DOI: 10.1017/psa.2024.53
This is a manuscript accepted for publication in *Philosophy of Science*.
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What Do Privileged Coordinates Tell Us About Structure?*

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Abstract

We examine whether the 'privileged coordinates' of a geometric space encode its 'amount of structure.' In doing so, we compare this coordinate approach to comparing amounts of structure to the more familiar automorphism approach. We first show that on a natural understanding of the former, it faces one of the same well known problems as the latter. We then capture a precise sense in which the two approaches are closely related to one another, and we conclude by discussing whether they might still prove useful in cases of philosophical interest, despite their shortcomings.

1 Introduction

It is sometimes the case that one geometric space posits less structure than another. For example, Newtonian spacetime posits all of the structure that Galilean spacetime does, but in addition it comes equipped with *absolute rest* structure. It allows one to distinguish between trajectories that are at rest and those that are moving at a constant (non-zero) velocity. Galilean spacetime does not have the conceptual resources to draw such a distinction, so the move from the Newtonian to the Galilean theory represents a move to a less structured spacetime.

The standard method of comparing amounts of structure has been called the "automorphism approach" (Barrett, 2021b). It appeals to the automorphisms or 'symmetries' of the objects under consideration. An automorphism of an object is a structure-preserving map from the object to itself. If an object admits more automorphisms, that suggests that the object has less structure that the automorphisms are being required to preserve. Conversely, fewer automorphisms suggests that the object has more structure that they must preserve. All symmetries of Newtonian spacetime are symmetries of Galilean spacetime, but *Galilean boosts* are symmetries of the latter but not the former. This indicates that Newtonian spacetime has more structure than Galilean spacetime.

*Thanks to David Malament, Henrique Gomes, Oliver Pooley, James Read and two anonymous referees for helpful discussion of this material.

The automorphism approach goes back at least to Earman's famous remark that "as the space-time structure becomes richer, the symmetries become narrower" (Earman, 1989, p. 36). North (2009, p. 87) echoes this thought when she writes that "stronger structure [...] admits a smaller group of symmetries" and again when she says that adding structure to an object means that the "associated group of structure-preserving transformations becomes narrower" (North, 2021, p. 50). The automorphism approach has been fruitfully applied in many cases; for example, see Barrett (2015a,b), Bradley (2021), and Barrett (2021b). But another way to compare amounts of structure has recently been proposed. Instead of looking to symmetries, one looks to the 'privileged coordinates' that the space admits. The idea is that the more privileged coordinates a space admits, the less structure it must have. This is best illustrated by an example (North, 2021, p. 17–26). The smooth manifold $\mathbb{R}^2$ admits many global coordinate charts. But suppose that one were to add to $\mathbb{R}^2$ the standard Euclidean metric g_{ab}. The metric g_{ab} ascribes 'distance structure' to $\mathbb{R}^2$. Some global coordinate charts on $\mathbb{R}^2$ will not adequately respect this structure. Some, for example, will have coordinate axes that are not orthogonal to one another. The 'rectilinear coordinates' — those obtained by rotating, translating, and reflecting the standard x-y coordinates — are the ones in which g_{ab} is most perspicuously presented. In this sense, laying down a metric on $\mathbb{R}^2$ reduces the class of 'privileged coordinates' on our geometric space.

Cases like this suggest that the privileged coordinates of a geometric space encode its amount of structure. When discussing the Euclidean plane, North (2021, p. 26) puts the idea as follows:

> the features or quantities that are agreed upon by all the different [privileged] coordinate systems we can use for the plane, the coordinate-independent, invariant features, correspond to the intrinsic nature of the plane, to aspects of the plane itself, apart from our descriptions of it — that is, to what I have been calling its *structure*.

If this idea is right, then the privileged coordinates of a geometric space are a good guide to its amount of structure. More privileged coordinates will mean fewer "features or quantities that are agreed upon" by them, and hence less structure. North (2021, Ch. 4) employs this reasoning in a concrete case. She argues that standard Newtonian mechanics admits fewer privileged coordinates than Lagrangian mechanics does. The former must therefore posit more structure, and hence the two theories must be inequivalent, a conclusion that dissents from the standard view. (See Barrett (2022) and Jacobs (2024) for further discussion.) Others have also stressed the structural significance of privileged coordinates. Fock (1964, p. 374) writes that "the existence of a preferred set of coordinates [...] reflects intrinsic properties of spacetime." And Wallace (2019) shows that one can present many geometric structures by singling out their privileged coordinates.

The aim of this paper is to examine the coordinate approach to comparing amounts of structure. First, we will show that there are geometric spaces that,

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on one natural account of privileged coordinates, admit the same privileged coordinates, but have different amounts of structure. This holds because there are geometric spaces whose collections of privileged coordinates are 'as small as can be.' Adding structure to such spaces cannot further pare down these collections. This implies that these privileged coordinates do not provide a perfect guide to amounts of structure. Second, we step back and compare the coordinate approach to the automorphism approach more generally. The two approaches turn on the same core idea related to implicit definability. We will therefore conclude by discussing implicit definability and assessing whether these approaches to comparing amounts of structure might, despite their shortcomings, still be employed in cases of philosophical interest.

2 What Are Privileged Coordinates?

In order to discuss the coordinate approach, one needs an account of what the privileged coordinates of a geometric space might be. We will use the framework of locally G-structured spaces, recently presented by Wallace (2019), to provide one such account. We will briefly review this framework here, but refer the reader to Barrett and Manchak (2024a) for details.

Preliminaries

We begin with some preliminaries. The automorphism group $\text{Aut}(X)$ of a mathematical object X is the collection of bijective structure-preserving maps from X to itself. For example, if M is a smooth manifold with tensor fields $\alpha_1, \ldots, \alpha_n$, the automorphism group of the geometric space $(M, \alpha_1, \ldots, \alpha_n)$ is the collection of diffeomorphisms $f : M \to M$ such that $f^*(\alpha_i) = \alpha_i$ for each i. A **pseudogroup** is the 'local analogue' of an automorphism group. It is a collection of bijective structure-preserving maps between open subsets of the space that satisfy some basic algebraic conditions mirroring those that one requires of a group (Kobayashi and Nomizu, 1996, p. 1). We note that because not every pair of functions in a pseudogroup is composable, a pseudogroup is not a group. The **diffeomorphism pseudogroup** of a smooth manifold M is the class of diffeomorphisms $f : U \to V$ between open sets U and V of M. Recall that a **relativistic spacetime** is a pair (M, g_{ab}) where M is a smooth, n-dimensional (for $n \geq 2$), connected, Hausdorff manifold without boundary and g_{ab} is a smooth Lorentzian metric on M. The **isometry pseudogroup** of a relativistic spacetime (M, g_{ab}) is the class of diffeomorphisms $f : U \to V$ between open sets U and V of M such that $f^*(g_{ab}) = g_{ab}$. In general, if M is a smooth manifold with $\alpha_1, \ldots, \alpha_n$ smooth tensors of arbitrary index structure on M, then we will call the collection of diffeomorphisms $f : U \to V$ between open sets U and V of M such that $f^*(\alpha_i) = \alpha_i$ for each i the **automorphism pseudogroup** of the geometric space $(M, \alpha_1, \ldots, \alpha_n)$.

