternary form Q corresponding to each such vector is determined from (2.2), (2.3), and (2.4) and each eta quotient

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_8}(8z)\eta^{a_{12}}(12z)\eta^{a_{16}}(16z)$$

from the formulae for $a_1, a_2, a_3, a_4, a_6, a_8, a_{12}, a_{16}$ in terms of r, s, t, u, v, w, x, y given in the product-to-sum formula. The values are given in Table 1.

It is known from the work of Dickson and Jones (see, for example, [1, pp. 111–112]) that the positive integers n for which $Q(x_1, x_2, x_3) = n$ is not solvable in integers x_1, x_2, and x_3 are as given in Table 2, where $k, \ell \in \mathbb{N}_0$. Appealing to (2.5), Table 1 and Table 2, we obtain the following results.

Case 1 gives

$$\big[\eta^6(2z)\eta^{-1}(4z)\eta^2(8z)\big]_n = 0 \tag{2.6}$$

for $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, but this is not interesting, as clearly (2.6) holds for all $n \in \mathbb{N}$ with $n \equiv 0 \pmod 2$.

Case 2 gives

$$\big[\eta^2(z)\eta^{-1}(2z)\eta^6(4z)\big]_n = 0 \tag{2.7}$$

for $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$, which is again not interesting, as by Lemma 2.1(i) (2.7) holds for all $n \equiv 0, 3 \pmod 4$.

Case 3 gives

$$\big[\eta^2(z)\eta(2z)\eta^{-2}(3z)\eta(4z)\eta^5(6z)\eta^2(8z)\eta^{-2}(12z)\big]_n = 0$$

for $n \equiv 5 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(i).

Case 4 gives

$$\big[\eta^2(z)\eta(2z)\eta^{-1}(4z)\eta^7(8z)\eta^{-2}(16z)\big]_n = 0 \tag{2.8}$$

for $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, but this is not interesting, as (2.8) holds for all $n \equiv 0, 6 \pmod 8$ by Lemma 2.1(ii).

Case 5 gives

$$\big[\eta^{-2}(z)\eta^8(2z)\eta^{-2}(4z)\eta^3(6z)\big]_n = 0$$

for $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(ii).

Case 6 gives

$$\big[\eta(2z)\eta^5(4z)\eta^3(6z)\eta^{-2}(8z)\big]_n = 0 \tag{2.9}$$

for $n \equiv 5 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, which is not interesting, as (2.9) holds for all $n \equiv 0 \pmod 2$.

Case 7 gives

$$\big[\eta^3(2z)\eta^{-2}(3z)\eta^8(6z)\eta^{-2}(12z)\big]_n = 0$$

for $n \equiv 2 \cdot 3^{2e} \pmod{3^{2e+1}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(iii).

