SIMPLE ALGEBRAS OF TYPE (1, 1) ARE ASSOCIATIVE

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1. Introduction. In the classification of almost alternative algebras relative quasiequivalence by Albert two new classes of algebras of type (γ, δ) were introduced, namely those of type (1, 1) and (−1, 0) (**1**, equations (34), (35), and Theorem 6). Since rings of type (1, 1) and (−1, 0) are anti-isomorphic it suffices to consider those of type (1, 1). They may be defined as rings satisfying

(1) $$B(x, y, z) = (x, y, z) - (x, z, y) = 0,$$

and

(2) $$A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0,$$

for all elements x, y, and z of the ring, where the associator (a, b, c) is given by $(a, b, c) = (ab)c - a(bc)$.

Actually the identity

(2′) $$(x, x, x) = 0,$$

together with (1) implies (2) whenever the characteristic of the ring is different from 2. This may readily be verified by linearizing (2′). Consequently we may use (1) and (2′) as the defining relations for a ring of type (1, 1).

Additional results on rings of type (1, 1) were obtained by Kokoris (**3**; **4**) and the author (**2**). The main result of the present paper, which is stated in the title, draws upon these results. Let R be a ring of type (1, 1), u any element of R of the form $u = (x, y, x)$, and C the right ideal of R generated by u. Then $uC = 0 = Cu$ (Theorem 2). This turns out to be the key result in the structure theory for it assures the existence of an abundance of right ideals even under the assumption of simplicity (Theorems 6 and 8). In contrast to this, if R is also not associative then it has no proper left ideals (Theorem 4). Every minimal right ideal A of R has the property $A^2 = 0$ (Theorem 5). With the additional hypothesis of chain conditions on right ideals no maximal right ideal of R can be nil and the union of the minimal right ideals of R is contained in the intersection of the maximal right ideals (Theorem 8). By assuming either that R has an idempotent or that R is a finite dimensional algebra one can utilize the information about the right ideals of R in order to reach a contradiction. In fact even primitive rings and hence semi-simple rings of type (1, 1) turn out to be associative [Theorem 11 and its Corollaries]. The characteristic of R is assumed to be different from 2 and 3, and in §4 different from 5 as well.

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We also consider the more general question of rings of type (γ, δ). When $\gamma \neq 1, -1$ it turns out that a simple ring that is not associative, has no proper left or right ideals, and therefore the techniques developed for rings of type $(1, 1)$ are not applicable.

2. Identities.

Fundamental to all our results on rings of type $(1, 1)$ is Theorem 2, already mentioned in the Introduction, which permits the construction of right ideals. This result must be obtained through complicated computation. It is a more sophisticated version of the identity $(x, y, x)^2 = 0$, which constituted the main result of **(2)**. We shall save considerable time and effort by recalling the following identities that hold for all elements w, x, y, z of a ring of type $(1, 1)$. Except for (10), which is a specialization of (1) and (2), these identities may easily be located in **(2)**. The commutator (x, y) is defined by $(x, y) = xy - yx$.

(3) $$(x, (x, y, z)) = 0,$$

(4) $$C(w, x, y, z) = (w, (x, y, z)) + (x, (w, y, z)) = 0,$$

(5) $$D(x, y, z) = (x, yz) - y(x, z) - (x, y)z + (x, y, z) = 0,$$

(6) $$F(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z = 0,$$

(7) $$H(w, x, y, z) = (w, (w, x, y)z) - (w, (w, x, z)y) = 0,$$

(8) $$((x, y, z), x, x) = 0,$$

(9) $$((x, y, x), y, x) = ((x, y, x), y) = ((x, y, x), yx) = 0,$$

(10) $$-(y, x, x) = 2(x, y, x) = 2(x, x, y).$$

In addition to these identities we shall also make use of a result of Kokoris **(4)**, that every subring of a ring of type $(1, 1)$ that is generated by a single element is associative.

Our first objective is to establish the following generalization of (3).

THEOREM 1. *Let v be any element of the form $v = (w, x, y)$, where w, x, y are elements of a ring R of type* $(1, 1)$. *Then the right ideal D generated by v has the property that* $(w, D) = 0$.

Proof. Let $s = (y, x, x)$ and $u = (x, y, x)$. Then $-s = 2u$, as a result of (10). Since $(u, yx) = 0$ is implied by (9), we know that $(s, yx) = 0$. Also

$$0 = C(y, yx, x, x) = (y, (yx, x, x)) + (yx, s) = (y, (yx, x, x)).$$

But

$$\begin{aligned} 0 &= F(y, x, x, x) \\ &= (yx, x, x) - (y, x^2, x) + (y, x, x^2) - y(x, x, x) - (y, x, x)x \\ &= (yx, x, x) - (y, x, x)x, \end{aligned}$$

as a result of (1). Since $(y, (yx, x, x)) = 0$, we must also have $(y, (y, x, x)x) = 0$. Replacing x by $x + z$ in this last identity and utilizing (1) and (7), it must follow that $3(y, (y, x, x)z) + 3(y, (y, z, z)x) = 0$. Now replacing x by $-x$ in this last identity and adding one obtains $6(y, (y, x, x)z) = 0$. Because of our assumption on the characteristic of R we may divide by 6, so that $(y, (y, x, x)z) = 0$. At this point replace x by $x + w$. Then utilization of (1) results in $(y, (y, w, x)z) = 0$. In summary, we have shown that

$$(w, (w, x, y)p) = 0. \tag{11}$$

From here on we consider elements of the form

$$(w, x, y)R_p R_q R_{z_3} \ldots R_{z_n}$$

where R_k is the mapping $a \to ak$. Our inductive assumption will be that w commutes with all such elements for a given n and we shall attempt to prove this for $n + 1$. Incidentally (11) suffices to start off the induction. In case $n = 2$ we merely leave off

$$T = R_{z_3} R_{z_4} \ldots R_{z_{n+1}}.$$

Consider therefore $t = (w, [(w, x, y)p \cdot q]T)$. In attempting to show that $t = 0$, the first step consists of establishing that the value of t is unchanged under all permutations on x, y, p, q. That the interchange of x and y does not alter the value of t is a consequence of (1). Starting with

$$0 = F(w, x, y, p) = (wx, y, p) - (w, xy, p) + (w, x, yp) - w(x, y, p) - (w, x, y)p,$$

we apply the mapping $R_q T$ to this equation and commute the result with w. Because of the induction hypothesis we have

$$(w, (w, xy, p)R_q T) = 0 = (w, (w, x, yp)R_q T).$$

Therefore

$$(w, (wx, y, p)R_q T) - (w, [w(x, y, p)]R_q T) - (w, [(w, x, y)p \cdot q]T) = 0.$$

In the first two terms of this last identity one may interchange y and p without changing their values, hence this must be true of the third term. But that term is $-t$. Finally the induction hypothesis implies $(w, [(w, x, y) \cdot pq]T) = 0$, so that

$$t = (w, [(w, x, y)p \cdot q]T) = (w, ((w, x, y), p, q)T).$$

But in the last term p and q may be permuted without change in value, so that this must also be true of t. This suffices to demonstrate that every permutation of x, y, p, q leaves t unchanged. Suppose now that

$$t' = (w, [(w, x, x)x \cdot x]T).$$

Because of (8) and (10), $t' = (w, [(w, x, x) \cdot x^2]T)$, and the latter is zero as a result of the induction hypothesis. Therefore $t' = 0$. In t' replace x by $x + p$

and also by $x - p$ and add the two expressions. Now, utilizing the fact that every permutation of x, y, p, q in t does not alter the value of t, we see that $12(w, [(w, x, x)p \cdot p]T) = 0$, so that $(w, [(w, x, x)p \cdot p]T) = 0$. By replacing x by $x + y$ in the last identity and then replacing p by $p + q$ it becomes clear that $(w, [(w, x, y)p \cdot q]T) = 0$. This of course completes the induction. However, (w, D) consists of sums of elements that are of the same type as t, but where n is arbitrary. Consequently $(w, D) = 0$. This completes the proof of the theorem.