Let G be a pseudogroup on $\mathbb{R}^n$ that is contained in the diffeomorphism pseudogroup of $\mathbb{R}^n$. A **locally G-structured space** is a pair (S, C), where S is a

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set and C is a collection of injective partial functions $c : S \to \mathbb{R}^n$ that satisfies some intuitive conditions (Barrett and Manchak, 2024a, p. 6). There is a natural way to recover a geometric space from a locally G-structured space (S, C). First, if C^+ is the collection of all n-charts on S that are compatible with all the n-charts in C, then (S, C^+) is a smooth n-dimensional manifold (Barrett and Manchak, 2024a, Proposition 2.2.1). Various levels of geometric structure are then recovered on the manifold (S, C^+). The maps in C suffice to induce a pseudogroup Γ on (S, C^+). Intuitively, this **coordinate transformation pseudogroup** contains all of the maps between open subsets of S that 'transform' between privileged coordinate systems in C, i.e. those homeomorphisms between open sets of S generated by functions of the form $f^{-1} \circ g$, where f and g are in C (Barrett and Manchak, 2024a, Definition 2.2.2). The coordinate transformation pseudogroup Γ now allows one to recover geometric structures on (S, C^+). We will say that a smooth tensor field α (of arbitrary index structure) on a smooth manifold M is **implicitly defined** by a pseudogroup G on M just in case $h^*(\alpha) = \alpha$ for all $h : U \to V$ in G. We now simply equip (S, C^+) with those smooth tensor fields α that are implicitly defined by the coordinate transformation pseudogroup Γ. In this way, one recovers a geometric space from a locally G-structured space.

We need one last definition. Let (S, C) and (S', C') be locally G- and G'-structured spaces, respectively. An **isomorphism** $f : (S, C) \to (S', C')$ is a bijection $f : S \to S'$ such that

1. f is a diffeomorphism between (S, C^+) and (S', C'^+), and
2. the map $s \mapsto f \circ s \circ f^{-1}$ is a bijection between Γ and Γ', the pseudogroups associated with (S, C) and (S', C').

Isomorphisms must preserve the smooth manifold structure (condition 1) and the pseudogroups (condition 2) that the spaces inherit. For details see Barrett and Manchak (2024a, Proposition 3.2.2).

Privileged Coordinates

We now turn to the question of what the privileged coordinates of a geometric space might be. The case of two-dimensional Minkowski spacetime is instructive. Let $(\mathbb{R}^2, \eta_{ab})$ be such that $\eta_{ab} = d_a x^1 d_b x^1 - d_b x^2 d_b x^2$ is the Minkowski metric in standard coordinates. The 'Minkowskian charts' are those coordinate charts (U, ϕ) on $\mathbb{R}^2$ such that $\phi^*(\eta_{ab}) = \eta_{ab}$. It is natural to think of the collection C of Minkowskian charts as the privileged coordinates of Minkowski spacetime. It is worth cataloguing a few features that this collection C has.

Feature 1. $(\mathbb{R}^2, C)$ is a locally G-structured space (Barrett and Manchak, 2024a, Lemma 3.2.3).

Feature 2. The locally G-structured space $(\mathbb{R}^2, C)$ is such that the recovered manifold $(\mathbb{R}^2, C^+)$ is just the underlying manifold $\mathbb{R}^2$ of Minkowski

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spacetime, and the coordinate transformation pseudogroup Γ is the isometry pseudogroup of Minkowski spacetime (Barrett and Manchak, 2024a, Proposition 3.2.1).

Feature 3. They are those coordinates in which the metric η_{ab} 'takes a simple form.' If (U, ϕ) is a Minkowski chart, then $\eta_{ab} = d_a u^1 d_b u^1 - d_a u^2 d_b u^2$ everywhere on U, where u^i are the coordinate maps associated with (U, ϕ).

All of these potential features of privileged coordinates have been emphasized in the literature. Wallace (2019) discusses Feature 1 at length. The atlas of a manifold M forms a locally G-structured space, and it is natural to think of privileged coordinates as retaining this property. Regarding Feature 2, it is common to speak of the structures 'invariant under coordinate transformations' being the significant ones on a geometric space. One especially sees this in discussions of the 'Kleinian method' of presenting a space. For example, North (2021, p. 48) writes that "Klein suggested that any geometry can be identified by means of the transformations that preserve the structure, likewise by the quantities that are invariant under the group of those transformations." Norton (2002, p. 259) and Wallace (2019, p. 135) make similar remarks. One also often sees endorsements of Feature 3. For example, North (2021, p. 112) writes that "[a] preference for certain coordinates, in the sense that the laws take a simple or natural form in them, is indicative of, it is evidence for, underlying structure." Wallace (2019, p. 131) also emphasizes coordinate transformations that leave invariant "the form of the equations."

There are different ways to generalize an account of privileged coordinates from the case of Minkowski spacetime depending on which of these features one takes to be salient. We will describe one account that emphasizes Features 1 and 2, due to Barrett and Manchak (2024a), and then briefly discuss another later on. Neither is an account of privileged coordinates for arbitrary geometric spaces, but it is helpful to first focus on the more straightforward case of relativistic spacetimes in order to ascertain what a general account might look like. In order to pick out the Minkowski charts, one considered the class of isometries between open regions of our spacetime and open regions of a fixed spacetime with underlying manifold $\mathbb{R}^n$ (in this case, Minkowski spacetime itself). One can generalize the idea behind Minkowskian privileged coordinates to arbitrary relativistic spacetimes. Instead of requiring the privileged coordinates to be isometries to Minkowski spacetime, one simply allows them to be isometries to some other fixed spacetime with underlying manifold $\mathbb{R}^n$.

More precisely, one begins by showing that every relativistic spacetime has a representation (Barrett and Manchak, 2024a, Lemma 3.2.2). A **representation** of (M, g_{ab}) is a spacetime $(\mathbb{R}^n, g'_{ab})$ with underlying manifold $\mathbb{R}^n$ such that for every point $p \in M$, there are open sets $O \subset M$ and $O' \subset \mathbb{R}^n$ such that $p \in O$ and (O, g_{ab}) is isometric to (O', g'_{ab}). This fact provides a way to construct a locally G-structured space from a relativistic spacetime (M, g_{ab}). Let (M, g_{ab}) be a relativistic spacetime with $(\mathbb{R}^n, g'_{ab})$ a representation of it. One then defines the following:

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- Let $S = M$.
- Let C be the collection of isometries between open subsets of (M, g_{ab}) and open subsets of $(\mathbb{R}^n, g'_{ab})$, i.e. diffeomorphisms $c : U \to V$ where $U \subset M$ and $V \subset \mathbb{R}^n$ are open and $c^*(g'_{ab}) = g_{ab}|_U$.
- Let G be the isometry pseudogroup of $(\mathbb{R}^n, g'_{ab})$.