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case	(r,s,t,u,v,w,x,y)	$(a_1,a_2,a_3,a_4,a_6,a_8,a_{12},a_{16})$	$Q(x_1,x_2,x_3)$
1	$(1,0,0,0,1,0,0,0)$	$(0,6,0,-1,0,2,0,0)$	$x_1^2+x_2^2+2x_3^2$
2	$(0,1,0,0,1,0,0,0)$	$(2,-1,0,6,0,0,0,0)$	$x_1^2+2x_2^2+2x_3^2$
3	$(0,0,1,0,1,0,0,0)$	$(2,1,-2,1,5,2,-2,0)$	$x_1^2+2x_2^2+3x_3^2$
4	$(0,0,0,1,1,0,0,0)$	$(2,1,0,-1,0,7,0,-2)$	$x_1^2+2x_2^2+4x_3^2$
5	$(1,0,0,0,0,1,0,0)$	$(-2,8,0,-2,3,0,0,0)$	$x_1^2+x_2^2+3x_3^2$
6	$(0,1,0,0,0,1,0,0)$	$(0,1,0,5,3,-2,0,0)$	$x_1^2+2x_2^2+3x_3^2$
7	$(0,0,1,0,0,1,0,0)$	$(0,3,-2,0,8,0,-2,0)$	$x_1^2+3x_2^2+3x_3^2$
8	$(0,0,0,1,0,1,0,0)$	$(0,3,0,-2,3,5,0,-2)$	$x_1^2+3x_2^2+4x_3^2$
9	$(1,0,0,0,0,0,1,0)$	$(-2,5,0,4,0,0,0,0)$	$x_1^2+x_2^2+4x_3^2$
10	$(0,1,0,0,0,0,1,0)$	$(0,-2,0,11,0,-2,0,0)$	$x_1^2+2x_2^2+4x_3^2$
11	$(0,0,1,0,0,0,1,0)$	$(0,0,-2,6,5,0,-2,0)$	$x_1^2+3x_2^2+4x_3^2$
12	$(0,0,0,1,0,0,1,0)$	$(0,0,0,4,0,5,0,-2)$	$x_1^2+4x_2^2+4x_3^2$
13	$(1,0,0,0,0,0,0,1)$	$(2,7,0,2,0,0,0,0)$	$x_1^2+x_2^2+x_3^2$
14	$(0,1,0,0,0,0,0,1)$	$(4,0,0,9,0,-2,0,0)$	$x_1^2+x_2^2+2x_3^2$
15	$(0,0,1,0,0,0,0,1)$	$(4,2,-2,4,5,0,-2,0)$	$x_1^2+x_2^2+3x_3^2$
16	$(0,0,0,1,0,0,0,1)$	$(4,2,0,2,0,5,0,-2)$	$x_1^2+x_2^2+4x_3^2$

Table 1: Eta quotients and ternary forms corresponding to (r,s,t,u,v,w,x,y)

$Q(x_1,x_2,x_3)$	integers not represented by Q	$Q(x_1,x_2,x_3)$	integers not represented by Q
$x_1^2+x_2^2+x_3^2$	$4^k(8\ell+7)$	$x_1^2+2x_2^2+3x_3^2$	$4^k(16\ell+10)$
$x_1^2+x_2^2+2x_3^2$	$4^k(16\ell+14)$	$x_1^2+2x_2^2+4x_3^2$	$4^k(16\ell+14)$
$x_1^2+x_2^2+3x_3^2$	$9^k(9\ell+6)$	$x_1^2+3x_2^2+3x_3^2$	$9^k(3\ell+2)$
$x_1^2+x_2^2+4x_3^2$	$8\ell+3,\ 4^k(8\ell+7)$	$x_1^2+3x_2^2+4x_3^2$	$4\ell+2,\ 9^k(9\ell+6)$
$x_1^2+2x_2^2+2x_3^2$	$4^k(8\ell+7)$	$x_1^2+4x_2^2+4x_3^2$	$4\ell+2,\ 4\ell+3,\ 4^k(8\ell+7)$

Table 2: Integers not represented by ternary quadratic forms

Case 8 gives

$$[\eta^3(2z)\eta^{-2}(4z)\eta^3(6z)\eta^5(8z)\eta^{-2}(16z)]_n = 0 \tag{2.10}$$

for $n \equiv 2 \pmod 4$ and $n \equiv 2\cdot 3^{2e+1} \pmod{3^{2e+2}}$ for all $e \in \mathbb{N}_0$. Clearly (2.10) holds for all $n \equiv 0 \pmod 2$ so the former is not interesting while the latter is interesting only

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when $n \equiv 1 \pmod 2$, that is, when $n \equiv 5 \cdot 3^{2e+1} \pmod{2 \cdot 3^{2e+2}}$, which is Theorem 1.1(iv).

Case 9 gives

$$\left[\eta^{-2}(z)\eta^{5}(2z)\eta^{4}(4z)\right]_n = 0 \tag{2.11}$$

for $n \equiv 3 \pmod 8$ and $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$. But this is not interesting, as (2.11) holds for all $n \equiv 0, 3 \pmod 4$ by Lemma 2.1(iii).