At this point we are ready to prove the basic

THEOREM 2. *Let u be any element of the form $u = (x, y, x)$, where x and y are elements of a ring R of type* $(1, 1)$. *Then the right ideal C generated by u has the property that $uC = 0 = Cu$.*

Proof. Let c be an arbitrary element of C. Then Theorem 1 implies that $(c, x) = 0$. Because of (10), $(y, x, x) = -2(x, y, x) = -2u$, so that the right ideal generated by (y, x, x) must also be C. A second application of Theorem 1 yields that $(c, y) = 0$. Suppose that

$$T = R_{z_1} R_{z_2} \ldots R_{z_n}.$$

Then as a consequence of Theorem 1 it follows that $((r, a, b)T, r) = 0$. If we replace r by $r + s$ in this last identity it becomes clear that

$$((s, a, b)T, r) = -((r, a, b)T, s). \tag{12}$$

Suppose then that we set $r = yx$, $s = y$, and $a = b = x$ in (12). Then we obtain $((y, x, x)T, yx) = -((yx, x, x)T, y)$. However, it follows from

$$\begin{aligned} 0 &= F(y, x, x, x) \\ &= (yx, x, x) - (y, x^2, x) + (y, x, x^2) - y(x, x, x) - (y, x, x)x \\ &= (yx, x, x) - (y, x, x)x, \end{aligned}$$

that $[(y, x, x)x]T = (yx, x, x)T$. Since $[(y, x, x)x]T$ is an element of C and $(C, y) = 0$, it must be that $((xy, x, x)T, y) = 0$. But then it follows from above that $((y, x, x)T, yx) = 0$. In other words $(c, yx) = 0$, because every element of C may be written as a sum of elements of the form $(y, x, x)T$. We have already established that $(c, y) = (c, x) = 0$, so that

$$0 = D(c, y, x) = (c, yx) - y(c, x) - (c, y)x + (c, y, x) = (c, y, x).$$

But then $(c, x, y) = 0$, as a result of (1). Expanding

$$0 = D(c, x, y) = (c, xy) - x(c, y) - (c, x)y + (c, x, y) = (c, xy),$$

it also follows that $(c, xy) = 0$. In summary, we have shown that

$$(c, xy) = (c, yx) = (c, x) = (c, y) = (c, x, y) = (c, y, x) = 0. \tag{13}$$

Also setting $r = u$, $s = y$, and $a = b = x$ in (12) we find that $((y, x, x)T, u) =$

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$-((u, x, x)T, y)$. But $(u, x, x) = 0$, as a result of (8), so that $((y, x, x)T, u) = 0$. From this one may conclude as before that

$$(c, u) = 0. \tag{14}$$

Since $u = (xy)x - x(yx)$ it follows from (14) that $-(c, xy\cdot x) + (c, x\cdot yx) = 0$. But then

$$\begin{aligned} 0 &= -D(c, xy, x) + D(c, x, yx) \\ &= -(c, xy\cdot x) + xy\cdot(c, x) + (c, xy)x - (c, xy, x) \\ &\qquad + (c, x\cdot yx) - x(c, yx) - (c, x)\cdot yx + (c, x, yx) \\ &= (c, x, yx) - (c, xy, x), \end{aligned}$$

as a result of (13). But then the last identity may be used in

$$0 = F(c, x, y, x) = (cx, y, x) - (c, xy, x) + (c, x, yx) - c(x, y, x) - (c, x, y)x,$$

together with (13) and the observation that cx is an element of C, to establish $-c(x, y, x) = 0$. This implies that $cu = 0$. But then as a result of (14) we also must have $uc = 0$. This argument holds for every element c of C, so that $uC = 0 = Cu$. This completes the proof of the theorem.

3. The structure of left and right ideals. In this section R will be assumed to be a simple ring of type (1, 1), of characteristic different from 2 and 3, that is, not associative. In this connection simple means that the only two-sided ideals of R are either R or 0. This hypothesis on R may of course lead to a contradiction, in which case we would be justified in concluding that simple rings of type (1, 1) are associative. Indeed we obtain this result in §4 with the added assumption that R is a finite dimensional algebra. In the present section we shall adhere to the more general situation.

THEOREM 3. *Rings of type* (1, 1) *that have no proper right ideals are associative.*

Proof. Form $u = (x, y, x)$, for arbitrary elements x and y of R. Let C be the right ideal generated by u. Then either $C = 0$, in which case $u = 0$, or $C = R$. In the latter case we may make use of Theorem 2 in order to obtain that $uR = 0 = Ru$. The set of all elements q of R with the property that $qR = 0 = Rq$ may be verified to form a two-sided ideal of R. Since R is simple, either all such q are zero, or $R^2 = 0$. In the latter instance R would be associative, contrary to assumption. Therefore in all cases $u = (x, y, x) = 0$. But then (1) and (2) may be employed to establish that $(y, x, x) = 0$. Replacing x by $x + z$ in this last identity we are forced to concludeth at R is associative, a contradiction. The contradiction is the result of the assumption that R was not associative. This concludes the proof of the theorem.

The situation on left ideals is just the reverse.

THEOREM 4. *Simple rings of type* (1, 1) *that are not associative have no proper left ideals.*