Since different choices of representation will result in isomorphic locally G-structured spaces, (S, C) is the locally G-structured space determined by (M, g_{ab}) (Barrett and Manchak, 2024a, Proposition 3.2.3, Lemma 3.2.3). The manifold (S, C^+) is diffeomorphic to M, and the coordinate transformation pseudogroup that it induces is the isometry pseudogroup of (M, g_{ab}) (Barrett and Manchak, 2024a, Proposition 3.2.1). This means that this account of privileged coordinates for relativistic spacetimes has Features 1 and 2. It is a natural account to adopt if one wants to assert (without caveat) that the significant structures of a spacetime are those 'invariant under coordinate transformation.' And on this understanding, we have a guarantee (since Feature 2 holds) that a spacetime's underlying manifold and its metric — its "intrinsic nature" — are among "the coordinate-independent, invariant features" (North, 2021, p. 26), and hence among those structures recovered by the privileged coordinates in the manner discussed above. However, because the representations for some spacetimes will have metrics whose 'forms' vary, it seems that Feature 3 does not hold of this account. This suggests that other accounts might be possible. We return to this point in the conclusion.

One might be able to extend this account of privileged coordinates for relativistic spacetimes to arbitrary geometric spaces, but we will not pursue the details here. Rather, we will take inspiration from this case and make two natural assumptions about what a general account might look like. Let M be a smooth manifold with $\alpha_1, \ldots, \alpha_n$ smooth tensors on M. We make the following two assumptions about the locally G-structured space (S, C) that the geometric space $(M, \alpha_1, \ldots, \alpha_n)$ determines.

P1. (S, C^+) and M are diffeomorphic.

P2. The coordinate transformation pseudogroup Γ on (S, C) is the same as the automorphism pseudogroup of $(M, \alpha_1, \ldots, \alpha_n)$.

We are simply assuming that one can provide an account of privileged coordinates for arbitrary geometric spaces that has analogues of Features 1 and 2. The fact that $(M, \alpha_1, \ldots, \alpha_n)$ determines a locally G-structured space (S, C) is just to say that the account has Feature 1; assuming P1 and P2 means that the account has Feature 2.

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3 Do privileged coordinates determine amounts of structure?

We now ask whether these privileged coordinates of a geometric space encode its amount of structure. Barrett and Manchak (2024a) have shown that for spacetimes (M, g_{ab}) with particularly small isometry pseudogroups, there are non-isometric metrics on M that are implicitly defined by the resulting coordinate transformation pseudogroup Γ. The locally G-structured space (S, C) that they determine will therefore recover more than one metric, meaning that one cannot recover the entire structure of a relativistic spacetime (up to isometry) from (S, C).

One might still wonder, however, whether privileged coordinates tell us something weaker about a geometric space. Its 'amount of structure' is a natural candidate. This is in accord with the significance that North (2021) attributes to privileged coordinates in her discussion of Lagrangian and Newtonian mechanics; they allow one to compare amounts of structure between the theories. Indeed, it seems that one can often reasonably compare amounts of structure without knowing the exact details of the structures under consideration. For example, one is comfortable saying that a relativistic spacetime (M, g_{ab}) has more structure than its underlying manifold M, even without knowing exactly what the metric g_{ab} is. This suggests that one does not need to be able to recover the entire structure of (M, g_{ab}) (up to isometry) in order to compare its amount of structure to that of other spaces. The following example is perhaps even more suggestive. Let $(M, \alpha_1, \ldots, \alpha_n)$ be a geometric space with β some other smooth tensor field on M. If one assumes P1 and P2, then it seems that the privileged coordinates of $(M, \alpha_1, \ldots, \alpha_n)$ and $(M, \alpha_1, \ldots, \alpha_n, \beta)$ can help us to compare their amounts of structure. This is because one expects that the addition of the field β will result in $(M, \alpha_1, \ldots, \alpha_n, \beta)$ having a smaller automorphism pseudogroup than $(M, \alpha_1, \ldots, \alpha_n)$. P2 implies that $(M, \alpha_1, \ldots, \alpha_n, \beta)$ has a smaller coordinate transformation pseudogroup than $(M, \alpha_1, \ldots, \alpha_n)$. So $(M, \alpha_1, \ldots, \alpha_n, \beta)$ must have had fewer privileged coordinates to begin with, since more privileged coordinates will result in more coordinate transformations (and thus a larger coordinate transformation pseudogroup). This case mirrors some of the examples — in particular, the case of Newtonian and Galilean spacetime — that we discussed at the outset. One expects that this exact reasoning can be applied in those cases as well.

Barrett and Manchak (2024a, p. 21) ask the following question: Are there geometric spaces with different amounts of structure that determine the same locally G-structured space? We show that the answer is "yes." This implies that the amount of structure that a geometric space has is not always encoded by the locally G-structured space that it determines. In order to provide this affirmative answer, we need some further details. We will say that a pseudogroup on a manifold M is **trivial** if it only contains identity maps. A relativistic spacetime (M, g_{ab}) is **Heraclitus** if, for any open subsets $U, V \subset M$ and any isometry $\psi : U \to V$, it follows that (i) $U = V$ and (ii) ψ is the identity map.

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Manchak and Barrett (2024) show that a Heraclitus spacetime exists. One can easily verify that the isometry pseudogroup of (M, g_{ab}) is trivial if and only if (M, g_{ab}) is Heraclitus.

Let (M, g_{ab}) be a Heraclitus spacetime. The two geometric spaces that we will consider are (M, g_{ab}) and the geometric space (M, g_{ab}, λ), where λ is an arbitrary tensor field on M that is not 'constructible' in terms of the metric g_{ab}. So, for example, λ is not some scalar multiple of g_{ab}, the Riemannian curvature tensor associated with g_{ab}, etc. Note that (M, g_{ab}, λ) has a trivial automorphism pseudogroup since it must be contained in the isometry pseudogroup of (M, g_{ab}), which is itself trivial. We put forward the following claim:

P3. (M, g_{ab}, λ) has more structure than (M, g_{ab}).

(M, g_{ab}, λ) results from adding the structure λ to (M, g_{ab}). Since λ is not constructible from g_{ab}, it is a genuinely new level of structure on the space. There is thus a compelling sense in which P3 holds. We now have the following result.