Case 10 gives

$$\left[\eta^{-2}(2z)\eta^{11}(4z)\eta^{-2}(8z)\right]_n = 0 \tag{2.12}$$

for $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$. But this is not interesting as (2.12) holds for all $n \equiv 0 \pmod 2$.

Case 11 gives

$$\left[\eta^{-2}(3z)\eta^{6}(4z)\eta^{5}(6z)\eta^{-2}(12z)\right]_n = 0 \tag{2.13}$$

for $n \equiv 2 \pmod 4$ and $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ for all $e \in \mathbb{N}_0$. By Lemma 2.1(iv) (2.13) holds for all $n \equiv 2, 3 \pmod 4$. Thus (2.13) is only interesting for those n satisfying $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ and $n \equiv 0, 1 \pmod 4$, that is, for $n \equiv 8 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$ and $n \equiv 11 \cdot 3^{2e+1} \pmod{4 \cdot 3^{2e+2}}$. These are parts (v) and (vi) of Theorem 1.1, respectively.

Case 12 gives

$$\left[\eta^{4}(4z)\eta^{5}(8z)\eta^{-2}(16z)\right]_n = 0 \tag{2.14}$$

for $n \equiv 2 \pmod 4$, $n \equiv 3 \pmod 4$ and $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$, but again this is not interesting as (2.14) clearly holds for all $n \not\equiv 1 \pmod 4$.

Case 13 gives

$$[\eta^{2}(z)\eta^{7}(2z)\eta^{2}(4z)]_n = 0$$

for $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(vii).

Case 14 gives

$$\left[\eta^{4}(z)\eta^{9}(4z)\eta^{-2}(8z)\right]_n = 0$$

for $n \equiv 7 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(viii).

Case 15 gives

$$\left[\eta^{4}(z)\eta^{2}(2z)\eta^{-2}(3z)\eta^{4}(4z)\eta^{5}(6z)\eta^{-2}(12z)\right]_n = 0$$

for $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ for all $e \in \mathbb{N}_0$, which is Theorem 1.1(ix).

Finally, Case 16 gives

$$\left[\eta^{4}(z)\eta^{2}(2z)\eta^{2}(4z)\eta^{5}(8z)\eta^{-2}(16z)\right]_n = 0$$

for $n \equiv 3 \pmod 8$ and $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for all $e \in \mathbb{N}_0$. By Lemma 2.1(v) the equality holds for all $n \equiv 3 \pmod 4$. The latter congruence is Theorem 1.1(x). ∎

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3 Final Remarks

We briefly discuss whether or not the criteria in each of the ten parts of Theorem 1.1 form a complete description of the set of vanishing coefficients for the corresponding eta quotient.

(i) As

$$\left[\eta^2(z)\eta(2z)\eta^{-2}(3z)\eta(4z)\eta^5(6z)\eta^2(8z)\eta^{-2}(12z)\right]_n = 0$$

for $n = 6, 24, 29, 39, 54, 60, 78, \ldots$, the congruences $n \equiv 5 \cdot 2^{2e+1} \pmod{2^{2e+4}}$ do not form a complete description of when the coefficients of the eta quotient vanish and it does not seem easy to formulate such a description. We note that the data up to $n = 2000$ suggest the following conjecture.

Conjecture 3.1

$$\left[\eta^2(z)\eta(2z)\eta^{-2}(3z)\eta(4z)\eta^5(6z)\eta^2(8z)\eta^{-2}(12z)\right]_n = 0$$
$$\textit{for } n \equiv 2 \cdot 3^{2e+1} \pmod{2 \cdot 3^{2e+2}}.$$

(ii) The data up to $n = 1000$ suggest the following conjecture.

Conjecture 3.2 *If*

$$\left[\eta^{-2}(z)\eta^8(2z)\eta^{-2}(4z)\eta^3(6z)\right]_n = 0$$

then $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ *for some* $e \in \mathbb{N}_0$.

