

ON THE TOTAL COMPONENT OF THE PARTIAL SCHUR MULTIPLIER

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Abstract

In this paper we determine the structure of the total component of the Schur multiplier over an algebraically closed field of some relevant families of groups, such as dihedral groups, dicyclic groups, the infinite cyclic group and the direct product of two finite cyclic groups.

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1. Introduction

The partial Schur multiplier $pM(G)$ of a group G was introduced in [3] and [4], together with the notion of partial projective representations of a group over a field K . It is a generalization of the classical Schur multiplier $M(G)$ and appeared naturally in the attempt to introduce a new cohomological theory based on partial actions. In contrast to its classical version, $pM(G)$ is not a group, but it is a semilattice of abelian groups $pM_D(G)$ (called components), indexed by certain subsets $D \subseteq G \times G$ (see Theorem 2.11). Each component $pM_D(G)$ is formed by partially defined functions $\sigma : G \times G \rightarrow K$, having D as domain, the so-called partial factor sets of G . These are associated to the partial projective representations according to Definition 2.6.

It is known that the domains of the partial factor sets form a semilattice with respect to the set-theoretic intersection and inclusion, and they were characterized in [3, Corollary 7] as the \mathcal{T} -invariant subsets of $G \times G$, where \mathcal{T} is a monoid acting on $G \times G$ (see (2.2) and (2.3)). In [6], the authors described the structure of these domains, as well as the structure of the domains which are associated to certain partial representations called elementary (see also [9]).

The *total component* $pM_{G \times G}(G)$ of $pM(G)$ (corresponding to the totally defined factor sets) is particularly important, since it contains the usual Schur multiplier $M(G)$

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as a subgroup and, moreover, according to [5, Corollary 5.8(iv)], any component of $pM(G)$ is an epimorphic image of $pM_{G \times G}(G)$, provided that the base field is algebraically closed. Some recent works provided a description for $pM_{G \times G}(G)$ over algebraically closed fields, in the case that G is a finite cyclic group [5, Corollary 6.4] (see also [6, Proposition 6.3]), an elementary abelian 2-group C_2^n [8, Theorem 3.11] and the symmetric group $S_3 \simeq D_6$ [10, Lemma 3.10].

To continue the study of the total component (and consequently the description of the partial Schur multiplier), in this work we characterize $pM_{G \times G}(G)$ for some other important classes of groups.

The article is structured as follows. After the introduction, in Section 2 we provide all the necessary background, fix some notation and describe some properties satisfied by the (total) coboundaries. In Section 3 we give some technical results on the effective orbits of some specific groups, as well as a full set of representatives of effective orbits of a direct product of arbitrary groups. A generalization of [10, Lemma 3.10] for the dihedral groups D_{2m} , where $m \in \mathbb{N}$, will be obtained in Theorem 5.2. In Theorem 5.4, we deal with D_∞ . Furthermore, Theorem 6.3 in Section 6 describes the total component for dicyclic groups, which are generalizations of quaternion groups. Moreover, [5, Corollary 6.4] will be extended to the infinite cyclic group in Theorem 7.3 and to the direct product $C_m \times C_n$ of finite cyclic groups in Corollary 4.2.

2. Preliminaries

In order to define the partial Schur multiplier of a group, we need to recall some preliminary facts and definitions. Thus, we start with the next definition.

DEFINITION 2.1. For a field K , a semigroup S with 0 is said to be a K -semigroup if there is a map $K \times S \rightarrow S$ satisfying the following properties: $\alpha(\beta x) = (\alpha\beta)x$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$, $1_K x = x$ and $0_K x = 0$, for any $\alpha, \beta \in K$ and $x, y \in S$.

DEFINITION 2.2. A K -semigroup M is said to be K -cancellative if, for every $\alpha, \beta \in K$ and $x \in M \setminus \{0\}$, the equality $\alpha x = \beta x$ implies $\alpha = \beta$.

EXAMPLE 2.3. The monoid $\text{Mat}_n K$ formed by $n \times n$ matrices with entries in K , is a K -cancellative monoid.

In a K -cancellative monoid M , one can define a congruence λ as follows: $x\lambda y \Leftrightarrow x = \alpha y$, for some $\alpha \in K^*$.

Thus, one obtains the quotient semigroup $\text{Proj } M = M/\lambda$ and the canonical projection $\xi : M \rightarrow \text{Proj } M$. If $M = \text{Mat}_n K$, then $\text{Proj } M$ is precisely the space $\text{PMat}_n K$ of projective $n \times n$ matrices over K .

DEFINITION 2.4. A (unital) partial homomorphism of a group G with values in a monoid M is a map $\phi : G \rightarrow M$ preserving the unity and such that $\phi(g)\phi(h)\phi(h^{-1}) = \phi(gh)\phi(h^{-1})$ and $\phi(g^{-1})\phi(g)\phi(h) = \phi(g^{-1})\phi(gh)$, for all $g, h \in G$.

Partial projective representations of groups appeared in [3]. They naturally extend the concept of projective representations, as one may notice in the next definition.

DEFINITION 2.5. A partial projective representation of a group G on a K -cancellative monoid M is a function $\Gamma : G \rightarrow M$ such that the composition $\xi\Gamma : G \rightarrow \text{Proj } M$ is a partial homomorphism.

The treatment of partial projective representations depends essentially on projective representations of Exel’s semigroup $E(G)$, which controls the partial actions of G (see [7]) and whose semigroup algebra $KE(G)$, called a partial group algebra, is responsible for the partial representations of G (see [2]). Indeed, taking into consideration the semigroup $E(G)$, it is shown in [3, Proposition 1] that a map $\Gamma : G \rightarrow M$ is a partial projective representation exactly when Γ factors through some projective representation $\tilde{\Gamma} : E(G) \rightarrow M$. Using this characterization, the authors of [3, Theorem 3] showed that given a partial projective representation $\Gamma : G \rightarrow M$, there is a unique partially defined function $\sigma : G \times G \rightarrow K^*$ such that

$$\begin{aligned} \text{dom } \sigma &= \{(x, y) \mid \Gamma(x)\Gamma(y) \neq 0\}, \\ \Gamma(x^{-1})\Gamma(x)\Gamma(y) &= \Gamma(x^{-1})\Gamma(xy)\sigma(x, y) \end{aligned} \tag{2.1}$$

and

$$\Gamma(x)\Gamma(y)\Gamma(y^{-1}) = \Gamma(xy)\Gamma(y^{-1})\sigma(x, y),$$

for every $(x, y) \in \text{dom } \sigma$. For convenience, we define $\sigma(x, y) = 0$ when $(x, y) \notin \text{dom } \sigma$ (making σ totally defined) and keep the notation $\text{dom } \sigma$ for (2.1). Additionally, we assume (without loss of generality) that $\Gamma(1) = 1$.