Theorem 1. *If P1, P2, and P3, then there are geometric spaces with different amounts of structure that determine isomorphic locally G-structured spaces.*

Proof. We consider the two geometric spaces (M, g_{ab}) and (M, g_{ab}, λ). P3 implies that they have different amounts of structure. We need only show that they determine isomorphic locally G-structured spaces. Let (S, C) be the locally G-structured space determined by (M, g_{ab}) and (S', C') the locally G-structure space determined by (M, g_{ab}, λ). P1 implies that there is a diffeomorphism $f : (S, C^+) \to (S', C'^+)$, since both of those manifolds must be diffeomorphic to M. Since the automorphism pseudogroups of (M, g_{ab}) and (M, g_{ab}, λ) are trivial, P2 implies that the coordinate transformation pseudogroups Γ and Γ' are trivial too. We now show that $f : S \to S'$ must be an isomorphism between (S, C) and (S', C'). We know immediately that f satisfies condition 1 of the definition of an isomorphism. We show that f also satisfies condition 2. We need to show that the map $s \mapsto f \circ s \circ f^{-1}$ is a bijection from Γ to Γ'. Let $s, s' \in \Gamma$ and suppose that $f \circ s \circ f^{-1} = f \circ s' \circ f^{-1}$. Since $f : S \to S'$ is a bijection, it must be that $s = s'$. Hence our map $s \mapsto f \circ s \circ f^{-1}$ is injective. Now let $s' \in \Gamma'$, so s' is the identity map 1_O on some open set $O \subset S'$. We see that $f^{-1} \circ 1_O \circ f = 1_{f^{-1}[O]}$. Since f is a diffeomorphism, $f^{-1}[O]$ is an open subset of S, and hence $1_{f^{-1}[O]}$ must be in Γ. (This is because a pseudogroup must contain the identity map for every open subset (Kobayashi and Nomizu, 1996, p. 1).) Since $f \circ 1_{f^{-1}[O]} \circ f^{-1} = 1_O$, our map is bijective, f satisfies condition 2, and hence f is an isomorphism between (S, C) and (S', C'). □

Theorem 1 tells us that there are geometric spaces with different amounts of structure that nonetheless have isomorphic collections of privileged coordinates. This means that the privileged coordinates of a geometric space, insofar as P1 and P2 hold, do not provide a perfect guide to its amount of structure. One can know the privileged coordinates of a geometric space — in the form of the locally G-structured space that it determines — but not be able to assess how much structure it has. The problem that Theorem 1 poses for the coordinate approach

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is related to the one that Barrett and Manchak (2024a) pose for attempts to present a geometric space by appealing to its privileged coordinates. Both problems are generated by Heraclitus spacetimes, and both point to a central issue with implicit definability. We turn to this issue in the following section.

4 Symmetries, Coordinates, and Definability

Given that there are cases in which the coordinate approach does not work, two natural questions remain. First, one wonders exactly how the coordinate and automorphism approaches are related. One would like to know the comparative benefits and drawbacks of each. Second, one wonders whether these approaches might still be useful in cases of philosophical interest, despite their shortcomings. We will address these questions by first discussing implicit definability. This will allow us to diagnose exactly where the problems faced by the coordinate approach come from, and it will allow us to make precise the close relationship between the two approaches. It will also suggest how one might still judiciously employ reasoning about automorphisms and privileged coordinates to compare amounts of structure.

The following discussion has precedent in the recent literature. In particular, it synthesizes and builds upon the closely related results of Barrett (2018, 2021b), Barrett et al. (2023), and Manchak and Barrett (2024). It extends each of these papers in different ways. Barrett (2018, 2021b) and Barrett et al. (2023) focus on the automorphism approach (and the 'category approach,' which we will not discuss here). Here we extend that discussion to the coordinate approach. The results about implicit definability proven in those papers are all in the context of first-order theories. Here we extend some of those results to the context of theories formulated using the tools of differential geometry. (We note that discussions of implicit definability in the geometrical context have precedent. The famous result of Malament (1977) concerns which 'simultaneity relations' are implicitly definable on Minkowski spacetime. This case is discussed in detail by Winnie (1986).) Barrett (2018, 2021b) discusses the idea that some kind of definability tracks what structure an object comes equipped with. Here we return to this idea, but in addition to the 'global' varieties of implicit definability considered there, we consider a 'local' variety. Relatedly, those three papers discuss problems for the automorphism approach that are generated by the existence of objects that have small automorphism groups, but they do not touch on the more pernicious problems generated by the existence of objects with small automorphism *pseudogroups*. Here we extend the discussion to the latter case. In this sense, our results are building upon those of Manchak and Barrett (2024), who discuss both giraffe and Heraclitus spacetimes in detail, and comment briefly on the problems that they generate for the automorphism approach. Here we extend that discussion to the coordinate approach and trace the problems back to a more foundational problem with implicit definability.

Coordinates or Automorphisms?

Our first aim is to make precise the automorphism and coordinate approaches. In doing so, we begin to see the close relationship they bear to one another. The following criterion is representative of the automorphism approach.

SYM*. A mathematical object X has at least as much structure as a mathematical object Y if (and only if) $\text{Aut}(X) \subset \text{Aut}(Y)$.

The condition that $\text{Aut}(X) \subset \text{Aut}(Y)$ is one way to make precise the idea that X admits 'no more' automorphisms than Y does. SYM* works well in easy cases; see (Barrett, 2021b) for details. But it makes unsatisfactory verdicts in cases where the objects under consideration admit *few* symmetries (Barrett, 2021b; Barrett et al., 2023). This idea is familiar from above. Following Manchak and Barrett (2024), we will call a relativistic spacetime (M, g_{ab}) **giraffe** if it has a trivial isometry group, i.e. the only diffeomorphism $f : M \to M$ such that $f^*(g_{ab}) = g_{ab}$ is the identity map. Since every Heraclitus spacetime is giraffe, it follows that a giraffe spacetime (M, g_{ab}) exists. According to SYM*, the spacetime (M, g_{ab}) has at least as much structure as (M, g_{ab}, λ), for any tensor field λ on M, because the automorphism group of (M, g_{ab}) is already as small as can be. This strikes one as a bad verdict.

We need to make the coordinate approach precise in order to see whether it improves upon SYM*. Recall that the coordinate approach is based upon the idea that fewer privileged coordinates should indicate more structure. More precisely, the amount of structure that a geometric space has should be correlated with the size of the coordinate transformation group Γ that its privileged coordinates determine. After all, the coordinate transformation group is what one uses to recover tensor fields on the geometric space. The following criterion makes this idea precise.

COORD. X has at least as much structure as Y if (and only if) the coordinate transformation pseudogroup Γ that X determines is a subset of the coordinate transformation pseudogroup Γ' that Y determines.

The motivation for COORD is closely related to the motivation for SYM*. If Γ is contained in Γ', then — insofar as we equip X and Y with exactly those tensor fields 'invariant under coordinate transformations' — X will have at least as much structure as Y. COORD works in easy cases of structural comparison. For example, recall our comparison of the geometric spaces $(M, \alpha_1, \ldots, \alpha_n)$ and $(M, \alpha_1, \ldots, \alpha_n, \beta)$ above. Insofar as β is a 'genuinely new' level of structure on $(M, \alpha_1, \ldots, \alpha_n)$ and not constructible from $\alpha_1, \ldots, \alpha_n$, one expects that the latter will have at least as much structure as the former, but not vice versa, according to COORD.