If Conjecture 3.2 is true, then by Theorem 1.1(ii) we have

$$\left[\eta^{-2}(z)\eta^8(2z)\eta^{-2}(4z)\eta^3(6z)\right]_n = 0$$

if and only if $n \equiv 2 \cdot 3^{2e+1} \pmod{3^{2e+2}}$ for some $e \in \mathbb{N}_0$.

(iii) The data up to $n = 1000$ suggest the following conjecture.

Conjecture 3.3 *If*

$$\left[\eta^3(2z)\eta^{-2}(3z)\eta^8(6z)\eta^{-2}(12z)\right]_n = 0,$$

then $n \equiv 2 \cdot 3^{2e} \pmod{3^{2e+1}}$ *for some* $e \in \mathbb{N}_0$.

If Conjecture 3.3 is true then by Theorem 1.1(iii) we have

$$\left[\eta^3(2z)\eta^{-2}(3z)\eta^8(6z)\eta^{-2}(12z)\right]_n = 0$$

if and only if

$$n \equiv 2 \cdot 3^{2e} \pmod{3^{2e+1}} \quad \text{for some } e \in \mathbb{N}_0.$$

(iv) The data up to $n = 1000$ suggest the following conjecture.

Conjecture 3.4 *If*

$$\left[\eta^3(2z)\eta^{-2}(4z)\eta^3(6z)\eta^5(8z)\eta^{-2}(16z)\right]_n = 0$$

and n is odd, then $n \equiv 5 \cdot 3^{2e+1} \pmod{2 \cdot 3^{2e+2}}$ *for some* $e \in \mathbb{N}_0$.

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If Conjecture 3.4 is true, then by Theorem 1.1(iv) we have

$$\left[\eta^3(2z)\eta^{-2}(4z)\eta^3(6z)\eta^5(8z)\eta^{-2}(16z)\right]_n = 0$$

if and only if $n \equiv 0 \pmod 2$ or $n \equiv 5\cdot 3^{2e+1} \pmod{2\cdot 3^{2e+2}}$ for some $e \in \mathbb{N}_0$.

(v)(vi) The data up to $n = 1000$ suggest the following conjecture.

Conjecture 3.5 *If* $n \equiv 0,1 \pmod 4$ *and*

$$\left[\eta^{-2}(3z)\eta^6(4z)\eta^5(6z)\eta^{-2}(12z)\right]_n = 0,$$

then $n \equiv 8\cdot 3^{2e+1} \pmod{4\cdot 3^{2e+2}}$ *or* $n \equiv 11\cdot 3^{2e+1} \pmod{4\cdot 3^{2e+2}}$ *for some* $e \in \mathbb{N}_0$.

If Conjecture 3.5 is true, then by Theorem 1.1(v)(vi) and Lemma 2.1(iv) we have

$$\left[\eta^{-2}(3z)\eta^6(4z)\eta^5(6z)\eta^{-2}(12z)\right]_n = 0$$

if and only if $n \equiv 2 \pmod 4$, $n \equiv 3 \pmod 4$, $n \equiv 8\cdot 3^{2e+1} \pmod{4\cdot 3^{2e+2}}$ or $n \equiv 11\cdot 3^{2e+1} \pmod{4\cdot 3^{2e+2}}$ for some $e \in \mathbb{N}_0$.

(vii) The data up to $n = 1000$ suggest the following conjecture.

Conjecture 3.6 *If*

$$\left[\eta^2(z)\eta^7(2z)\eta^2(4z)\right]_n = 0,$$

then $n \equiv 7\cdot 2^{2e} \pmod{2^{2e+3}}$ *for some* $e \in \mathbb{N}_0$.

If Conjecture 3.6 is true, then by Theorem 1.1(vii) we have

$$\left[\eta^2(z)\eta^7(2z)\eta^2(4z)\right]_n = 0$$

if and only if $n \equiv 7\cdot 2^{2e} \pmod{2^{2e+3}}$ for some $e \in \mathbb{N}_0$.