DEFINITION 2.6. The function σ associated with a partial projective representation Γ as above is called a *factor set* of Γ or a *partial factor set* of G .

REMARK 2.7. Observe that according to [3, Corollary 5], the factor sets of partial projective representations of G form a commutative inverse monoid $pm(G)$, with respect to point-wise multiplication. Thus, by Clifford’s theorem [1], this semigroup is isomorphic to a semilattice of abelian groups. Therefore, it is useful to pay attention to the idempotents of $pm(G)$, which are obviously the partial factor sets whose values are 0 or 1, so we must obtain a description of their domains.

We recall from [3] the following proposition.

PROPOSITION 2.8. *Let G be a group. Then:*

- [3, Proposition 4] *if σ is a factor set of some partial projective representation of G and D its domain, then*

$$\begin{aligned} (x, y) \in D &\Leftrightarrow (xy, y^{-1}) \in D \Leftrightarrow (x^{-1}, xy) \in D \Leftrightarrow (y, y^{-1}x^{-1}) \in D \\ &\Leftrightarrow (y^{-1}, x^{-1}) \in D \Leftrightarrow (y^{-1}x^{-1}, x) \in D; \end{aligned}$$

- [3, Proposition 5 and Corollary 6] *let D be as above. Then*

$$(x, 1) \in D \Leftrightarrow (x^{-1}, x) \in D \Leftrightarrow (1, x^{-1}) \in D \Leftrightarrow (1, x) \in D.$$

Moreover, if a map $\tau : G \times G \rightarrow K$ satisfies $\tau(1, 1) = 1$, then τ is an idempotent factor set of some partial projective representation of G , provided that its values are 0 and 1 and, for any $(x, y) \in \text{dom } \tau$,

$$(xy, y^{-1}), (y^{-1}, x^{-1}), (x, 1) \in \text{dom } \tau.$$

2.1. The semigroup \mathcal{T} and the partial Schur multiplier. Using semigroup actions, we shall obtain a better description of domains of partial factor sets. Indeed, Proposition 2.8 motivates us to consider the following maps on $G \times G$:

$$u : (x, y) \mapsto (xy, y^{-1}), \quad v : (x, y) \mapsto (y^{-1}, x^{-1}), \quad t : (x, y) \mapsto (x, 1).$$

These transformations satisfy the equalities

$$u^2 = v^2 = (uv)^3 = 1, \quad t^2 = t, \quad ut = t, \quad tuv t = tvuv, \quad tv t = 0, \quad (2.2)$$

where 0 stands for the map $(x, y) \mapsto (1, 1)$.

In [3, Section 6], there was introduced the abstract monoid \mathcal{T} generated by symbols u, v and t with relations (2.2). Then there is a disjoint union

$$\mathcal{T} = \mathcal{S} \cup t\mathcal{S} \cup vt\mathcal{S} \cup uv t\mathcal{S} \cup 0,$$

where $\mathcal{S} = \langle u, v \mid u^2 = v^2 = (uv)^3 = 1 \rangle$ is a group isomorphic to the symmetric group S_3 .

Given an arbitrary group G , there is a left action of \mathcal{T} on $G \times G$ defined by the following transformations:

$$t(x, y) = (x, 1), \quad u(x, y) = (xy, y^{-1}) \quad \text{and} \quad v(x, y) = (y^{-1}, x^{-1}), \quad (2.3)$$

for any $x, y \in G$.

REMARK 2.9. Using Proposition 2.8 and the construction of \mathcal{T} , we get that the \mathcal{T} -invariant subsets D of $G \times G$, that is, the elements of $C(G) = \{D \subseteq G \times G \mid \mathcal{T}D \subseteq D\}$, are precisely the domains of the partial factor sets of G . Then they form a semilattice with respect to the set-theoretic inclusion and intersection.

It follows from (2.2) and (2.3) that $0(x, y) = (1, 1)$, for any $x, y \in G$, and there is an action of S_3 in $G \times G$ induced by the action of \mathcal{T} . Thus, the orbit $S_3(x, y)$ of a pair $(x, y) \in G \times G$ is

$$\{(x, y), (xy, y^{-1}), (y, y^{-1}x^{-1}), (y^{-1}, x^{-1}), (y^{-1}x^{-1}, x), (x^{-1}, xy)\}. \quad (2.4)$$

Consequently, each S_3 -orbit contains one, two, three or six elements (see [5, page 216], where the S_3 -orbit containing (a, b) is denoted by $A_{(a,b)}$). As in [8], the orbits with two or six elements are called *effective orbits*. Hence, the noneffective orbits are of the form

$$\{(1, y), (y, y^{-1}), (y^{-1}, 1)\}, \quad y \in G. \quad (2.5)$$

Apart from this characterization of the domains as \mathcal{T} -invariant subsets of $G \times G$, we have the following result.

DEFINITION 2.10. The *partial Schur multiplier* of G is the quotient semigroup $pm(G) = pm(G)/\sim$, where the equivalence \sim is given by

$$\sigma \sim \tau \Leftrightarrow \sigma(x, y) = \eta(x)\eta(xy)^{-1}\eta(y)\tau(x, y), \quad x, y \in G,$$

for some function $\eta : G \rightarrow K^*$.

THEOREM 2.11 [3, Theorem 5]. *The semigroups $pm(G)$ and $pM(G)$ are semilattices of abelian groups:*

$$pm(G) = \bigcup_{D \in C(G)} pm_D(G), \quad pM(G) = \bigcup_{D \in C(G)} pM_D(G),$$

where $C(G)$ is the semilattice of \mathcal{T} -invariant subsets of $G \times G$ with respect to the intersection of sets.

Assuming that the field K is algebraically closed, there is another characterization of the partial factor sets of G .

THEOREM 2.12 [5, Theorem 5.6]. *If τ is a partial factor set of G with domain D , then there is a partial factor set $\sigma \sim \tau$, satisfying*

$$\sigma(a, b)\sigma(b^{-1}, a^{-1}) = 1_K, \tag{2.6}$$

$$\sigma(a, b) = \sigma(b^{-1}a^{-1}, a) = \sigma(b, b^{-1}a^{-1}), \tag{2.7}$$

$$\sigma(a, 1) = 1_K, \tag{2.8}$$

for any $(a, b) \in D$. Conversely, if $\sigma : G \times G \rightarrow K$ is a partially defined map with $\text{dom } \sigma \in C(G)$ such that (2.6)–(2.8) are satisfied for any $(a, b) \in D$, then σ is a partial factor set of G .