There is a sense in which COORD is worse than SYM*, and another sense in which it is better. It is worse because it is only applicable to geometric spaces, not arbitrary mathematical objects that one might use to formulate a physical theory. It makes sense to discuss the automorphisms of any mathematical object; it does not always make sense to discuss an object's privileged coordinates

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or its coordinate transformation group. At best, that will only make sense for geometric spaces. On the other hand, COORD is better than SYM* because it does not run into difficulty with mere giraffe spacetimes. Compare again the giraffe spacetime (M, g_{ab}) with (M, g_{ab}, λ). The fact that (M, g_{ab}) and (M, g_{ab}, λ) have the same trivial automorphism group does not imply that they have the same coordinate transformation *pseudogroups*. For this reason, COORD does not run into problems with all giraffe spacetimes. Not every giraffe spacetime is Heraclitus (Manchak and Barrett, 2024). And so if (M, g_{ab}) is giraffe but not Heraclitus, then while its automorphism group is as small as can be, its isometry pseudogroup is not. The addition of a tensor field λ may further reduce the automorphism pseudogroup, and hence the coordinate transformation pseudogroup Γ' determined by (M, g_{ab}, λ) can be properly contained in that of (M, g_{ab}). Thus it can be that (M, g_{ab}) does not have at least as much structure as (M, g_{ab}, λ) according to COORD. Of course, Theorem 1 shows that Heraclitus spacetimes generate problems for COORD, but mere giraffe spacetimes do not, and in this sense COORD represents an improvement upon SYM*.

These considerations suggest the following improvement of SYM*.

SYM*2. A mathematical object X has at least as much structure as a mathematical object Y if and only if the automorphism pseudogroup of X is contained in the automorphism pseudogroup of Y.

P2 implies that SYM*2 and COORD are equivalent. SYM*2 therefore inherits all the benefits and drawbacks of COORD. It does not necessarily struggle with giraffe spacetimes. It does struggle with Heraclitus spacetimes and is only applicable to objects that have pseudogroups (and one can only define a pseudogroup on objects that have at least topological structure). On the other hand, it improves upon COORD since one does not need an account of privileged coordinates to apply SYM*2. One simply considers the automorphism pseudogroups of the geometric spaces. The equivalence of SYM*2 and COORD (assuming P2) shows how closely related the automorphism and coordinate approaches are. This is intuitive; singling out a collection of privileged coordinates is just another way of singling out a collection of symmetries. Since automorphism pseudogroups do not perfectly encode amounts of structure, neither do privileged coordinates.

The Argument from Definability

Having catalogued the basic relationship between the coordinate and automorphism approaches, we now want to isolate precisely where their shortcomings come from. We will do so by presenting an argument for SYM*, SYM*2, and COORD. This argument will not succeed, but the precise way in which it fails will suggest how to salvage something from these approaches. Following Barrett (2021b), we will call this the "argument from definability." (See Winnie (1986), Halvorson (2019), and the references therein for philosophically motivated discussions of definability.) In brief, the argument points out that the

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automorphism and coordinate criteria SYM*, SYM*2, and COORD track facts about implicit definability. It is natural to think that a mathematical object comes equipped with those structures that it implicitly defines. If so, the automorphism and coordinate approaches track facts about which structures an object comes equipped with, and this explains how they encode amounts of structure.

Suppose that we have an object X and a collection of maps from X to itself. A structure is implicitly defined on X by this collection of maps if the maps 'preserve' that structure. In the context of geometric spaces, one can make this precise in the following two ways. Let M be a smooth manifold with G a group of diffeomorphisms $f : M \to M$. We will say that a smooth tensor field λ on M is **globally implicitly defined** by G if $f^*(\lambda) = \lambda$ for every $f \in G$. Following our discussion in section 2, if Γ is a pseudogroup on M, we will say that a smooth tensor field λ on M is **locally implicitly defined** by Γ if $f^*(\lambda) = \lambda$ for every $f \in \Gamma$. If G is the automorphism group of $(M, \alpha_1, \ldots, \alpha_n)$ and G globally implicitly defines λ, we will say simply that λ is globally implicitly defined by $(M, \alpha_1, \ldots, \alpha_n)$. Similarly, if Γ is the automorphism pseudogroup of $(M, \alpha_1, \ldots, \alpha_n)$ and Γ locally implicitly defines λ, we will say that λ is locally implicitly defined by $(M, \alpha_1, \ldots, \alpha_n)$.

These two varieties of implicit definability are related to one another exactly as one would expect. Let $(M, \alpha_1, \ldots, \alpha_n)$ be a geometric space.

Proposition 1. *If λ is locally implicitly defined by $(M, \alpha_1, \ldots, \alpha_n)$, then it is globally implicitly defined by $(M, \alpha_1, \ldots, \alpha_n)$; the converse does not hold.*

Proof. It follows easily from definitions that if λ is locally implicitly defined by $(M, \alpha_1, \ldots, \alpha_n)$, then it is globally implicitly defined by $(M, \alpha_1, \ldots, \alpha_n)$. Let $(\mathbb{R}^2, \eta_{ab})$ be Minkowski spacetime, and consider the spacetime (M, η_{ab}) where $M = \{(t, x) : 0 < t < 1, 0 < x, x^2 < t^2\}$. Manchak and Barrett (2024, Example 6) show that this spacetime is giraffe but not Heraclitus. Since (M, η_{ab}) is giraffe, every tensor field λ on M is globally implicitly defined by (M, η_{ab}). One shows, however, that $(\frac{\partial}{\partial t})^a$ is not locally implicitly defined on (M, η_{ab}). □

The basic idea behind this result is easy to appreciate. There are geometric spaces with trivial automorphism groups that do not have trivial automorphism pseudogroups. Every tensor field on such a space will be globally implicitly defined, despite some of those fields not being preserved by the richer collection of maps in the automorphism pseudogroup.

These two varieties of implicit definability provide the core mechanisms by which SYM*, SYM*2, and COORD function. We begin with the cases for SYM* and SYM*2.

Proposition 2. *Let $(M, \alpha_1, \ldots, \alpha_m)$ and $(M, \beta_1, \ldots, \beta_n)$ be geometric spaces. The following are equivalent.*

1. *The automorphism group of $(M, \alpha_1, \ldots, \alpha_m)$ is a subset of the automorphism group of $(M, \beta_1, \ldots, \beta_n)$.*

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2. *The space* $(M, \alpha_1, \ldots, \alpha_m)$ *globally implicitly defines all of the tensors that* $(M, \beta_1, \ldots, \beta_n)$ *globally implicitly defines.*

Proof. Assume 1 and let λ be a tensor that $(M, \beta_1, \ldots, \beta_n)$ globally implicitly defines. Since the automorphism group of $(M, \alpha_1, \ldots, \alpha_m)$ is contained in the automorphism group of $(M, \beta_1, \ldots, \beta_n)$, the former globally implicitly defines λ too. Now assume 2 and suppose for contradiction that f is in the automorphism group of $(M, \alpha_1, \ldots, \alpha_m)$ but not in the automorphism group of $(M, \beta_1, \ldots, \beta_n)$. This means that $f : M \to M$ is a diffeomorphism but that there is some β_j such that $f^*(\beta_j) \neq \beta_j$. This means that $(M, \alpha_1, \ldots, \alpha_m)$ does not globally implicitly define β_j. This contradicts 2 since $(M, \beta_1, \ldots, \beta_n)$ clearly does globally implicitly define β_j. □

An analogous result holds about SYM*2. We leave the proof to the reader since it is essentially the same as that of Proposition 2.

