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SUBLATTICES OF MODULAR LATTICES OF FINITE LENGTH

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It is well-known that the join-irreducible elements $J(L)$ and the meet-irreducible elements $M(L)$ of a lattice L of finite length play a central role in its arithmetic and, especially, in the case that L is distributive. In [3] it was shown that the *quotient set* $Q(L)=\{b/a \mid a \in J(L), b \in M(L), a \leq b\}$ plays a somewhat analogous role in the study of the sublattices of L. Indeed, *in a lattice L of finite length, if S is a sublattice of L then $S=L-\bigcup_{b/a\in A}[a, b]$ for some $A\subseteq Q(L)$*. Furthermore, the converse actually characterizes finite distributive lattices [3].

On the other hand, the arithmetical theory of a modular lattice of finite length in terms of its join-irreducible and meet-irreducible elements is far more involved than it is for a finite distributive lattice; consequently, it is not unexpected that the study of the structure of sublattices of a modular lattice of finite length is also more involved than it is for a finite distributive lattice. The purpose of this paper is to introduce and investigate some new concepts useful in the general study of sublattices of a lattice of finite length and particularly, in the case that the lattice is modular. The author acknowledges with gratitude the helpful comments of K. M. Koh in the preparation of this paper.

Preliminaries. Let $J(L)$ and $M(L)$ denote the sets of join-irreducible and meet-irreducible elements of a lattice L, respectively. For $x, y \in L$, $x \parallel y$ if x is incomparable with y and $x\succ y$ (or $y\prec x$) if x covers y in L. A subset A of L is *connected* if, for every $a, b \in A$, there is a sequence $a=x_0, x_1, \ldots, x_n=b$ of elements in A such that either $x_i\succ x_{i-1}$ or $x_i\prec x_{i-1}$ for every $i=1, 2, \ldots, n$; thus, every subset of a finite lattice can be partitioned into components, that is, maximal connected subsets. It is a simple matter to verify that if L is a lattice of finite length and M is a maximal proper sublattice of L then $L-M$ is a connected subset of L. Finally, for a subset A of L and an element $a \in A$ we define

$$A_*(a) = |\{x \in L-A \mid x \prec a\}|$$

and

$$A^*(a) = |\{x \in L-A \mid x \succ a\}|.$$

For all further terminology we refer to [1].

Sublattices of modular lattices. Clearly, if $L-A$ is a sublattice of L then $A_*(a)\leq 1$ and $A^*(a)\leq 1$ for every $a \in A$. In the case that L is modular of finite length we can recover a partial converse.

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PROPOSITION 1. *Let L be a modular lattice of finite length and let A be a subset of L satisfying the following conditions:* (i) $A_*(a)\leq 1$ *and* $A^*(a)\leq 1$ *for every* $a\in A$; (ii) *A is a convex subset of L. Then $L-A$ is a sublattice of L.*

Proof. Let us suppose that there exist $x, y\in L-A$ such that $x\wedge y\in A$. In this case we may choose a maximal element $a\in A$ such that there exist $x, y\in L-A$ with $x\wedge y=a$. In view of (i) x and y cannot both cover a so that we may furthermore assume that there exists $z\in A$ such that $y\succ z>a$. Now if $x\vee z=x\vee y$ then by the modularity of L, $y=z$; thus, $x\vee z<x\vee y$. By virtue of (ii) $x\vee z\in L-A$. But, since $y\succ z$ and $x\vee z\ngeq y$ we have that $(x\vee z)\wedge y=z\in A$ contradicting the maximality of a. □

We are now in a position to describe at least one method of generating maximal proper sublattices of a modular lattice of finite length.

COROLLARY 2. *Let L be a modular lattice of finite length and let A be a subset of L satisfying the following conditions:* (i) $A_*(a)=1=A^*(a)$ *for every* $a\in A$; (ii) *A is a convex subset of L;* (iii) *A is a connected subset of L. Then $L-A$ is a maximal proper sublattice of L.*

Proof. By the Proposition $L-A$ is a sublattice of L. If $L-A$ is not a maximal proper sublattice of L then there exists $\phi\neq A'\subset A$ such that $M=(L-A)\cup A'$ is a maximal proper sublattice of L. In view of (iii) there exist $a_1\in A'$ and $a_2\in A-A'$ such that either $a_1\succ a_2$ or $a_2\succ a_1$. We may suppose that $a_2\succ a_1$. By (i) there exists $a_3\in L-A$ such that $a_2\succ a_3$. Obviously, $a_1\parallel a_3$, $a_1, a_3\in M$ and $a_2=a_1\vee a_3$ which must then lie in M although $a_2\in A-A'$. □

For finite distributive lattices we have already seen in [3] that the conditions (i)–(iii) of Corollary 2 characterize maximal proper sublattices; whether this extends to modular lattices of finite length seems much more difficult to settle. Unfortunately, as to properties of $L-M$, where M is a maximal proper sublattice of a modular lattice L, very little apart from the next proposition is available. It will be convenient to keep in mind the following property concerning irreducible elements in a modular lattice L of finite length: *for* $a, b, c\in L$, *if* $a\in M(L)$ *and* $b>a\geq b\wedge c$ *then* $a\geq c$; *if* $a\in J(L)$ *and* $b<a\leq b\vee c$ *then* $a\leq c$.