For every $D \in C(G)$, the subgroup of $pm_D(G)$ formed by all the maps $\sigma : G \times G \rightarrow K$ satisfying (2.6)–(2.8) will be denoted by $pm'_D(G)$.

REMARK 2.13. It follows from the proof of Theorem 2.12 that a factor set $\sigma \in pm'_D(G)$ is completely determined by its values in a full set of representatives of the effective orbits of D .

Now we recall the next result.

COROLLARY 2.14 [5, Corollary 5.8]. *Let $D \in C(G)$. Then:*

- (1) every partial factor set of $pm_D(G)$ is equivalent to some element of $pm'_D(G)$;
- (2) the kernel $N_D = \{\sigma \in pm'_D(G) \mid \sigma \sim 1\}$ of the natural epimorphism of $pm'_D(G) \rightarrow pM_D(G)$ consists of those $\sigma : G \times G \rightarrow K$ for which there is $\rho : G \rightarrow K^*$ satisfying the following conditions:

$$\rho(1) = 1_K, \quad \rho(a)\rho(a^{-1}) = 1, \tag{2.9}$$

for any $a \in G$ with $(a, 1) \in D$, and

$$\sigma(a, b) = \begin{cases} \rho(a)\rho(b)\rho(ab)^{-1} & \text{if } (a, b) \in D, \\ 0 & \text{if } (a, b) \notin D; \end{cases} \tag{2.10}$$

- (3) let $s = s(G, D)$ be the cardinality of the set of effective S_3 -orbits of D and $\{(a_i, b_i)\}_{1 \leq i \leq s}$ a full set of representatives of these orbits. Then the map

$$\phi : (K^*)^s \ni x \mapsto \sigma_x \in pm'_D(G),$$

in which $x = (x_i)_{1 \leq i \leq s}$ and $\sigma_x(a_i, b_i) = x_i$, is an isomorphism of multiplicative groups;

- (4) for every domain $Y \in C(G)$ such that $Y \supseteq D$, there is an epimorphism $\psi_D^Y : pM_Y(G) \rightarrow pM_D(G)$. In particular, $pM_D(G)$ is an epimorphic image of the total component $pM_{G \times G}(G)$.

DEFINITION 2.15. The partial factor sets σ in N_D are called coboundaries in $pm'_D(G)$. In this case we write $\sigma = \partial\rho$, where ρ is a function satisfying (2.9) and (2.10).

Setting $L_D = \{x \in (K^*)^s \mid \sigma_x \sim 1\} = \{x \in (K^*)^s \mid \sigma_x \in N_D\}$, one has the following result.

THEOREM 2.16 [5, Theorem 5.9]. If s is the cardinality of the set of effective S_3 -orbits of D as in Corollary 2.14, then

$$pM_D(G) \simeq pm'_D(G)/N_D \simeq (K^*)^s/L_D.$$

It follows from Corollary 2.14(4) that each component of $pM(G)$, and thus the structure of $pM(G)$, depends on the total component $pM_{G \times G}(G)$; hence, a first step to study the semigroup $pM(G)$ is to obtain a description of its total component. For this, one proceeds as follows: by Corollary 2.14(3), we need first to find the cardinality s of the set of effective S_3 -orbits of $G \times G$. Then, according to item (4) of the same corollary, we get a group epimorphism

$$\psi : (K^*)^s \ni x \mapsto \text{cls}(\sigma_x) \in pM_{G \times G}(G), \tag{2.11}$$

for which $\ker(\psi) = L_{G \times G}$. Here, $\text{cls}(\sigma_x)$ denotes the class of σ_x in the component $pM_{G \times G}(G)$, and the calculation of $pM_{G \times G}(G)$ will be completed when determining the quotient group $(K^*)^s / \ker(\psi)$.

Let $\sigma \in pm'_{G \times G}(G)$. Throughout this work, we will use the following notation.

- Fixing an element $x \in G$, set

$$\pi_j = \pi_j(x) = \sigma(x, x)\sigma(x, x^2) \cdots \sigma(x, x^{j-1}), \tag{2.12}$$

for each $j \in \mathbb{N}$, $j \geq 2$.

- More generally, given arbitrary elements $x, y \in G$ and $j \in \mathbb{N}$, $j \geq 1$, let

$$\sigma_j(x, y) = \sigma(x, y)\sigma(x, xy)\sigma(x, x^2y) \cdots \sigma(x, x^{j-1}y).$$

Set also $\sigma_0(x, y) = 1$ and notice that $\sigma_{j-1}(x, x) = \pi_j(x)$.

Some properties of the coboundaries in $pm'_{G \times G}(G)$ are given in the next result.

LEMMA 2.17. Given $\sigma \in pm'_{G \times G}(G)$, if there is $\rho : G \rightarrow K^*$ such that $\sigma = \partial\rho$, then

$$\sigma_j(x, y) = \frac{\rho^j(x)\rho(y)}{\rho(x^jy)}, \tag{2.13}$$

$$\sigma(x^i, x^k y) = \frac{\sigma_{i+k}(x, y)\sigma(x^i, y)}{\sigma_i(x, y)\sigma_k(x, y)}, \tag{2.14}$$

for any $i, j, k \geq 1$ and every $x, y \in G$. Moreover,

$$\sigma(x^i, x^k) = \frac{\sigma(x^i, y)\sigma(x^k, x^i y)}{\sigma(x^{i+k}, y)}, \tag{2.15}$$

for every $x, y \in G$ and any $i, k \in \mathbb{Z}$.