Proposition 3. *Let* $(M, \alpha_1, \ldots, \alpha_m)$ *and* $(M, \beta_1, \ldots, \beta_n)$ *be geometric spaces. The following are equivalent.*

1. *The automorphism pseudogroup of* $(M, \alpha_1, \ldots, \alpha_m)$ *is a subset of the automorphism pseudogroup of* $(M, \beta_1, \ldots, \beta_n)$.

2. *The space* $(M, \alpha_1, \ldots, \alpha_m)$ *locally implicitly defines all of the tensors that* $(M, \beta_1, \ldots, \beta_n)$ *locally implicitly defines.*

Propositions 2 and 3 show that SYM* and SYM*2 perfectly track implicit definability. The first condition of Proposition 2 says that $(M, \alpha_1, \ldots, \alpha_m)$ has at least as much structure as $(M, \beta_1, \ldots, \beta_n)$ according to SYM*. Hence SYM* says that X has at least as much structure as Y just in case X globally implicitly defines all of the structures of Y. The first condition of Proposition 3 says that $(M, \alpha_1, \ldots, \alpha_m)$ has at least as much structure as $(M, \beta_1, \ldots, \beta_n)$ according to SYM*2. Hence SYM*2 says that X has at least as much structure as Y just in case X locally implicitly defines all of the structures of Y.

COORD follows the same pattern. We again leave the straightforward proof to the reader.

Proposition 4. *Let* $(M, \alpha_1, \ldots, \alpha_m)$ *and* $(M, \beta_1, \ldots, \beta_n)$ *be geometric spaces with coordinate transformation pseudogroups* Γ *and* Γ'*, respectively. If P2 holds, then the following are equivalent.*

1. $\Gamma \subset \Gamma'$.

2. *The space* $(M, \alpha_1, \ldots, \alpha_m)$ *locally implicitly defines all of the tensors that* $(M, \beta_1, \ldots, \beta_n)$ *locally implicitly defines.*

Since the first condition says that $(M, \alpha_1, \ldots, \alpha_m)$ has at least as much structure as $(M, \beta_1, \ldots, \beta_n)$ according to COORD, we see that COORD is also tracking implicit definability.

We need one further thought about implicit definability in order to state the argument from definability. It is natural to take those structures that are

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'invariant under the symmetries' of a mathematical object to be part of the genuine structure of that object. This was the core idea behind the 'Kleinian method' discussed above. Invariance under symmetry is often taken to indicate that, in some sense, the structure 'comes for free' given the basic structures on the object. For example, a metric space (X, d) comes equipped with its metric topology τ, despite the fact that τ is not explicitly appealed to in the presentation of (X, d). One way of accounting for this is to notice that every symmetry of (X, d) — that is, every distance-preserving bijection from X to itself — preserves τ in the sense that it is a homeomorphism with respect to τ. Hence τ is invariant under the symmetries of (X, d). If implicit definability tracks which structures an object comes equipped with, then we have an account of why (X, d) comes equipped with its metric topology.

One can make this idea precise in the following two ways; each corresponds to one of our varieties of implicit definability.

Global P4. A geometric space $(M, \alpha_1, \ldots, \alpha_n)$ comes equipped with all and only the structures that it globally implicitly defines.

Local P4. A geometric space $(M, \alpha_1, \ldots, \alpha_n)$ comes equipped with all and only the structures that it locally implicitly defines.

Local P4 implies that a geometric space will (in general) come equipped with fewer structures than Global P4 implies it will. This is because, by Proposition 1, fewer tensor fields will be locally implicitly defined than will be globally implicitly defined.

These principles form the crucial premises in the arguments from definability for SYM*, SYM*2, and COORD. Suppose that Global P4 is true. This means that the second condition of Proposition 2 is saying that $(M, \alpha_1, \ldots, \alpha_m)$ comes equipped with all of the structures that $(M, \beta_1, \ldots, \beta_n)$ comes equipped with. This is a particularly natural way in which the former might have at least as much structure as the latter, and it is (by Proposition 2) equivalent to SYM* making that verdict. Global P4 therefore leads us to SYM*. Similarly, Local P4 implies that the second conditions of Propositions 3 and 4 are saying that $(M, \alpha_1, \ldots, \alpha_m)$ comes equipped with all of the structures that $(M, \beta_1, \ldots, \beta_n)$ comes equipped with. Since these second conditions are (by Propositions 3 and 4) equivalent to $(M, \alpha_1, \ldots, \alpha_m)$ having at least as much structure as $(M, \beta_1, \ldots, \beta_n)$ according to SYM*2 and COORD, Local P4 leads us to those criteria. We therefore have arguments from definability for SYM*, SYM*2, and COORD.

The triviality problem

The problem with these arguments is that both Global P4 and Local P4 seem false. The case of Global P4 is straightforward and mirrors the arguments above. Let (M, g_{ab}) be a giraffe spacetime. Since it is giraffe, every smooth tensor field λ on M is globally implicitly defined by (M, g_{ab}). But most of these fields are not in any sense 'constructible' from g_{ab}; they are simply arbitrary tensor

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fields on M. So one does not want to say that (M, g_{ab}) comes equipped with them. Indeed, in many cases they will not even be locally implicitly definable by (M, g_{ab}). To put the point acutely, an arbitrary *metric* on M is globally implicitly defined by (M, g_{ab}), and since most of these metrics will not be related to g_{ab} in any interesting sense, one certainly does not want to say that (M, g_{ab}) comes equipped with them in the same sense as it comes equipped with g_{ab}. It therefore seems that Global P4 cannot be true. A geometric space does not always come equipped with all the structures it globally implicitly defines.

The failure of Global P4 has been appreciated. Barrett (2021b) calls it the 'triviality problem'; it is discussed in detail by Barrett et al. (2023). North (2021, p. 117) points to it when she writes that there are geometric spaces that "lie beyond the scope of Klein's program," and Torretti (2016) explicitly mentions the problem of trivial isometry groups faced by Kleinian methods. But one might expect that the prospects are better for Local P4. Indeed, by Proposition 1, Local P4 will imply that (M, g_{ab}) comes equipped with fewer structures than Global P4 implies it does, and so perhaps the triviality problem can be avoided. Unfortunately, this is not the case. We now let (M, g_{ab}) be a Heraclitus spacetime. Every smooth tensor field λ on M is locally implicitly defined by (M, g_{ab}). One again has the strong feeling that the vast majority of these fields are not in any sense 'constructible' from g_{ab}, and so one does not want to say that (M, g_{ab}) comes equipped with them. If so, Local P4 also cannot be right. A geometric space does not always seem to come equipped with all the structures it locally implicitly defines.