(viii) The data up to $n = 1000$ suggest the following conjecture.

Conjecture 3.7 *If*

$$\left[\eta^4(z)\eta^9(4z)\eta^{-2}(8z)\right]_n = 0,$$

then $n \equiv 7\cdot 2^{2e+1} \pmod{2^{2e+4}}$ *for some* $e \in \mathbb{N}_0$.

If Conjecture 3.7 is true, then by Theorem 1.1(viii) we have

$$\left[\eta^4(z)\eta^9(4z)\eta^{-2}(8z)\right]_n = 0$$

if and only if $n \equiv 7\cdot 2^{2e+1} \pmod{2^{2e+4}}$ for some $e \in \mathbb{N}_0$.

(ix) Theorem 1.1(ix) and the data up to $n = 1000$ suggest the following conjecture.

Conjecture 3.8

$$\left[\eta^4(z)\eta^2(2z)\eta^{-2}(3z)\eta^4(z)\eta^5(6z)\eta^{-2}(12z)\right]_n = 0$$

if and only if $n \equiv 2\cdot 3^{2e+1} \pmod{3^{2e+2}}$ *or* $n = 3^{2e+1}$ *for some* $e \in \mathbb{N}_0$.

(x) The data up to $n = 1000$ suggest the following conjecture.

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Conjecture 3.9 *If* $n \not\equiv 3 \pmod 4$ *and*

$$\left[\eta^4(z)\eta^2(2z)\eta^2(4z)\eta^5(8z)\eta^{-2}(16z)\right]_n = 0$$

then $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ *for some* $e \in \mathbb{N}$.

If Conjecture 3.9 is true, then Lemma 2.1(v) and Theorem 1.1(x) give

$$\left[\eta^4(z)\eta^2(2z)\eta^2(4z)\eta^5(8z)\eta^{-2}(16z)\right]_n = 0$$

if and only if $n \equiv 3 \pmod 4$ or $n \equiv 7 \cdot 2^{2e} \pmod{2^{2e+3}}$ for some $e \in \mathbb{N}$.

Over the past twenty years or so, many new results concerning the representability and non-representability of positive integers by ternary quadratic forms have been proved by a number of authors, for example, Berkovich, Bhargava, Duke, Jagy, Kaplansky, and Oh, as well as many others. However, as the product-to-sum formula used in this paper applies only to the ten ternaries $x_1^2 + ax_2^2 + bx_3^2$ with $1 \le a \le b \le 4$, we cannot use these new results in conjunction with the product-to-sum formula to obtain further results like those in Theorem 1.1.

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References

[1] L. E. Dickson, *Modern elementary theory of numbers*. University of Chicago Press, Chicago, IL, 1939.

[2] B. Gordon and S. Robins, *Lacunarity of Dedekind η-products*. Glasgow Math. J. **37**(1995), no. 1, 1–14.
http://dx.doi.org/10.1017/S0017089500030329

[3] L. J. P. Kilford, *Modular forms: a classical and computational introduction*. Imperial College Press, London, 2008.

[4] D. H. Lehmer, *The vanishing of Ramanujan's function* $\tau(n)$. Duke Math. J. **14**(1947), 429–433.
http://dx.doi.org/10.1215/S0012-7094-47-01436-1

[5] V. K. Murty, *Lacunarity of modular forms*. J. Indian Math. Soc. **52**(1987), 127–146.

[6] J.-P. Serre, *Sur la lacunarité des puissances de η*. Glasgow Math. J. **27**(1985), 203–221.
http://dx.doi.org/10.1017/S0017089500006194

[7] K. S. Williams, *Quadratic forms and a product-to-sum formula*. Acta Arith. **158**(2013), no. 1, 79–97.
http://dx.doi.org/10.4064/aa158-1-5

Centre for Research in Algebra and Number Theory, School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6
e-mail: williams@math.carleton.ca