PROOF. Given i, j, k, x and y as above,

$$\sigma_j(x, y) = \frac{\rho(x)\rho(y)}{\rho(xy)} \frac{\rho(x)\rho(xy)}{\rho(x^2y)} \frac{\rho(x)\rho(x^2y)}{\rho(x^3y)} \dots \frac{\rho(x)\rho(x^{j-1}y)}{\rho(x^jy)} = \frac{\rho^j(x)\rho(y)}{\rho(x^jy)},$$

which gives (2.13). Consequently,

$$\begin{aligned} \sigma(x^i, x^k y) &= \frac{\rho(x^i)\rho(x^k y)}{\rho(x^{i+k} y)} = \frac{\rho^{i+k}(x)\rho(y)}{\rho(x^{i+k} y)} \frac{\rho(x^i)\rho(y)}{\rho(x^i y)} \frac{\rho(x^i y)}{\rho^i(x)\rho(y)} \frac{\rho(x^k y)}{\rho^k(x)\rho(y)} \\ &= \frac{\sigma_{i+k}(x, y)\sigma(x^i, y)}{\sigma_i(x, y)\sigma_k(x, y)}, \end{aligned}$$

and one obtains (2.14). Moreover,

$$\begin{aligned} \sigma(x^i, x^k) &= \frac{\rho(x^i)\rho(x^k)}{\rho(x^{i+k})} = \frac{\rho(x^i)\rho(y)}{\rho(x^i y)} \frac{\rho(x^k)\rho(x^i y)}{\rho(x^{i+k} y)} \frac{\rho(x^{i+k} y)}{\rho(x^{i+k})\rho(y)} \\ &= \frac{\sigma(x^i, y)\sigma(x^k, x^i y)}{\sigma(x^{i+k}, y)}, \end{aligned}$$

proving (2.15). □

We will denote by C_m the cyclic group of order $m \in \mathbb{N}$. Given two elements a, b of G , the commutator $aba^{-1}b^{-1}$ of a and b will be denoted by $[a, b]$. Finally, given two semigroups S_1, S_2 , we write $S_1 \leq S_2$ to indicate that S_1 is a subsemigroup of S_2 .

3. Some remarks on effective orbits

As recalled in Remark 2.13, a partial factor set $\sigma \in pm'_{G \times G}(G)$ is completely determined by its values in a full set of representatives of the effective S_3 -orbits of G . Moreover, according to [6, Theorem 6.2], the number $s(G, G \times G)$ of distinct effective S_3 -orbits of G equals the number of \mathcal{T} -orbits of the form $\mathcal{T}(a, b)$, where $1 \notin \{a, b, ab\}$ and it is given by

$$s(G, G \times G) = \frac{\binom{|G|-1}{2} + |G_{(3)}|}{3},$$

where $G_{(3)}$ denotes the set of elements of order three in G .

EXAMPLE 3.1. Suppose that $G = D_{2m} = \langle a, b \mid a^m = b^2 = (ab)^2 = 1 \rangle$ or $G = C_m \times C_2 = \langle a, b \mid a^m = b^2 = [a, b] = 1 \rangle$. Then $|o_3(G)| = 2$ if $|G| \equiv 0 \pmod 3$ and $|o_3(G)| = 0$ otherwise. Consequently,

$$s(G, G \times G) = \begin{cases} \frac{(|G| - 1)(|G| - 2) + 4}{6} & \text{if } |G| \equiv 0 \pmod 3, \\ \frac{(|G| - 1)(|G| - 2)}{6} & \text{if } |G| \not\equiv 0 \pmod 3. \end{cases} \tag{3.1}$$

Given $\sigma \in pm'_{G \times G}(G)$, equality (2.7) implies

$$\sigma(a^i b, a^j) = \sigma(a^k b, a^i b) = \sigma(a^j, a^k b),$$

where $k = i - j$ if $G = D_{2m}$ and $k = -i - j$ if $G = C_m \times C_2$. It follows that σ is completely determined by its values on pairs whose first coordinate is an element of the cyclic subgroup $C_m = \langle a \rangle < G$.

From now on in this work all partial factor sets have their values in an algebraically closed field K .

In particular, there is a group isomorphism

$$\frac{K^*}{\{1, -1\}} \simeq K^*. \tag{3.2}$$

LEMMA 3.2. *If $\sigma \in pm'_{C_m \times C_m}(C_m)$ and $m \geq 3$, then σ is uniquely determined by its values in the set S_{C_m} given by*

$$\{(a^i, a^j) \mid 1 \leq i \leq \lfloor (m-1)/3 \rfloor \text{ and } i \leq j \leq m-2i-1\} \cup Z_{a,m}, \tag{3.3}$$

where $Z_{a,m} = \{(a^{m/3}, a^{m/3})\}$ if $m \equiv 0 \pmod 3$ and $Z_{a,m} = \emptyset$ otherwise. Moreover, these values can be chosen arbitrarily in K^* .

PROOF. The proof of [5, Proposition 6.1] implies that for every $m \geq 3$, the set

$$\{S_3(a^i, a^j) \mid 1 \leq i \leq \lfloor m/3 \rfloor, i \leq j \leq m-2i\} \tag{3.4}$$

contains all the effective S_3 -orbits of C_m . Moreover, since $S_3(a^i, a^i) = S_3(a^i, a^{m-2i})$, one gets that for each integer i satisfying $i < m-2i$, there are two representatives for the same S_3 -orbit in (3.4). In these cases $i < m/3$ and it is enough to consider those $S_3(a^i, a^j)$ for which $j \leq m-2i-1$ to get a complete set of representatives of the effective S_3 -orbits. \square

Let $G = C_m \times C_2$ or $G = D_{2m}$ as in Example 3.1 and $m \geq 3$. We will use the following notation:

$$\sigma_{ij} = \sigma(a^i, a^j) \quad \text{and} \quad \tau_{ij} = \sigma(a^i, a^j b), \tag{3.5}$$

where $0 \leq i, j \leq m-1$.

LEMMA 3.3. *Let $G = C_m \times C_2$ or $G = D_{2m}$, $m \geq 3$ and S_{C_m} as in Lemma 3.2. Then any $\sigma \in pm'_{G \times G}(G)$ is completely determined by its values on the set S_G given by*

$$S_{C_m} \cup \left\{ (a^k, a^l b) \mid 1 \leq k \leq \left\lfloor \frac{m-1}{2} \right\rfloor \text{ and } 0 \leq l \leq m-1 \right\} \cup Z_{a,b,m}, \tag{3.6}$$

where $Z_{a,b,m} = \{(a^{m/2}, a^l b) \mid 0 \leq l \leq (m/2) - 1\}$ if $m \equiv 0 \pmod 2$ and $Z_{a,b,m} = \emptyset$ otherwise. Moreover, these values can be chosen arbitrarily in K^* .

PROOF. By Lemma 3.2, the restriction of σ to $C_m \times C_m$ is completely determined by its values σ_{ij} , where $(a^i, a^j) \in S_{C_m}$. Consequently, it remains to show that each of the values τ_{kl} is determined by those values whose indexes satisfy the inequalities in (3.6). Since $\sigma(1, a^j b) = 1$, for every j , it is enough to consider $1 \leq k \leq m - 1$.

It follows from (2.4) that $(a^k, a^l b)$ and $(a^{m-k}, a^{k+l} b)$ are the only pairs of $S_3(a^k, a^l b)$ whose first coordinate is a power of a . Therefore, one may assume without loss of generality that the representative $(a^k, a^l b)$ was chosen in such a way that $k \leq m - k$. Consequently, it suffices to consider $1 \leq k \leq m/2$.