Proof. Let B be a proper left ideal of R. An element s of B will be defined to be special (relative to B) in case sR is always contained in B. It is easy to verify that the set S of special elements is closed under subtraction. It turns out we can even show that the special elements form a two-sided ideal of R. Select arbitrary elements x, y, z in R, a, b in B, and s in S. Then $(x, y, b) = (xy)b - x(yb)$ is an element of the left ideal B. Then because of (1), (x, b, y) must also be in B. But then (b, x, y) is also in B, as a result of (2). Since s is in B and B is a left ideal of R it follows that xs is in B. On the other hand $(xs)y = (x, s, y) + x(sy)$. But it follows from the definition of S that sy must be in B, so that $x(sy)$ is also in B. Since s is in B it follows from previous discussion that (x, s, y) is in B. As a result both xs and $(xs)y$ are in B, hence xs is special. Then we know that S is a left ideal of R. In a similar manner sx is in B because of the definition of S, while $(sx)y = (s, x, y) + s(xy)$ is also in B. Then S is a two-sided ideal of R. Since S is contained in B and B is a proper left ideal of R, we must have $S \neq R$. But R is simple, so that $S = 0$. This is very useful information since we can show that various elements of R must be special. Finally we shall be able to deduce that $B = 0$, which is the desired contradiction. To begin with (a, b, x) is clearly in B. Also

$$0 = F(a, b, x, y) = (ab, x, y) - (a, bx, y) + (a, b, xy) - a(b, x, x) - (a, b, x)y,$$

so that $-(a, b, x)y$ is also in B. Hence (a, b, x) is special and consequently $(a, b, x) = 0$. In other words $(B, B, R) = 0$. Then from (1) and (2) it follows that also $(B, R, B) = 0$ and $(R, B, B) = 0$. For this reason

$$\begin{aligned} 0 &= F(x, y, b, b) \\ &= (xy, b, b) - (x, yb, b) + (x, y, b^2) - x(y, b, b) - (x, y, b)b \\ &= (x, y, b^2) - (x, y, b)b, \end{aligned}$$

so that $(x, y, b^2) = (x, y, b)b$. On the other hand

$$\begin{aligned} 0 &= F(x, b, b, y) + F(x, b, y, b) \\ &= (xb, b, y) - (x, b^2, y) + (x, b, by) - x(b, b, y) - (x, b, b)y \\ &\qquad + (xb, y, b) - (x, by, b) + (x, b, yb) - x(b, y, b) - (x, b, y)b \\ &= -(x, b^2, y) + (x, b, by) - (x, by, b) - (x, b, y)b. \end{aligned}$$

But $(x, b, by) - (x, by, b) = 0$ because of (1), while $(x, b^2, y) = (x, y, b^2)$, and $(x, b, y)b = (x, y, b)b$, for the same reason. But then $(x, y, b^2) = -(x, y, b)b$, whereas we have already noted that $(x, y, b^2) = (x, y, b)b$. Therefore $(x, y, b^2) = (x, y, b)b = 0$. The nucleus N of R is defined as the set of all elements n in R such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. Because of (1) and (2), b^2 must be in N. Let n be an arbitrary element of N. Then

$$0 = C(n, x, y, z) = (n, (x, y, z)) + (x, (n, y, z)) = (n, (x, y, z)),$$

so that $(n, (R, R, R)) = 0$. Similarly (12) implies that $(n, (R, R, R)R) = 0$. However, the set of finite sums of elements that are of the form (R, R, R)

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and of the form $(R, R, R)R$ form a two-sided ideal in an arbitrary ring R. If that ring is also simple and not associative then of course it follows that this ideal must be the whole ring. The conclusion we can draw in the present situation is that $(n, R) = 0$, so that $(b^2, R) = 0$. Since b^2 is an element of B and now also $b^2x = xb^2$ is an element of B we conclude that b^2 must be special, hence $b^2 = 0$. If we replace b by $a + b$ then also $ab + ba = 0$. Since (x, y, b) is an element of B it must then follow that $(x, y, b)a = -a(x, y, b)$. Also

$$\begin{aligned} 0 &= C(a, x, y, b) \\ &= (a, (x, y, b)) + (x, (a, y, b)) = (a, (x, y, b)) \\ &= a(x, y, b) - (x, y, b)a. \end{aligned}$$

But then $a(x, y, b) = (x, y, b)a = 0$. This may be used in the expansion of

$$0 = F(x, y, b, a) = (xy, b, a) - (x, yb, a) + (x, y, ba) - x(y, b, a) - (x, y, b)a,$$

to show that $(x, y, ba) = 0$. As before this implies ba is in the nucleus, and hence commutes with every element of R. Consequently one can show that ba is special and so $ba = 0$. In other words $B^2 = 0$. Form $I = B + BR$. Then $(bx)y = (b, x, y) + b(xy)$. We have already noted that (b, x, y) is an element of B, and hence of I. Then I must be a right ideal of R. Similarly $y(bx) = -(y, b, x) + (yb)x$, where $-(y, b, x)$ is an element of B and $(yb)x$ is an element of BR. Thus $y(bx)$ is an element of I. This suffices to show that I is a two-sided ideal of R. If $I = 0$, then $B = 0$, contrary to assumption. If on the other hand $I = R$, then $BI = B(B + BR) = 0$, so that $BR = 0$. But then $I = B$, and so $R = B$, contrary to assumption. In either case we have reached a contradiction. Consequently there can exist no proper left ideal B in R. This completes the proof of the theorem.

If Theorem 4 were true for right ideals also, then of course this would prove simple rings of type (1, 1) to be associative, which is the strongest possible result one could hope to get. Arguments of the type used in the proof of Theorem 4 seem inadequate for this purpose. With some effort one can obtain a construction for a right ideal of R, properly contained within any non-zero right ideal A of R. Except when A is minimal, this construction does not seem to be especially enlightening, and since we can get some information on minimal right ideals more directly, we shall not go into this construction.

THEOREM 5. *If R is a simple ring of type* (1, 1) *that is not associative, and if A is a minimal right ideal of R, then $A^2 = 0$.*

Proof. Let $t = (a, x, x)$, where a is an arbitrary element of A and x an arbitrary element of R. Let C denote the right ideal generated by t. Since t is an element of A, C must also be contained in A. From the minimality of A as a right ideal it follows that either $C = 0$, or that $C = A$. If $C = 0$ then $t = 0$. On the other hand Theorem 2 implies that $Ct = 0 = tC$, so that if

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$C = A$, then $At = 0 = tA$. In either case we may conclude that $At = 0 = tA$. Replacing x by $x + y$ in this last identity and using (1) we conclude that $A(a, x, y) = 0 = (a, x, y)A$, where y is an arbitrary element of R. In other words $A(A, R, R) = 0 = (A, R, R)A$. Let P be the set of all elements p in A with the property that $Ap = A(pR) = 0$. P is obviously closed under subtraction. With the last identity we shall be able to show that P is in fact a right ideal of R. Select arbitrary elements p in P, a, b, d in A, and x, y, z in R. Since p is in A, so is px. From the definition of P it follows that $a(px) = 0$. Also $a(px \cdot y) = a(p, x, y) + a(p \cdot xy) = a(p, x, y)$. Since $0 = A(A, R, R)$, we have $a(p, x, y) = 0$, and thereby $a(px \cdot y) = 0$. Consequently P is a right ideal of R and contained in A. Let us assume that $A^2 \neq 0$. Then $P \neq A$. From the minimality of A as a right ideal it follows that $P = 0$. Our next objective will be to show that (A, A, R) is contained in P. We have seen previously that (A, A, R) is contained in A and also that $A(A, R, R) = 0$. But then

$$\begin{aligned} 0 &= aF(b, d, x, y) \\ &= a(bd, x, y) - a(b, dx, y) + a(b, d, xy) - a[b(d, x, y)] - a[(b, d, x)y] \\ &= -a[(b, d, x)y], \end{aligned}$$

and so (A, A, R) is contained in P. But then $(A, A, R) = 0$. As a consequence of the last identity

$$0 = F(a, b, x, y) = (ab, x, y) - (a, bx, y) + (a, b, xy) - a(b, x, y) - (a, b, x)y = (ab, x, y).$$

In other words $(A^2, R, R) = 0$. As in the proof of Theorem 4 we can now use $0 = C(ab, x, y, z)$ to show that $(A^2, (R, R, R)) = 0$, and (12) to show that $(A^2, (R, R, R)R) = 0$. Since the two-sided ideal generated by all associators must be all of R, we have demonstrated that $(A^2, R) = 0$. But then $x(ab) = (ab)x = (a, b, x) + a(bx) = a(bx)$, proving that A^2 is a two-sided ideal of R. Since we have assumed that $A^2 \neq 0$, it must be the case that $A^2 = R$. But A^2 is contained in A, so that $A = R$. This contradicts the assumption that A is a minimal right ideal of R. Consequently $A^2 = 0$. This completes the proof of the theorem.