We have therefore identified exactly where the argument from definability fails. The shortcomings of the automorphism and coordinate approaches trace back to a basic limitation faced by notions of implicit definability that look to 'invariance under symmetry.' In general, more structures will be invariant under symmetry than one wants to countenance among the genuine structures of the object.

5 Conclusion

We conclude with two remarks. First, we make a few suggestions about how one might respond to our results. Second, we use our discussion of implicit definability to salvage an idea from the automorphism and coordinate approaches. Along the way, we catalogue a number of interesting questions for future work.

Responses

We will mention two interesting ways in which one might respond to the results presented in sections 3 and 4. The first concerns a subtlety related to P3 and Local P4. In brief, there may be room for one to argue that Local P4 is true. If so, then one would have a compelling argument against P3. The second concerns whether one might provide another account of 'privileged coordinates'

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that better suits the purposes of the coordinate approach. We take these two points in turn.

First, one might ask the following question (Barrett and Manchak, 2024a, p. 19).

Question 1. *Is there an interesting account of 'explicit definability' in the context of geometric spaces?*

Suppose that one answers Question 1 in the affirmative and formulates an interesting variety of explicit definability, capturing a sense in which a field λ is 'constructible' from the fields on a geometric space $(M, \alpha_1, \ldots, \alpha_n)$. Such an account would come to bear on the issues discussed here. It would be natural to then consider the following revision of Global P4 and Local P4.

P5. A geometric space $(M, \alpha_1, \ldots, \alpha_n)$ comes equipped with all and only the structures that it explicitly defines.

The same kinds of examples that motivated Global P4 and Local P4 could be used to motivate P5. Suppose, for example, that one has a vector space with inner product $(V, \langle,\rangle)$. It is natural to think that this object comes equipped with a norm $||\cdot||$, which assigns to a vector $v \in V$ its 'length' $||v||$. Not only is the norm invariant under the symmetries of V, it is directly constructible from the inner product. One 'explicitly defines' the norm in terms of the inner product by letting $||v|| = \langle v, v\rangle$. So P5 would explain why we are inclined to say that $(V, \langle,\rangle)$ comes equipped with a norm $||\cdot||$.

One might, however, have better conceptual reasons to adopt P5 than Local P4 or Global P4. In particular, if some structure is explicitly definable on $(M, \alpha_1, \ldots, \alpha_n)$, that would capture a sense in which the basic structures of $(M, \alpha_1, \ldots, \alpha_n)$ suffice to 'construct' or 'build' that new structure. And this would perhaps provide a more compelling reason to think this new structure 'comes for free' given the basic structures on $(M, \alpha_1, \ldots, \alpha_n)$ than mere implicit definability provides.

We will assume that if an interesting variety of explicit definability for geometric spaces could be made precise, then it would entail local implicit definability. This parallels the state of affairs in first-order logic where these notions are well understood. It is then natural to ask the following question.

Question 2. *Does local implicit definability entail explicit definability?*

It is well known that in the first-order context, there are some (particularly strong) varieties of implicit definability that entail explicit definability. Beth's Theorem provides a famous example of this. An affirmative answer to Question 2 would therefore not be entirely without precedent.

If the answer to Question 2 is "yes" and P5 holds, then one would have an argument for Local P4 and against P3. The argument for Local P4 would be precisely the same as the argument that one provides for P5. Indeed, P5 and Local P4 would be equivalent, since local implicit definability and explicit definability would themselves be equivalent. The argument against P3 would

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note that since (M, g_{ab}) is Heraclitus, it locally implicitly defines every tensor field λ on M. The affirmative answer to Question 2 would then imply that λ is explicitly definable by (M, g_{ab}). P5 would entail that (M, g_{ab}) comes equipped with λ, and hence (M, g_{ab}, λ) would not have more structure than (M, g_{ab}).

On the other hand, if the answer to Question 2 is "no," then insofar as P5 holds, Local P4 would be false. The negative answer to Question 2 would imply that local implicit definability and explicit definability are inequivalent, and hence one cannot both adopt P5 and Local P4. In particular, there would be some geometric space $(M, \alpha_1, \ldots, \alpha_n)$ and smooth tensor λ on M such that $(M, \alpha_1, \ldots, \alpha_n)$ locally implicitly defines λ but does not explicitly define λ. P5 would imply that $(M, \alpha_1, \ldots, \alpha_n)$ does not come equipped with λ, contradicting Local P4. And moreover, one would have a correspondingly more robust argument for P3. The addition of a tensor field λ to a geometric space that does not explicitly define λ will result in a space that comes equipped with more structure.

It is therefore important to examine Question 2 and P5 further. We will leave careful investigation to future work, but it is worth mentioning one example here. Suppose that (M, g_{ab}) is a Heraclitus spacetime. We know that every derivative operator ∇ on M is locally implicitly defined by (M, g_{ab}). (Of course, strictly speaking ∇ is not a tensor field on M, but by slightly extending our terminology, we can still speak of it being locally implicitly defined by (M, g_{ab}), in the sense that all maps in the isometry pseudogroup of (M, g_{ab}) preserve ∇. See Weatherall (2016) or Barrett (2015b, 2021a) for a precise account.) Despite the fact that all of these derivative operators are locally implicitly defined by (M, g_{ab}), one is tempted to say that (M, g_{ab}) only genuinely comes equipped with one of them: the unique derivative operator that is compatible with g_{ab}, i.e. the Levi-Civita derivative operator of (M, g_{ab}). If this is right, then the answer to Question 2 will be "no" for any variety of explicit definability for which P5 holds. For if P5 holds and local implicit definability entail explicit definability, then a Heraclitus spacetime (M, g_{ab}) will come equipped with all of the derivative operators on M. The central question here concerns what kind of definability (if any) best captures which structures a geometric space comes equipped with. We have seen reasons to think that this is neither global nor local implicit definability. It remains to be seen whether there is a better candidate.

We now turn to the second way in which one might respond to the results in sections 3 and 4. We pose the following question, which is closely related to what Barrett and Manchak (2024a) call "Revision 1."

Question 3. *Is there another account of privileged coordinates that is better suited to comparing amounts of structure?*

The account suggested in section 2 emphasized Features 1 and 2 that the natural account of privileged coordinates for Minkowski spacetime exhibited. Theorem 1 implies that all such accounts will struggle with Heraclitus spacetimes. One might suspect that there is an account of privileged coordinates for arbitrary geometric spaces that emphasizes Feature 3 instead. And if so, it is worth examining whether such an account would improve the coordinate approach.

We here make a few preliminary remarks on Question 3. A natural place to start is by taking the 'Lorentz normal coordinates' of a relativistic spacetime (M, g_{ab}) to be its privileged coordinates. (See further discussion in Barrett and Manchak (2024a,b).) There is a sense in which this account would have Feature 3; in Lorentz normal coordinates about p the metric takes the simple Minkowskian form at the point p. But this account of privileged coordinates for (M, g_{ab}) does not have Features 1 or 2. In general, the Lorentz normal coordinates for (M, g_{ab}) do not form a locally G-structured space. (In essence, it is the fact that Lorentz normal coordinates must map to open sets surrounding the origin that prevents them from forming a locally G-structured space.) And moreover, one can easily see that Lorentz normal coordinates do not have Feature 2 (Barrett and Manchak, 2024a, p. 20). This means that one could not maintain (without caveat) that the structures of (M, g_{ab}) are those 'invariant under coordinate transformations.' But it also suggests that Theorem 1 would not apply to this account.