If there are $(a^k, a^l b) \neq (a^{k'}, a^{l'} b)$ in the same S_3 -orbit, with $1 \leq k \leq k' \leq m/2$ and $0 \leq l, l' \leq m - 1$, then $a^{k'} = a^{m-k}$ implies $k' = k = m/2$, and $a^{l'} b = a^{m/2+l} b$ yields $l' \equiv m/2 + l \pmod m$. In this case, for any $l \geq m/2$, we have $l' = l - m/2 \leq m/2 - 1$. Therefore, when $k = m/2$ is an integer, it is possible to choose $0 \leq l \leq m/2 - 1$. This completes the proof. \square

We proceed with a simple result, which is obtained in the proof of [5, Proposition 6.3].

LEMMA 3.4. *Let $C_m = \langle a \mid a^m = 1 \rangle$ and $\sigma = \partial\rho \in pm'_{C_m \times C_m}(C_m)$, for some $\rho : G \rightarrow K^*$. Then $\pi_j(a) = \rho^j(a)\rho(a^j)^{-1}$ and*

$$\sigma(a^i, a^j) = \frac{\pi_{i+j}}{\pi_i \pi_j} = \frac{\sigma(a, a^i) \dots \sigma(a, a^{i+j-1})}{\sigma(a, a) \dots \sigma(a, a^{j-1})},$$

for any $(a^i, a^j) \in S_{C_m}$ such that $i, j \geq 2$. Moreover, $\sigma(a, a^j) = \sigma(a, a^{m-j-1})$, for all $(a, a^j) \in S_{C_m}$ satisfying $\lfloor (m + 1)/2 \rfloor \leq j \leq m - 3$.

3.1. Orbits of a direct product of groups. Given a group G , an action of the semigroup \mathcal{T} on $G \times G$ was defined by means of the transformations given in (2.3). Recall also that $S_3 \simeq \langle u, v \rangle \leq \mathcal{T}$.

Denote by GH the direct product of the groups G and H , and by $(g_1 h_1, g_2 h_2)$ an arbitrary element of $GH \times GH$, where $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We have the following result.

LEMMA 3.5. *For any groups G and H , and arbitrary elements $g_i, g'_i \in G, h_i, h'_i \in H, x \in S_3$,*

$$(g'_1 h'_1, g'_2 h'_2) = x(g_1 h_1, g_2 h_2) \Leftrightarrow (g'_1, g'_2) = x(g_1, g_2) \quad \text{and} \quad (h'_1, h'_2) = x(h_1, h_2).$$

PROOF. It is enough to prove the equivalence above for the generators u and v of S_3 . We have

$$u(g_1 h_1, g_2 h_2) = ((g_1 h_1)(g_2 h_2), (g_2 h_2)^{-1}) = ((g_1 g_2)(h_1 h_2), g_2^{-1} h_2^{-1})$$

and consequently $(g'_1 h'_1, g'_2 h'_2) = u(g_1 h_1, g_2 h_2)$ if and only if

$$g'_1 = g_1 g_2, \quad g'_2 = g_2^{-1}, \quad h'_1 = h_1 h_2 \quad \text{and} \quad h'_2 = h_2^{-1}.$$

Similarly, thanks to

$$v(g_1h_1, g_2h_2) = ((g_2h_2)^{-1}, (g_1h_1)^{-1}) = (g_2^{-1}h_2^{-1}, g_1^{-1}h_1^{-1}),$$

it follows that $(g'_1h'_1, g'_2h'_2) = v(g_1h_1, g_2h_2)$ if and only if

$$g'_1 = g_2^{-1}, \quad g'_2 = h_2^{-1}, \quad h'_1 = g_1^{-1} \quad \text{and} \quad h'_2 = h_1^{-1}.$$

The lemma is now clear. □

Given a group G , let T_G be a transversal of the action of S_3 in $G \times G$, and S_G be the subset of T_G formed by those elements which correspond to the effective orbits.

From now on, T_G will always be chosen in such a way that $T_G \setminus S_G = \{1\} \times G$. This is possible because by (2.4) the noneffective orbits are of the form $\{(1, a), (a, a^{-1}), (a^{-1}, 1)\}$, $a \in G$. In particular, for $G = C_m$, we will always choose S_{C_m} as in Lemma 3.2, for any $m \geq 3$ and $S_{C_m} = \emptyset$, if $m \in \{1, 2\}$.

In order to calculate effective orbits of a direct product of groups, we need to introduce some more notation. Let $G_{(k)} = \{g \in G \mid \text{ord}(g) = k\}$, for any $k \in \mathbb{N}$ and $G^* = G \setminus \{1\}$. If $G_{(k)} \neq \emptyset$, for some $k > 2$, we take a subset $X(G)$ of G such that $G \setminus (G_{(1)} \cup G_{(2)}) = X(G) \cup X(G)^{-1}$ and $X(G) \cap X(G)^{-1} = \emptyset$. Otherwise, we set $X(G) = \emptyset$.

Finally, if there exists $e \in G_{(2)}$, we denote by $Y(e)$ a subset of G such that $G = Y(e) \cup eY(e)$ and $Y(e) \cap eY(e) = \emptyset$.

EXAMPLE 3.6. If $G = C_m = \langle a \mid a^m = 1 \rangle$ and $m \geq 3$, we take

$$X(C_m) = \{a, a^2, \dots, a^{\lfloor (m-1)/2 \rfloor}\}.$$

Now, if m is even, then $a^{m/2}$ is the only element of order two in C_m , and a natural choice is

$$Y(a^{m/2}) = \{1, a, \dots, a^{m/2-1}\}.$$

For $\mathbb{Z} = \langle a \rangle$, we choose $X(\mathbb{Z}) = \{a^i \mid i > 0\} \simeq \mathbb{N}$.

The subsets $X(G)$ and $Y(e)$, with $e \in G_{(2)}$, play a fundamental role in the calculation of a full set of representatives for the orbits of a direct product of groups, as we notice in the following result.