The next result plays a very important part in the Main Theorem.

THEOREM 6. *Let R be a simple ring of type* (1, 1), *with chain conditions on right ideals, that is not associative. Then the number of maximal right ideals and the number of minimal right ideals are both greater than one.*

Proof. The existence of at least one maximal right ideal and of at least one minimal right ideal are insured by the chain conditions and Theorem 3. Suppose that R has only one maximal right ideal. Call it B. Consider an arbitrary element u of the form $u = (y, x, x)$, and let C be the right ideal generated by u. Then, because of Theorem 2, $uC = 0 = Cu$. If u is not an

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element of B, then C is not contained in B. But B is the unique maximal right ideal of R and there is only one right ideal not contained in it, namely R. Thence $uR = 0 = Ru$. But the absolute divisors of zero of R form a two-sided ideal of R, which cannot be all of R. Consequently $u = 0$, contrary to assumption. Thus u is an element of B for all x and y in R. Replace x by $x + z$ in u.

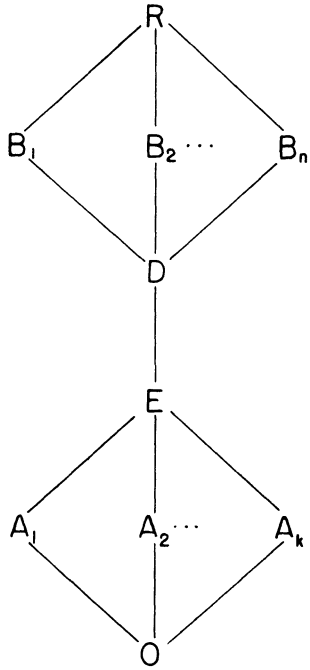

FIG. 1

Then as a result of (1) we note that all associators of R must be contained in B and thereby also all right multiples of associators. But then $B = R$, a contradiction since B was assumed to be a maximal right ideal. Because of this contradiction B cannot be the unique maximal right ideal and consequently R must have at least two maximal right ideals. This completes the first half of the theorem. Suppose now that A is the only minimal right ideal of R. Define u as before, as well as C, so that $uC = 0 = Cu$. We see at once that either C contains A or C must be zero. In the latter case $u = 0$. In the former $uA = 0 = Au$. But then we may conclude that $uA = 0 = Au$ in either case. Replacing x by $x + z$ in the last identity and using (1) we obtain $(R, R, R)A = 0 = A(R, R, R)$. Let w, x, y, z be arbitrary elements of R and a an arbitrary element of A. Then

$$\begin{aligned} 0 &= F(w, x, y, z)a \\ &= (wx, y, z)a - (w, xy, z)a + (w, x, yz)a - [w(x, y, z)]a - [(w, x, y)z]a \\ &= -[w(x, y, z)]a - [(w, x, y)z]a. \end{aligned}$$

Therefore in $q = [w(x, y, z)]a$ all permutations of x, y, and z do not alter the value of q. But $0 = (x, y, z) + (y, z, x) + (z, x, y)$, because of (2). Therefore $3q = 0$, so that $q = 0$. But then $[(w, x, y)z]a = 0$. In summary, we have shown that $(R, R, R)A = 0$, and that $[(R, R, R)R]A = 0$. As before we can deduce from this $RA = 0$, so that $A = 0$. However, A was chosen to be a minimal right ideal and this is clearly a contradiction. Hence R has at least two minimal right ideals. This concludes the proof of the theorem.

The last result seems to leave open the possibility that a maximal right ideal might be a minimal right ideal. However we shall see later that this cannot happen. In fact every minimal right ideal will be seen to be contained in every maximal right ideal, and any such pair are always separated by at least one intermediate right ideal (Theorem 8).

THEOREM 7. *Let R be a simple ring of type* (1, 1). *Suppose A and B are right ideals of R such that $A^2 = 0$, $A + B = R$, and $B \neq R$. Then R is associative.*

Proof. Since $A^2 = 0$, we see that $(A, A, R) = 0$. But then $(A, R, A) = 0$ and $(R, A, A) = 0$ as a result of (1) and (2). Also (B, R, R) is contained in B and therefore so is (R, B, B), as a result of (2). Expanding

$$\begin{aligned}(R, R, R) &= (A + B, A + B, A + B)\\ &= (A, A, A) + (B, B, B) + (A, B, B) + (B, A, B) + (B, B, A)\\ &\qquad + (B, A, A) + (A, B, A) + (A, A, B),\end{aligned}$$

it becomes evident that (R, R, R) is contained in B. But then also $(R, R, R)R$ is contained in B. Since $B \neq R$, the only two-sided ideal of R that is contained in B is zero. But we have just seen that the ideal generated by all associators is contained in B. Therefore R must be associative. This completes the proof of the theorem.

COROLLARY. *Let R be a simple ring of type* (1, 1) *that is not associative. If A is a minimal right ideal of R and B a maximal right ideal of R then A is contained in B.*

Proof. Suppose that A is not contained in B. Since B is a maximal right ideal of R, $A + B = R$. Since A is a minimal right ideal of R, we may use Theorem 5 to obtain $A^2 = 0$. But then the hypothesis of Theorem 7 is satisfied, so that R must be associative. From this contradiction one deduces that A is contained in B. This completes the proof of the corollary.

THEOREM 8. *Let R be a simple ring of type* (1, 1) *with unit element and chain conditions on right ideals that is not associative. Let B be any maximal right ideal of R, A any minimal right ideal of R, D the intersection of all the maximal right ideals of R, and E the union of all the minimal right ideals of R. Then B is is not nil and*

$$0 \subset A \subset E \subseteqq D \subset B \subset R.$$

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Proof. Suppose that B is nil (that means every element of B is nilpotent). Theorem 6 implies the existence of another maximal right ideal $B' \neq B$. Therefore $B + B' = R$. Since the unit element 1 is in R, there must exist elements x in B and y in B', such that $1 = x + y$. Then $1 - x = y$. Suppose that $x^n = 0$. Let $s = 1 + x + \ldots + x^{n-1}$. Then $(1 - x)s = 1 = ys$. But this implies that 1 is in B', so that $B' = R$, contrary to assumption. Thus B cannot be nil. On the other hand Theorem 5 tells us that $A^2 = 0$, so that A is certainly nil. Therefore $B \neq A$. Because of Theorem 6, $E \neq A$ and $D \neq B$. Clearly A is contained in E, and D is contained in B. From the Corollary to Theorem 7 it follows that A is contained in B and hence in D. But then E must be contained in B. So far all the inclusions have been proper. However the best we can say about E and D is that E is contained in D, but in this case we are unable to eliminate the possibility that $E = D$. This completes the proof of the theorem.