In order to answer Question 3, one would have to generalize this account from relativistic spacetimes to arbitrary geometric spaces, and then prove that such coordinates always exist. (The basic idea would be to have privileged coordinates about the point p be those in which the structures of the geometric space at p is well reflected by 'natural' structures on $\mathbb{R}^n$. The less structure the geometric space has, therefore, the less stringent this requirement would become.) One could only then examine whether the resulting account dodges the triviality problems we have considered. Regardless, one conjectures that there would again be a close relationship between the coordinate approach and the automorphism approach, and that some kind of implicit definability would be mediating this connection. Further work remains to be done on Question 3, but for now, it is worth acknowledging that there are different things one might mean by the 'privileged coordinates' of a geometric space. There is value in investigating all of the possibilities.

Prospects

We conclude by assessing the prospects moving forward for the automorphism and coordinate approaches. As we have discussed at length, these approaches struggle with giraffe and Heraclitus spacetimes. But giraffe and Heraclitus spacetimes are extreme cases, and as such one might think that they are rare. If so, these approaches might still work in most cases. This strikes us as implausible. To the contrary, one conjectures that almost all relativistic spacetimes are giraffe and Heraclitus, in the sense that they are *generic*. It is worth explicitly posing the following questions:

Question 4. *Are giraffe spacetimes generic? Are Heraclitus spacetimes generic?*

It is likely that both answers are "yes." Steps have been taken toward proving the genericity of giraffe and Heraclitus spacetimes — for example, see

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Ebin (1968), Fischer (1970), Sunada (1985, Proposition 1), and Mounoud (2015, Theorem 1) — but we are not aware of a full proof in either case. That said, it has been claimed that"everyone knows" that giraffe spacetimes are generic (D'Ambra and Gromov, 1991, p. 21). Presumably this holds for Heraclitus spacetimes as well. If the answers are in fact "yes," then this implies that the automorphism and coordinate approaches will struggle in almost all cases. Given this, one might worry that all we have is some simple cases where the coordinate and automorphism approaches work — like the transition from Newtonian spacetime to Galilean spacetime — and many more simple cases where they do not — involving giraffe spacetimes and Heraclitus spacetimes. One might then wonder whether appeals to automorphisms or privileged coordinates to compare amounts of structure are ever justified.

We would like to suggest that one can still use these tools responsibly. We begin with the following question.

Question 5. *Is there an interesting class of geometric spaces for which Global P4 holds? Is there an interesting class of geometric spaces for which Local P4 holds?*

If the answer to Question 4 is "yes," then these interesting classes cannot be particularly large. But one does suspect that for spaces with 'many symmetries,' Global P4 or Local P4 is true. For geometric spaces with rich automorphism (pseudo)groups, implicit definability should track which structures the object comes equipped with. For example, Barrett and Manchak (2024a, Proposition 4.1.1) show that the only metrics globally implicitly defined on Minkowski spacetime are isometric to the Minkowski metric. Global P4 and Local P4 may thus be true of Minkowski spacetime.

Further work on Question 5 is required. In the meantime, however, one can draw conclusions about amounts of structure from facts about automorphisms or privileged coordinates, so long as one is careful. The precise way in which the argument from definability failed points to a way forward. In particular, it is natural to adopt the following weaker versions of Global P4 and Local P4 related to 'Padoa's method' (Hodges, 2008, Lemma 2.1.1, p. 65, and p. 302).

Global P6. A geometric space $(M, \alpha_1, \ldots, \alpha_n)$ comes equipped with only those structures that it globally implicitly defines.

Local P6. A geometric space $(M, \alpha_1, \ldots, \alpha_n)$ comes equipped with only those structures that it locally implicitly defines.

The existence of giraffe and Heraclitus spacetimes undermines the idea that a geometric space comes equipped with *all* the structures it globally or locally implicitly defines. But global and local implicit definability still seem to be necessary conditions on a geometric space coming equipped with some structure. If we can exhibit a (local or global) symmetry of a space that does not preserve some structure, we can conclude that the space does not come equipped with that structure.

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Global P6 and Local P6 provide a 'partial' argument from definability. In particular, one can verify that they lead to the 'only if' directions of SYM*, SYM*2, and COORD in the same way as Global P4 and Local P4 lead to (both directions of) those criteria. Global P6 and Local P6 do not imply the 'if' directions of SYM*, SYM*2, and COORD, but they nonetheless suffice to generate some interesting results. Two examples will be helpful. First, consider the case of Newtonian and Galilean spacetime. One shows that the Newtonian standard of rest is not globally implicitly defined by Galilean spacetime. Global P6 implies that Galilean spacetime does not come equipped with that structure. It is easy to see by inspection that Newtonian spacetime comes equipped with all of the basic structures on Galilean spacetime — the temporal metric, spatial metric, and derivative operator. One therefore draws the correct conclusion: Newtonian spacetime has more structure than Galilean spacetime, in the sense that it comes equipped with all of the Galilean spacetime structure and, in addition, some structure that Galilean spacetime lacks. Second, compare Galilean and Minkowski spacetime (Barrett, 2015b). One can show that the Galilean temporal metric is not globally implicitly defined by Minkowski spacetime. And conversely, the Minkowski spacetime metric is not globally implicitly defined by Galilean spacetime. Hence, by Global P6 each comes equipped with a structure that the other lacks, and so they have 'incomparable' amounts of structure. The important thing to note about this example is that it is perfectly clear that Galilean spacetime comes equipped with its temporal metric and that Minkowski spacetime comes equipped with its spacetime metric. One need not rely on implicitly definability to demonstrate this. And indeed, one cannot rely on such reasoning, insofar as Global P4 and Local P4 are false.

Altogether, these considerations point to a middle way. We have seen that in general automorphisms and privileged coordinates do not provide a perfect guide to amounts of structure, and so we should not unreflectively apply criteria like SYM*, SYM*2, and COORD. There are many cases where they can lead one astray; they may even lead one astray in almost all cases (see Question 4). At the same time, we need not entirely avoid reasoning about automorphisms and privileged coordinates in debates about structure. Careful consideration of automorphisms and privileged coordinates can allow one to fruitfully compare amounts of structure. But as the two examples above demonstrate, it is really judicious reasoning about implicit definability that does the conceptual work. Criteria like SYM*, SYM*2, and COORD are best thought of as helpful heuristics that allow one to perform such reasoning. We are not the first to suggest a middle way on these issues. Indeed, after presenting similar concerns about structural comparisons, North (2021, p. 51) remarks that "[n]one of this means these comparisons [of structure] are without value." What it does mean, however, is that these comparisons must be made with great care.

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