THEOREM 3.7. *Given groups G and H , fix $S_G, S_H, T_G, X(G)$ and $Y(e)$ (if $e \in G_{(2)}$) as above. Then the set*

$$\begin{aligned} & S_G \cup (X(G) \times GH^*) \cup \bigcup_{e \in G_{(2)}} \{e\} \times Y(e)H^* \\ & \cup \{(g_1h_1, g_2h_2) \mid g_1, g_2 \in G \text{ and } (h_1, h_2) \in S_H, \\ & \quad \text{where } (h_1, h_2) \notin H_{(3)} \times H_{(3)} \text{ or } h_1 \neq h_2\} \\ & \cup \{(g_1h, g_2h) \mid (g_1, g_2) \in T_G \cup v(S_G), (h, h) \in S_H \cap (H_{(3)} \times H_{(3)})\} \end{aligned}$$

contains exactly one representative of each effective orbit of S_3 on $GH \times GH$.

PROOF. Consider the following decomposition:

$$GH \times GH = (G \times G) \cup (GH^* \times G) \cup (G \times GH^*) \cup (GH^* \times GH^*).$$

By (2.4), the orbit $S_3(g_1h_1, g_2h_2)$ of any $(g_1h_1, g_2h_2) \in GH \times GH$ is of the form

$$\{(g_1h_1, g_2h_2), (g_2^{-1}g_1^{-1}h_2^{-1}h_1^{-1}, g_1h_1), (g_2h_2, g_2^{-1}g_1^{-1}h_2^{-1}h_1^{-1}), \\ (g_2^{-1}h_2^{-1}, g_1^{-1}h_1^{-1}), (g_1^{-1}h_1^{-1}, g_1g_2h_1h_2), (g_1g_2h_1h_2, g_2^{-1}h_2^{-1})\}.$$

In particular, if $h_1 = h_2 = 1$, then $S_3(g_1, g_2) \subseteq G \times G$. It follows that S_G contains exactly one representative of each of these effective orbits. Now take one of the remaining S_3 -orbits, that is, the orbit of a pair from $(GH \times GH) \setminus (G \times G)$. There are some cases to be considered.

Case 1. If the orbit contains a pair $(g_1h_1, g_2) \in GH^* \times G$, it also has a representative of the form $(g_2^{-1}, g_1^{-1}h_1^{-1}) \in G \times GH^*$. Therefore, one can choose pairs from $G \times GH^*$ instead of pairs in $GH^* \times G$.

Case 2. Suppose that the orbit is effective and has a representative $(g_1, g_2h_2) \in G \times GH^*$. Notice that $(g_1^{-1}, g_1g_2h_2)$ is also in $S_3(g_1, g_2h_2)$.

If $\text{ord}(g_1) \neq 1, 2$, then $g_1 \neq g_1^{-1}$ and it suffices to choose one of the (distinct) pairs (g_1, g_2h_2) or $(g_1^{-1}, g_1g_2h_2)$. Thus, we may assume without loss of generality that the pair (g_1, g_2h_2) is such that $g_1 \in X(G)$, which yields $(g_1, g_2h_2) \in X(G) \times GH^*$.

On the other hand, if $g_1 \in G_{(2)}$, then $g_1^{-1} = g_1 \neq 1$ (because the orbit $S_3(g_1, g_2h_2)$ is effective) and $(g_1, g_1g_2h_2) \neq (g_1, g_2h_2)$. In this case, to avoid having two representatives for the same S_3 -orbit, without loss of generality one may suppose that g_2 belongs to $Y(g_1)$, that is, $(g_1, g_2h_2) \in \{g_1\} \times Y(g_1)H^*$.

Case 3. Suppose that the orbit contains some $(g_1h_1, g_2h_2) \in GH^* \times GH^*$ as a representative. If the orbit intersects $G \times GH^*$, we choose a representative as in the previous case. Otherwise, $h_1h_2 \neq 1$ (thanks to the form of the S_3 -orbit) and $S_3(h_1, h_2)$ is an effective orbit (see (2.5)). Consequently, $(h_1, h_2) = x(h'_1, h'_2)$, for some $x \in S_3$ and some $(h'_1, h'_2) \in S_H$. Let $(g'_1, g'_2) = x^{-1}(g_1, g_2)$. Then Lemma 3.5 implies that $(g_1h_1, g_2h_2) = x(g'_1h'_1, g'_2h'_2)$. Therefore, one may suppose that (g_1h_1, g_2h_2) belongs to

$$(G \times G)S_H = \{(g_1h_1, g_2h_2) \mid (g_1, g_2) \in G \times G, (h_1, h_2) \in S_H\}. \tag{3.7}$$

If two different pairs $(g'_1h'_1, g'_2h'_2)$ and $(g''_1h''_1, g''_2h''_2)$ from (3.7) are in the orbit $S_3(g_1h_1, g_2h_2)$, then, by Lemma 3.5, there is $y \in S_3 \setminus \{1\}$ such that $(g'_1, g'_2) = y(g''_1, g''_2)$ and $(h'_1, h'_2) = y(h''_1, h''_2)$. In that case, since S_H contains only one pair from each effective orbit, it follows that $(h'_1, h'_2) = (h''_1, h''_2)$. By inspecting the elements of the effective orbit $S_3(h''_1, h''_2)$, given by (2.4), one sees that

$$(h''_1, h''_2) = y(h''_1, h''_2) \Leftrightarrow y \in \{uv, (uv)^2\}, \quad h''_1 = h''_2 \text{ and } \text{ord}(h''_1) = 3. \tag{3.8}$$

We conclude that $h = h''_1 = h''_2 \in H_{(3)}$, and this leads to $(h, h) = (h''_1, h''_2) \in S_H \cap (H_{(3)} \times H_{(3)})$. Now we consider the orbit $S_3(g_1, g_2)$.

Case 3.1. If $S_3(g_1, g_2)$ is effective, then $(g_1, g_2) = z(g'_1, g'_2)$, for some $(g'_1, g'_2) \in S_G$ and $z \in S_3$. If $z \in \{1, uv, (uv)^2\}$, then, using (3.8), one gets $(h, h) = z(h, h)$ and consequently $(g_1h, g_2h) = z(g'_1h, g'_2h)$ thanks to Lemma 3.5. Otherwise, $z = vw$ with $w \in \{1, uv, (uv)^2\}$ and

$$(g_1, g_2) = vw(g'_1, g'_2) = w^{-1}v(g'_1, g'_2) = w^{-1}(g_2^{-1}, g_1^{-1}),$$

where $(g_2^{-1}, g_1^{-1}) = v(g'_1, g'_2) \in v(S_G)$. Using (3.8), we get $w^{-1}(h, h) = (h, h)$; thus, Lemma 3.5 implies that $(g_1h, g_2h) = w^{-1}(g_2^{-1}h, g_1^{-1}h)$. Hence, for any orbit $S_3(g_1h, g_2h)$ such that (g_1, g_2) is effective and $(h, h) \in S_H \cap (H_{(3)} \times H_{(3)})$, it can be assumed that $(g_1, g_2) \in S_G \cup v(S_G) \subseteq T_G \cup v(S_G)$.