Fig. 1 indicates the simplest possible structure of any ring R satisfying the hypothesis of Theorem 8, if indeed such a ring exists. The B_i denote maximal right ideals and the A_j minimal right ideals. D and E are defined in the statement of Theorem 8.

4. Main section. We shall make use of the following theorem, whose proof appears in **(4)**.

THEOREM 9 (Kokoris). *Let R be a simple ring of type* (1, 1) *that is not associative, and e any idempotent of R. Then e must be the unit element* 1 *of R.*

There appears to be a minor gap in Kokoris' proof, but fortunately a simple permutation of the facts already proved in **(4)** can be used to prove Theorem 9. We proceed with the details. In the proof of his Lemma 3, the element $a = xy$, where x is in R_{10} and y is in R_{11} is not the most general element of G_0. Rather G_0 consists of sums of such elements and hence one can only say that G_0 is the sum of nilpotent elements rather than that G_0 is nil. Let us consider the case when R is simple and $H = R$. Then $R_{11} = R_{01}R_{00}$. Moreover, it is proved that R_{11} commutes with $R_{01}R_{00}$, so that R_{11} is commutative. Now the fact that R_{11} is the sum of nilpotent elements suffices to establish that R_{11} is nil, and this of course contradicts the fact that e is in R_{11}.

We shall also make use of the following result about algebras of type (1, 1) (understood to be finite dimensional), whose proof may be found in **(1)**.

THEOREM 10 (Albert). *A nil algebra of type* (1, 1) *is nilpotent. With this background we are ready to prove the result stated in the title of the present paper.*

MAIN THEOREM. *Simple algebras of type* (1, 1) *are associative.*

Proof. Let R be a simple algebra of type (1, 1) that is not associative. We shall attempt to show that R satisfies the hypothesis of Theorem 8, but not one of the conclusions, thus obtaining the necessary contradiction. If R were nil then it would be nilpotent. Since R^2 is an ideal of R, either $R^2 = R$ or $R^2 =$

0. If $R^2 = 0$, then R would be associative, contrary to assumption. On the other hand $R^2 = R$ would contradict the fact that R is nilpotent. Therefore R is not nil. Suppose x is some element of R that is not nilpotent. The subalgebra S of R that is generated by x therefore cannot be nil. Since S is a finite dimensional, associative algebra it must contain an idempotent e. But then Theorem 9 implies that $e = 1$. Thus R contains a unit element. Since R is a finite dimensional algebra with unit element it clearly has ascending and descending chain conditions on right ideals. Thus R satisfies the hypothesis of Theorem 8. Let B be any maximal right ideal of R. If 1 were an element in B then we would have $B = R$, a contradiction. Hence 1 is not an element of B. Let y be an arbitrary element of B and T the subalgebra generated by y. If T were not nil then it would have to contain an idempotent. However, that is impossible since Theorem 9 limits any idempotent in B to be 1, and we have already seen that 1 is not in B. Thus T is nilpotent, which implies that B is nil. We have reached a contradiction since one of the conclusions of Theorem 8 states that B cannot be nil. The contradiction arose from the assumption that R was not associative. Therefore R is associative. This concludes the proof of the theorem.

Once it is known that simple algebras of type (1, 1) are associative, it is easy to extend this result to semi-simple algebras. The radical may be defined as the maximal nil ideal. One such argument follows closely the one given in **(3)** for algebras of type (γ, δ), where $\gamma \neq 1, -1$, and need not be repeated here.

At this point we shall demonstrate how the main theorem carries over to a large extent to rings without finiteness assumptions. This also results in a second and somewhat more direct proof of the main theorem (Corollary 3 of Theorem 11). As usual a ring R is defined to be primitive in case it has a maximal right ideal A, which contains no two-sided ideal of R other than the zero ideal and in case there exists an element e in R such that $ex - x$ is always in A for all x in R.

THEOREM 11. *If R is a primitive ring of type* (1, 1) *then R is associative.*

COROLLARY 1. *If R is a semi-simple ring of type* (1, 1) *then R is associative.*

COROLLARY 2. *If R is a simple ring of type* (1, 1) *and contains an idempotent then R is associative.*

COROLLARY 3. *If R is a simple, finite dimensional algebra of type* (1, 1) *then R is associative.*

Proof. Let A be a regular maximal right ideal of R which contains no two-sided ideal of R other than the zero ideal and assume that R is not associative. We assert that there exists at least one element u of the form $u = (x, y, x)$ which is not contained in A. For assume otherwise. Then (y, x, x) must also be in A. Replacing x by $x + z$ it then follows that $2(y, z, x)$ is in A, for all elements x, y and z in R. Now it is well known, and can easily be verified directly, that in an arbitrary ring all finite sums of elements of the form (R, R, R) and $(R, R, R)R$ form a two-sided ideal I of R. In this instance I would be contained in A. But then by assumption we would have $I = 0$, and R would be associative. This is clearly a contradiction. Hence there must exist an element $u = (x, y, x)$ which is not in A. Let C be the right ideal generated by u. Since A is a maximal right ideal it follows that $A + C = R$. Then we can find an element a in A and an element c in C such that $e = a + c$. Forming $eu - u = au + cu - u$, we note that $cu = 0$ as a result of Theorem 2, while $eu - u$ is in A. Therefore $au - u$ must be an element of A. Since A is a right ideal au belongs to A, hence u must also be in A. But this is clearly a contradiction, since we deliberately chose u not in A. Hence R must have been associative to begin with. This completes the proof of the theorem.

Making use of the Jacobson-Brown radical of a ring it is clear that a semi-simple ring is a subdirect sum of primitive rings, so that Corollary 1 follows at once from the theorem.

If R is a simple, non-associative ring of type (1, 1) and contains an idempotent, then as a result of Theorem 9 R must contain a unit element 1. But then form a maximal right ideal not containing 1. This must indeed be a regular, maximal right ideal of R. It contains no ideal of R other than zero since R is simple. Therefore R is primitive, and hence associative as a result of the theorem. This is a contradiction. Hence R must have been associative to begin with. This establishes Corollary 2.

If R is a simple, finite dimensional algebra of type (1, 1) then, as in the early part of the proof of the main theorem, R is either associative or contains an idempotent. Then one may use Corollary 2 in order to establish Corollary 3.

Rings of type (1, 1) with radical need not be associative, of course. In fact it is not difficult to construct finite dimensional algebras of type (1, 1) which are not associative. It is worth noting that there exists a division ring of characteristic 2 which satisfies both (1) and (2′), yet is not associative **(5)**.