PROPOSITION 3. *Let L be a modular lattice of finite length and let M be a maximal proper sublattice of L. If $a, b\in L-M$ and $b\succ a$ in L then either b is join-reducible in L or a is meet-reducible in L.*

Proof. Let us suppose that $b\in J(L)$ and $a\in M(L)$ and set $M'=M\cup\{x\in L \mid x\geq b\}$. Therefore, the sublattice in L generated by M' is L. On the other hand, M' is a join-subsemilattice of L; hence, there exist $y\in M-\{x\in L \mid x\geq b\}$ and $z\geq b$ such that $y\wedge z\in L-M'$ and $y\wedge z\ngeq b$. But $a\in M(L)$ and L is modular so that $y\wedge z\nleq a$ and $a\vee(y\wedge z)=z\wedge(a\vee y)$. Furthermore, $z\geq b$ and $a\vee y=b\vee y\geq b$

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(since $b \succ a$ and $a \in M(L)$), so that $a \vee (y \wedge z) \geq b$. Finally, since $b \in J(L)$ we have that $y \wedge z \geq b$, contradicting our assumption. $\square$

The lattice of Figure 1 illustrates the necessity of modularity in Proposition 3.

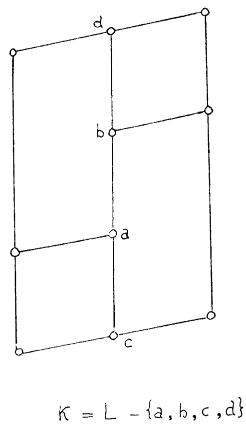

FIGURE 1

In [3] we have shown that *in a lattice* L *of finite length, if* S *is a sublattice of* L *then* $S = L - \bigcup_{b/a \in A} [a, b]$ *for some* $A \subseteq Q(L)$. Slightly more information can be obtained in the case that L is modular.

PROPOSITION 4. *Let* L *be a modular lattice of finite length and let* S *be a sublattice of* L. *Then, for every* $x \in L - S$ *there exists* $b/a \in Q(L)$ *such that* (i) $x \in [a, b] \subseteq L - S$, (ii) $(a, x) \subseteq L - J(L)$, *and* (iii) $(x, b) \subseteq L - M(L)$.

Proof. In view of the remark above and duality it suffices to show that there exists $b \in M(L)$ such that $x \leq b$ and $(x, b) \subseteq (L - S) \cap (L - M(L))$. We may assume that $x \notin M(L)$ so that there exists an integer n and a subset $\{b_i \mid 1 \leq i \leq n\}$ of $M(L)$ irredundant with respect to $\bigwedge (b_i \mid 1 \leq i \leq n) = x$. If, for every i, there exists $y_i \in [x, b_i] \cap S$ then $x = \bigwedge (y_i \mid 1 \leq i \leq n) \in S$. Hence, there exists $j \in \{ 2, \ldots, 1., n\}$

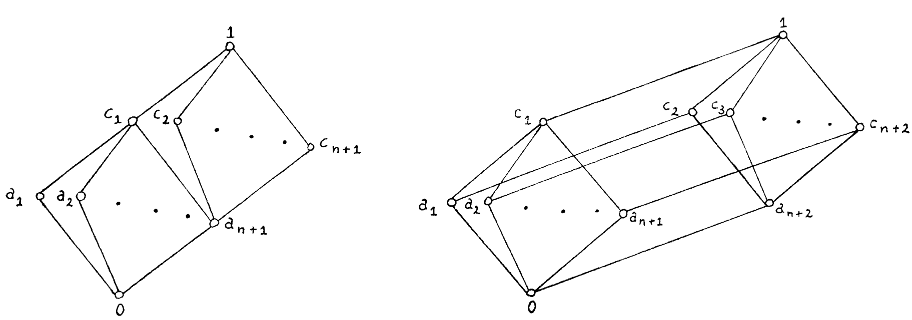

FIGURE 2

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such that $[x, b_j] \subseteq L-S$. Let $z = \bigwedge (b_i \mid 1 \le i \le n, i \ne j)$. Clearly, $z \wedge b_j = x$ and, since $\{b_i \mid 1 \le i \le n\}$ is an irredundant meet representation of x, $z \not\le b_j$. Thus, for every $u \in (x, b_j)$, $z \wedge b_j < u$ which, since $b_j \in M(L)$ and L is modular, implies that $u \in L-M(L)$. Choosing $b = b_j$ completes the proof. □

Let M be a maximal proper sublattice of a finite lattice L. If L is Boolean then $|M/L| = \frac{3}{4}$ and, if L is distributive then $|M/L| \ge \frac{2}{3}$ (cf. [2]). However, if L is modular there is, in general, no non-zero constant k such that $|M/L| \ge k$. In fact, B. Wolk has pointed out (and it is straightforward to verify) that, if P_n denotes the lattice of subspaces of a projective plane of order n, then a maximal proper sublattice M of P_n satisfies either $|M| = 2n+4$ or $|M| = 2n+6$ so that $\lim_{n\to\infty} |M/P_n| = 0$. Figure 2 illustrates the two possible maximal proper sublattices of P_n.

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