Case 3.2. If $S_3(g_1, g_2)$ is not effective, then $(g_1, g_2) = z(g'_1, g'_2)$, for some $(g'_1, g'_2) \in T_G \setminus S_G$ and $z \in S_3$. Since $v(S_3(g_1, g_2)) = S_3(g_1, g_2)$, it is possible to choose $z \in \{1, uv, (uv)^2\}$ and get $(g_1h, g_2h) = z(g'_1h, g'_2h)$ by (3.8), for any $(h, h) \in S_H \cap (H_{(3)} \times H_{(3)})$.

To finish the proof, it is enough to observe that if $(G \times G)S_H$ contains exactly one element of $S_3(g_1h_1, g_2h_2)$, then $(h_1, h_2) \notin H_{(3)} \times H_{(3)}$ or $h_1 \neq h_2$. □

Using Theorem 3.7, we obtain as an example a full set of representatives for the effective orbits of products of finite cyclic groups. In fact, Theorem 3.7, and in particular the construction of the subsets $X(G)$, were inspired by this particular case.

EXAMPLE 3.8. Let $G = C_m \times C_n = \langle a, b \mid a^m = b^n = [a, b] = 1 \rangle$, for some $m, n \in \mathbb{N}$. Using (3.3) and Example 3.6, set

$$\begin{aligned} S_{C_m \times C_n} &= S_{C_m} \cup X(C_m) \times C_m C_n^* \cup \{a^{m/2}\} \times Y(C_m) C_n^* \\ &\quad \cup (C_m \times C_m)(S_{C_n} \setminus \{(b^{n/3}, b^{n/3})\}) \\ &\quad \cup ((\{1\} \times C_m) \cup S_{C_m} \cup v(S_{C_m}))\{(b^{n/3}, b^{n/3})\}, \end{aligned}$$

where $Y(C_m) = Y(a^{m/2})$ if $m \equiv 0 \pmod 2$, $Y(C_m) = \emptyset$ if $m \not\equiv 0 \pmod 2$ and $\{(b^{n/3}, b^{n/3})\} = \emptyset$ if $n \not\equiv 0 \pmod 3$. Then $S_{C_m \times C_n}$ contains precisely one representative of each effective S_3 -orbit. In particular, any $\sigma \in pm'_{G \times G}(G)$ is uniquely determined by its values in the elements of $S_{C_m \times C_n}$, and these values can be chosen arbitrarily from K^* .

4. Direct product of two cyclic groups

4.1. Product of cyclic groups. Now we calculate $pM_{G \times G}(G)$ for $G = C_m \times C_n, m, n \in \mathbb{N}$. We start with the next result.

PROPOSITION 4.1. *Let $G = C_m \times C_n, m, n \in \mathbb{N}$ and $\sigma \in pm'_{G \times G}(G)$ be such that $\sigma \sim 1$. Then σ is uniquely determined by its values on the pairs*

$$(a, a^k b^l), \quad \text{where } 0 \leq k \leq m - 1, 1 \leq l \leq \lfloor (n - 1)/2 \rfloor, \tag{4.1}$$

$$(a, a^k b^{n/2}), \quad \text{where } 0 \leq k \leq \lfloor (m - 1)/2 \rfloor \text{ (if } n \text{ is even)}, \tag{4.2}$$

$$(a^i, b), \quad \text{where } 2 \leq i \leq \lfloor m/2 \rfloor, \tag{4.3}$$

$$(b, b^l), \quad \text{where } 1 \leq l \leq \lfloor (n - 1)/2 \rfloor \text{ (if } n \geq 3), \tag{4.4}$$

and these values can be chosen arbitrarily in K^* .

PROOF. Assume that $G = C_m \times C_n$ is generated by a and b as in Example 3.8. Since $\sigma \sim 1$, there is a map $\rho : G \rightarrow K^*$ satisfying (2.9) such that $\sigma(x, y) = \partial\rho(x, y)$, for any $x, y \in G$. By (2.15) and Example 3.8, the value of σ on any $(a^i, a^k) \in S_{C_m}$ is determined by its values on elements from

$$B_1 = (X(C_m) \times C_m C_n^*) \cup (\{a^{m/2}\} \times Y(C_m) C_n^*) \\ = \{(a^i, a^k b^l) \mid 1 \leq i \leq \lfloor (m-1)/2 \rfloor, 0 \leq k \leq m-1 \text{ and } 1 \leq l \leq n-1\} \\ \cup \{(a^{m/2}, a^k b^l) \mid 0 \leq k \leq m/2-1 \text{ and } 1 \leq l \leq n-1\}.$$

By (2.14), if $i, k \geq 1$, then

$$\sigma(a^i, a^k b^l) = \frac{\sigma_{i+k}(a, b^l) \sigma(a^i, b^l)}{\sigma_i(a, b^l) \sigma_k(a, b^l)} = \frac{\sigma(a, a^i b^l) \dots \sigma(a, a^{i+k-1} b^l) \sigma(a^i, b^l)}{\sigma(a, b^l) \dots \sigma(a, a^{k-1} b^l)}.$$

Moreover, since $\sigma(a^i, b^l) \stackrel{(2.7)}{=} \sigma((a^i b^l)^{-1}, a) \stackrel{(2.6)}{=} \sigma(a^{m-i}, a^i b^l)^{-1}$, we see that for any $i \geq 2$ the value of $\sigma(a^i, a^k b^l)$ is determined by the values $\sigma(a, a^k b^l)$ and $\sigma(a^{i'}, b^{l'})$ such that $1 \leq k' \leq m-1, 1 \leq l' \leq n-1$ and $1 \leq i' \leq \lfloor m/2 \rfloor$.

Now we will consider the values $\sigma(a, a^k b^l)$ and $\sigma(a^i, b^l)$. In the first case,

$$\sigma(a, a^k b^l) = \frac{\rho(a) \rho(a^k b^l)}{\rho(a^{k+1} b^l)} = \frac{\rho(a) \rho(a^{m-1-k} b^{n-l})}{\rho(a^{m-k} b^{n-l})} = \sigma(a, a^{m-1-k} b^{n-l}),$$

so we can choose $1 \leq l \leq n/2$. Further, if n is even and $l = n/2$, then $\sigma(a, a^k b^{n/2}) = \sigma(a, a^{m-1-k} b^{n/2})$, and it can be supposed that $k \leq m-1-k$, that is, $k \leq \lfloor (m-1)/2 \rfloor$.