5. Rings of type (γ, δ), $\gamma \neq 1, -1$**.** A ring is said to be of type (γ, δ) in case identities (2), and (15), which follows, hold. Identity (15) is given by

$$J(x, y, z) = \gamma(x, z, y) + \delta(y, z, x) + (z, x, y) = 0, \tag{15}$$

where x, y, z are arbitrary elements of the ring and γ, δ are constant scalar elements. One may also assume that $\gamma^2 = \delta^2 - \delta + 1$, for otherwise one can readily verify the ring to be associative. Therefore the condition that $\gamma \neq 1$, -1 is equivalent to the condition that $\delta \neq 0.1$. In the remainder of this section we shall consider rings R of type (γ, δ), $\gamma \neq 1, -1$ whose characteristic is different from 2 and 3. We shall first develop some essential identities. As was shown in **(2)**,

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$$(16)\quad G(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)) = 0,$$

and since this may be proved in a ring satisfying (2) only, it must be satisfied by all elements of R. From

$$\begin{aligned} 0 &= G(y, x, x, x) + (x, A(x, y, x)) \\ &= (y, (x, x, x)) - (x, (x, x, y)) + (x, (x, y, x)) - (x, (y, x, x)) \\ &\quad + (x, (x, y, x) + (x, (y, x, x)) + (x, (x, x, y)) \\ &= 2(x, y, x)), \end{aligned}$$

it follows that $(x, (x, y, x)) = 0$. But then

$$\begin{aligned} 0 &= (x, J(x, x, y)) \\ &= \gamma(x, (x, y, x)) + \delta(x, (x, y, x)) + (x, (y, x, x)) \\ &= (x, (y, x, x)). \end{aligned}$$

But then

$$\begin{aligned} 0 &= (x, A(x, y, x)) \\ &= (x, (x, y, x)) + (x, (y, x, x)) + (x, (x, x, y)) \\ &= (x, (x, x, y)). \end{aligned}$$

If in the last two identities we replace x by $x + z$ and $x - z$ and add, we obtain

$$(17)\quad K(x, y, z) = (x, (y, x, z)) + (x, (y, z, x)) + (z, (y, x, x)) = 0,$$

and

$$(18)\quad L(x, y, z) = (x, (z, x, y)) + (x, (x, z, y)) + (z, (x, x, y)) = 0.$$

Now let

$$t = (x, (x, y, z)) + (x, (x, z, y)), \text{ and } u = (x, (z, y, x)) + (x, (y, z, x)).$$

Then

$$\begin{aligned} 0 &= G(x, x, y, z) - K(x, z, y) + L(x, y, z) \\ &= (x, (x, y, z)) - (x, (y, z, x)) + (y, (z, x, x)) - (z, (x, x, y)) \\ &\quad - (x, (z, x, y)) - (x, (z, y, x)) - (y, (z, x, x)) + (x, (z, x, y)) \\ &\quad + (x, (x, z, y)) + (z, (x, x, y)) \\ &= (x, (x, y, z)) + (x, (x, z, y)) - (x, (z, y, x)) - (x, (y, z, x)) \\ &= t - u. \end{aligned}$$

Consequently $t = u$. On the other hand

$$\begin{aligned} 0 &= J(y, x, z) + J(z, x, y) \\ &= \gamma(y, z, x) + \delta(x, z, y) + (z, y, x) + \gamma(z, y, x) + \delta(x, y, z) + (y, z, x) \\ &= (\gamma + 1)[(z, y, x) + (y, z, x)] + \delta[(x, y, z) + (x, z, y)]. \end{aligned}$$

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Commuting both sides with x one obtains $(\gamma + \delta + 1)t = 0$, since $t = u$. However

$$\begin{aligned} 0 &= J(x, y, z) + J(x, z, y) - A(x, y, z) - A(x, z, y) \\ &= \gamma(x, z, y) + \delta(y, z, x) + (z, x, y) + \gamma(x, y, z) + \delta(z, y, x) \\ &\qquad + (y, x, z) - (x, y, z) - (y, z, x) - (z, x, y) - (x, z, y) \\ &\qquad - (z, y, x) - (y, x, z) \\ &= (\gamma - 1)[(x, y, z) + (x, z, y)] + (\delta - 1)[(z, y, x) + (y, z, x)]. \end{aligned}$$

Commuting both sides with x one obtains $(\gamma + \delta - 2)t = 0$. Since both $(\gamma + \delta + 1)t = 0$, and $(\gamma + \delta - 2)t = 0$, it must be that $3t = 0$, and so $t = 0$. Since $u = t$, we also have $u = 0$. We have shown that

$$(x, (x, y, z)) + (x, (x, z, y)) = 0 = (x, (z, y, x)) + (x, (y, z, x)). \tag{19}$$

Incidentally up to this point we have made no use of the restriction on γ. However, the next result makes use of this assumption.

THEOREM 12. *Let R be a simple ring of type (γ, δ), $\gamma \neq 1, -1$ that is not associative. Then R has no proper left or right ideals.*

Proof. Let B be any proper right ideal of R. Define S as the set of all elements s of B with the property that Rs is always contained in B. Let x, y, z denote arbitrary elements of R, a, b arbitrary elements of B and s an arbitrary element of S. Since B is a right ideal of R, (b, x, y) must be an element of B. But then

$$\begin{aligned} 0 &= J(b, y, x) - A(b, y, x) \\ &= \gamma(b, x, y) + \delta(y, x, b) + (x, b, y) - (b, y, x) - (y, x, b) - (x, b, y) \\ &= (\delta - 1)(y, x, b) + \gamma(b, x, y) - (b, y, x), \end{aligned}$$

so that $(\delta - 1)(y, x, b)$ is in B. Since $\delta \neq 1$, this implies that (y, x, b) is in B. Expanding $0 = A(b, x, y)$, we note that also (y, b, x) is in B. Clearly S is closed under subtraction. We now show that in fact S is an ideal of B. Since s is in B, sy will be also. Then $z(sy) = -(z, s, y) + (zs)y$. We have already noted that (z, s, y) is in B. Also it follows from the definition of S that zs is in B. Since B is a right ideal of R, $(zs)y$ must be in B. Thereby $z(sy)$ is also in B. But this implies that sy is in S, so that S is a right ideal of R. Again from the definition of S it follows that ys is in B. Then $z(ys) = -(z, y, s) + (zy)s$, and so $z(ys)$ is also in B. But then ys is in S and therefore S is a two-sided ideal of R. However, B is a proper right ideal of R and S is contained in B. Consequently, since R is simple, $S = 0$. Next we proceed to show that a number of elements are zero by virtue of the fact that we can prove they are contained in S. Thus $0 = F(x, y, a, b) = (xy, a, b) - (x, ya, b) + (x, y, ab) - x(y, a, b) - (x, y, a)b$, implies that $-x(y, a, b)$ is contained in B. But then (y, a, b) is contained in S and hence $(R, B, B) = 0$. But then

$$\begin{aligned} 0 &= J(x, a, b) - A(x, a, b) \\ &= \gamma(x, b, a) + \delta(a, b, x) + (b, x, a) - (x, a, b) - (a, b, x) - (b, x, a) \\ &= (\delta - 1)(a, b, x). \end{aligned}$$

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Since $\delta \neq 1$, $(B, B, R) = 0$. At this point

$$0 = A(x, a, b) = (x, a, b) + (a, b, x) + (b, x, a) = (b, x, a),$$

so that $(B, R, B) = 0$. In summary, we have shown that

$$(B, B, R) = (B, R, B) = (R, B, B) = 0. \tag{20}$$

Set $x = b$, $y = z = x$ in (19). Then we obtain $(b, (b, x, x)) + (b, (b, x, x)) = 0$. Hence $(b, (b, x, x)) = 0$. We shall now establish that

$$(b^2, x, x) = b(b, x, x) = (b, x, x)b. \tag{21}$$

So far we have been able to show that the second and third terms of (21) are equal.

$$\begin{aligned} 0 &= F(b, b, x, x) \\ &= (b^2, x, x) - (b, bx, x) + (b, b, x^2) - b(b, x, x) - (b, b, x)x \\ &= (b^2, x, x) - b(b, x, x), \end{aligned}$$

because of (20). Thus the first term of (21) is equal to the second term. This establishes (21). Since

$$\begin{aligned} 0 &= J(x, y, x) \\ &= \gamma(x, x, y) + \delta(y, x, x) + (x, x, y) \\ &= (\gamma + 1)(x, x, y) + \delta(y, x, x) \end{aligned}$$

and $\gamma \neq -1$, we have

$$(x, x, y) = -\left(\frac{\delta}{\gamma + 1}\right)(y, x, x).$$

But then substituting $y = b^2$ we obtain

$$(x, x, b^2) = -\left(\frac{\delta}{\gamma + 1}\right)(b^2, x, x).$$

Similarly, substituting $y = b$,

$$(x, x, b) = -\left(\frac{\delta}{\gamma + 1}\right)(b, x, x)$$

and so

$$(x, x, b)b = -\left(\frac{\delta}{\gamma + 1}\right)(b, x, x)b.$$

But we have already seen in (21) that $(b^2, x, x) = (b, x, x)b$. Then we may conclude that $(x, x, b^2) = (x, x, b)b$. Expanding

$$\begin{aligned} 0 &= F(x, x, b, b) \\ &= (x^2, b, b) - (x, xb, b) + (x, x, b^2) - x(x, b, b) - (x, x, b)b, \end{aligned}$$

we see that $-(x, xb, b) = 0$, as a result of the previous identity and (20). But then

$$\begin{aligned} 0 &= J(b, x, xb) \\ &= \gamma(b, xb, x) + \delta(x, xb, b) + (xb, b, x) \\ &= \gamma(b, xb, x) + (xb, b, x). \end{aligned}$$

However,

$$\begin{aligned} 0 &= F(b, x, b, x) \\ &= (bx, b, x) - (b, xb, x) + (b, x, bx) - b(x, b, x) - (b, x, b)x \\ &= -(b, xb, x) - b(x, b, x). \end{aligned}$$

This implies that $(b, xb, x) = -b(x, b, x)$. But then $-\gamma b(x, b, x) + (xb, b, x) = 0$. From

$$\begin{aligned} 0 &= F(x, b, b, x) \\ &= (xb, b, x) - (x, b^2, x) + (x, b, bx) - x(b, b, x) - (x, b, b)x \\ &= (xb, b, x) - (x, b^2, x) \end{aligned}$$

it follows that $(xb, b, x) = (x, b^2, x)$. Thus $-\gamma b(x, b, x) + (x, b^2, x) = 0$. In

$$\begin{aligned} 0 &= J(x, x, y) \\ &= \gamma(x, y, x) + \delta(x, y, x) + (y, x, x) \\ &= (\gamma + \delta)(x, y, x) + (y, x, x), \end{aligned}$$

substitute $y = b^2$ to obtain $(\gamma + \delta)(x, b^2, x) + (b^2, x, x) = 0$, and also $(\gamma + \delta)b(x, b, x) + b(b, x, x) = 0$. We have already established in (21) that $(b^2, x, x) = b(b, x, x)$, so that $(\gamma + \delta)(x, b^2, x) = (\gamma + \delta)b(x, b, x)$. If $\gamma + \delta = 0$, then substituting in $\gamma^2 = \delta^2 - \delta + 1$ we see that $\delta = 1$, contrary to assumption. Therefore $(x, b^2, x) = b(x, b, x)$. Since $-\gamma b(x, b, x) + (x, b^2, x) = 0$ has already been established, we combine the last two identities and get $(1 - \gamma) b(x, b, x) = 0$. But $\gamma \neq 1$, so that $b(x, b, x) = 0$, and hence all the terms in (21) are zero. Replacing x by $x + y$ in our last identity we see that

$$b(x, b, y) = -b(y, b, x). \tag{22}$$

We showed that $(x, xb, b) = 0$ earlier in the proof. On the other hand

$$\begin{aligned} 0 &= F(x, b, b, x) \\ &= (xb, b, x) - (x, b^2, x) + (x, b, bx) - x(b, b, x) - (x, b, b)x \\ &= (xb, b, x). \end{aligned}$$

But then

$$\begin{aligned} 0 &= J(xb, b, x) - \delta A(b, x, xb) \\ &= \gamma(xb, x, b) + \delta(b, x, xb) + (x, xb, b) - \delta(b, x, xb) - \delta(x, xb, b) \\ &\qquad - \delta(xb, b, x) \\ &= \gamma(xb, x, b). \end{aligned}$$

Since $\gamma \neq 0$, $(xb, x, b) = 0$. Substituting $x + y$ for x in this last identity we get

(23) $$(xb, y, b) = -(yb, x, b).$$

In the second part of (19) replace x by $w + x$, so that

$$(w, (z, y, x)) + (x, (z, y, w)) + (x, (y, z, w)) + (w, (y, z, x)) = 0.$$

Now let $w = z = b$. Then because of (20), $(b, (b, y, x)) + (b, (y, b, x)) = 0$. From (19) and (2) one proves that $(z, (y, z, x)) + (z, (x, z, y)) = 0$. Then if we let $z = b$, $(b, (y, b, x)) + (b, (x, b, y)) = 0$. But then

$$\begin{aligned} 0 &= (b, J(x, y, b)) \\ &= (b, \gamma(x, b, y) + \delta(y, b, x) + (b, x, y)) \\ &= (\gamma - \delta - 1)(b, (x, b, y)). \end{aligned}$$

If $\gamma = \delta + 1$ and $\gamma^2 = \delta^2 - \delta + 1$ then $3\delta = 0$ so that $\delta = 0$, contrary to assumption. Therefore $(b, (x, b, y)) = 0$. From this it follows readily that

(24) $$(b, (b, y, x)) = 0 = (b, (y, b, x)).$$

Then

$$\begin{aligned} 0 &= F(b, y, x, b) \\ &= (by, x, b) - (b, yx, b) + (b, y, xb) - b(y, x, b) - (b, y, x)b \\ &= (b, y, xb) - b(y, x, b) - (b, y, x)b. \end{aligned}$$

Now using (24),

$$b(y, x, b) + (b, y, x)b = b(y, x, b) + b(b, y, x) = bJ(y, x, b) - b(x, b, y).$$

Hence $(b, y, xb) = -b(x, b, y) = b(y, b, x)$, using (22). We have demonstrated that

(25) $$(b, y, xb) = b(y, b, x).$$

Now

$$\begin{aligned} 0 &= F(y, b, x, b) \\ &= (yb, x, b) - (y, bx, b) + (y, b, xb) - y(b, x, b) - (y, b, x)b \\ &= (yb, x, b) + (y, b, xb) - (y, b, x)b. \end{aligned}$$

Therefore $(yb, x, b) + (y, b, xb) = (y, b, x)b = b(y, b, x)$, as a result of (24). But then $(y, b, xb) = -(yb, x, b) + b(y, b, x) = (xb, y, b) - b(x, b, y)$, because of (23) and (22). We have demonstrated that

(26) $$(y, b, xb) = (xb, y, b) - b(x, b, y).$$

Now

$$\begin{aligned} 0 &= F(b, x, b, y) + A(xb, y, b) \\ &= (bx, b, y) - (b, xb, y) + (b, x, by) - b(x, b, y) \\ &\qquad - (b, x, b)y + (xb, y, b) + (y, b, xb) + (b, xb, y) \\ &= -b(x, b, y) + (xb, y, b) + (y, b, xb). \end{aligned}$$

But then comparison of the last identity with (26) shows that

(27) $$(xb, y, b) = b(x, b, y),$$

and

(28) $$(y, b, xb) = 0.$$

From $0 = J(b, xb, y) = \gamma(b, y, xb) + \delta(xb, y, b) + (y, b, xb)$, we get $\gamma(b, y, xb) + \delta(xb, y, b) = 0$, using (28). But then as a result of (22), (25), and (27) $(\gamma - \delta)b(y, b, x) = 0$. Since $\gamma \neq \delta$, then $b(y, b, x) = 0$. As before one may also deduce $b(b, x, y) = 0$, by use of (2) and (15). But then

$$\begin{aligned} 0 &= F(b, b, x, y) \\ &= (b^2, x, y) - (b, bx, y) + (b, b, xy) - b(b, x, y) - (b, b, x)y \\ &= (b^2, x, y). \end{aligned}$$

Again using (2) and (15) one may deduce that $(x, y, b^2) = (y, b^2, x) = 0$. In other words b^2 must lie in the nucleus N of R. Furthermore it follows from an argument presented in the Appendix of **(4)** that therefore $(b^2, R) = 0$. But then we have b^2 in B and $xb^2 = b^2x$ is also in B, so that b^2 is in S. Therefore $b^2 = 0$. Replacing b by $a + b$ we see that

(29) $$ab + ba = 0.$$

Now

$$\begin{aligned} 0 &= K(x, a, b) \\ &= (x, (a, x, b)) + (x, (a, b, x)) + (b, (a, x, x)) \\ &= (b, (a, x, x)). \end{aligned}$$

But then $(b, (x, a, x)) = 0$, as a result of (2) and (15). On the other hand (x, a, x) is in B, so that $b(x, a, x) = -(x, a, x)b$, using (29). Consequently

(30) $$b(x, a, x) = 0 = (x, a, x)b.$$

Then

$$\begin{aligned} 0 &= F(x, a, x, b) \\ &= (xa, x, b) - (x, ax, b) + (x, a, xb) - x(a, x, b) - (x, a, x)b \\ &= (xa, x, b) + (x, a, xb). \end{aligned}$$

As a result of substituting $y = x$ in (28) we obtain $(x, b, xb) = 0$. At this point replace b by $a + b$ in the last identity. Then one gets $(x, a, xb) = -(x, b, xa)$. Therefore $(xa, x, b) = (x, b, xa)$. Adding to the last identity

$$0 = A(xa, x, b) = (xa, x, b) + (x, b, xa) + (b, xa, x),$$

we get

$$2(xa, x, b) + (b, xa, x) = 0.$$

From

$$\begin{aligned} 0 &= F(b, x, a, x) \\ &= (bx, a, x) - (b, xa, x) + (b, x, ax) - b(x, a, x) - (b, x, a)x \\ &= -(b, xa, x) \end{aligned}$$

one may now deduce that $2(xa, x, b) = 0$, so that $(xa, x, b) = 0$. But we have noted previously that $(xa, x, b) + (x, a, xb) = 0$. Hence

$$(x, a, xb) = 0. \tag{31}$$

We note that

$$\begin{aligned} 0 &= F(x, a, y, b) \\ &= (xa, y, b) - (x, ay, b) + (x, a, yb) - x(a, y, b) - (x, a, y)b \\ &= (xa, y, b) + (x, a, yb) - (x, a, y)b. \end{aligned}$$

However, replacing x by $x + y$ in (31) we see that $(x, a, yb) = -(y, a, xb)$. Replacing b by $a + b$ in (28) shows that $-(y, a, xb) = (y, b, xa)$. Therefore $(x, a, yb) = (y, b, xa)$. Consequently $(xa, y, b) + (y, b, xa) = (x, a, y)b$. Subtracting $0 = A(xa, y, b) = (xa, y, b) + (y, b, xa) + (b, xa, y)$ from the last equation we see that $-(b, xa, y) = (x, a, y)b$. Because of (29) it follows that $(x, a, y)b = -b(x, a, y)$, and thereby $(b, xa, y) = b(x, a, y)$. Comparing the last identity with

$$\begin{aligned} 0 &= F(b, x, a, y) \\ &= (bx, a, y) - (b, xa, y) + (b, x, ay) - b(x, a, y) - (b, x, a)y \\ &= -(b, xa, y) - b(x, a, y), \end{aligned}$$

we conclude that $(b, xa, y) = b(x, a, y) = 0$. Consequently $(B, RB, R) = 0$. Define $D = B + RB$. It is a simple matter to verify that D is a two-sided ideal of R. Since D contains B, a non-trivial right ideal of R, it must be that $D = R$. Therefore $(B, RB, R) = 0$ and $(B, B, R) = 0$ imply $(B, R, R) = (B, D, R) = 0$. But then B is contained in the nucleus N of R and as before then $(B, R) = 0$ follows from **(2)**. This, however, suffices to show that B is contained in S, so that $B = 0$. But this is clearly a contradiction. It arose out of the original assumption that B was a proper right ideal of R. Therefore R can have no proper right ideals. The argument that R can have no proper left ideals follows from the fact that a ring of type (γ, δ) is anti-isomorphic to one of type $(-\gamma, 1 - \delta)$ (**4**, Theorem 1). This completes the proof of the theorem.

We have purposely omitted from our discussion the rings of type $(-1, 1)$ and their anti-isomorphic copies, the rings of type $(1, 0)$. The former are right alternative rings that satisfy (2). From the structure theory of right alternative algebras it follows that simple algebras of type $(-1, 1)$ whose characteristic is different from 2 and 3 are associative. This result is analogous to our Main Theorem. Rings of type $(-1, 1)$ have been considered by Maneri for his PhD. dissertation. His results will be published elsewhere.

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