With respect to the values $\sigma(a^i, b^l)$,

$$\frac{\sigma(a^i, b^l)}{\sigma_i(a, b^l)} = \frac{\rho(a^i) \rho(b^l)}{\rho(a^i b^l)} \frac{\rho(a^i b^l)}{\rho^i(a) \rho(b^l)} = \frac{\rho(a^i)}{\rho^i(a)},$$

for every l and $i \geq 1$. In particular, taking $l = 1$, it follows that

$$\frac{\sigma(a^i, b^{l'})}{\sigma_i(a, b^{l'})} = \frac{\sigma(a^i, b)}{\sigma_i(a, b)},$$

for any $i, l' \geq 2$ and, consequently,

$$\sigma(a^i, b^l) = \sigma(a^i, b) \frac{\sigma_i(a, b^l)}{\sigma_i(a, b)},$$

for all $i, l \geq 2$. We conclude that the values of σ on elements of B_1 are determined by

$$\sigma(a, a^k b), \sigma(a, a^k b^2), \dots, \sigma(a, a^k b^{\lfloor (n-1)/2 \rfloor}), \quad \text{where } 0 \leq k \leq m-1, \quad (4.5)$$

$$\sigma(a^i, b), \quad \text{where } 2 \leq i \leq \lfloor m/2 \rfloor, \quad (4.6)$$

and (if n is even)

$$\sigma(a, a^k b^{n/2}), \quad \text{where } 0 \leq k \leq \lfloor (m-1)/2 \rfloor. \quad (4.7)$$

It follows from Lemma 3.2, applied to $C_n = \langle b \rangle$, that the set

$$B_2 = (C_m \times C_m)(S_{C_n} \setminus \{(b^{n/3}, b^{n/3})\}) \cup ((\{1\} \times C_m) \cup S_{C_m} \cup \nu(S_{C_m}))\{(b^{n/3}, b^{n/3})\}$$

from Example 3.8 is nonempty if $n \geq 3$, where $\{(b^{n/3}, b^{n/3})\} = \emptyset$ if $n \not\equiv 0 \pmod 3$. Furthermore, for all $i, j, k, l \in \mathbb{Z}$,

$$\begin{aligned} \sigma(a^i b^j, a^k b^l) &= \frac{\rho(a^i b^j) \rho(a^k b^l)}{\rho(a^{i+k} b^{j+l})} \\ &= \frac{\rho(a^i) \rho(b^j) \rho(a^k b^l)}{\sigma(a^i, b^j) \rho(a^{i+k} b^{j+l})} \\ &= \frac{\sigma(a^i, a^k b^l) \rho(a^{i+k} b^l) \rho(b^j)}{\sigma(a^i, b^j) \rho(a^{i+k} b^{j+l})} \\ &= \frac{\sigma(a^i, a^k b^l) \rho(a^{i+k}) \rho(b^l) \rho(b^j)}{\sigma(a^i, b^j) \sigma(a^{i+k}, b^l) \rho(a^{i+k} b^{j+l})} \\ &= \sigma(b^j, b^l) \frac{\sigma(a^i, a^k b^l) \sigma(a^{i+k}, b^{j+l})}{\sigma(a^i, b^j) \sigma(a^{i+k}, b^l)}. \end{aligned}$$

Therefore, the values of σ on any $(a^i b^j, a^k b^l) \in B_2$ are determined by the values specified in (4.5)–(4.7) and $\sigma(b^j, b^l)$, where $(b^j, b^l) \in S_{C_n}$. Applying Lemma 3.4 to C_n , one concludes that the values $\sigma(b^j, b^l)$, $(b^j, b^l) \in S_{C_n}$ are determined by

$$\sigma(b, b^l), \quad \text{where } 1 \leq l \leq \lfloor (n - 1)/2 \rfloor,$$

by means of

$$\sigma(b^j, b^l) = \frac{\pi_{j+l}}{\pi_j \pi_l} = \frac{\sigma_{j+l-1}(b, b)}{\sigma_{j-1}(b, b) \sigma_{l-1}(b, b)},$$

where $\pi_j = \pi_j(b)$ is given by (2.12), and by

$$\sigma(b, b^l) = \sigma(b, b^{n-l-1}).$$

Conversely, denote by I the list of pairs in (4.1)–(4.4) and fix a family $(\nu(x, y))_{(x,y) \in I}$ of elements in K^* . In what follows, our intention is to show that these values determine a partial factor set $\sigma \sim 1$. By Corollary 2.14(2), it is enough to construct a map $\rho : G \rightarrow K^*$ satisfying (2.9) and (2.10) such that $\sigma(x, y) = \partial \rho(x, y)$, for any $(x, y) \in I$.

Let $\nu_0(x, y) = 1$ and $\nu_j(x, y) = \nu(x, y) \nu(x, xy) \dots \nu(x, x^{j-1}y)$, $x, y \in G$.

First we want to define $\rho(b), \dots, \rho(b^{\lfloor (n+1)/2 \rfloor})$ in such a way that ρ satisfies (2.9) and

$$\nu_j(b, b) = \frac{\rho^{j+1}(b)}{\rho(b^{j+1})}, \tag{4.8}$$

for any $1 \leq j \leq \lfloor (n - 1)/2 \rfloor$ (this is already true for $j = 0$). The values of ρ on the other powers of b (if $n \geq 3$) will be defined using (2.9). Since $n = \lfloor n/2 \rfloor + \lfloor (n + 1)/2 \rfloor$, this

is possible only if

$$\begin{aligned} \rho^n(b) &= \rho^{\lfloor n/2 \rfloor}(b) \rho^{\lfloor (n+1)/2 \rfloor}(b) \stackrel{(4.8)}{=} (v_{\lfloor n/2 \rfloor - 1}(b, b) \rho(b^{\lfloor n/2 \rfloor})) (v_{\lfloor (n-1)/2 \rfloor}(b, b) \rho(b^{\lfloor (n+1)/2 \rfloor})) \\ &\stackrel{(2.9)}{=} v_{\lfloor n/2 \rfloor - 1}(b, b) v_{\lfloor (n-1)/2 \rfloor}(b, b). \end{aligned}$$

Thus, we define

$$\rho(b) = \frac{1}{\rho(b^{n-1})} = \omega_2, \quad \text{where } \omega_2^n = v_{\lfloor n/2 \rfloor - 1}(b, b) v_{\lfloor (n-1)/2 \rfloor}(b, b). \tag{4.9}$$

For $2 \leq j \leq \lfloor n/2 \rfloor$, let

$$\rho(b^j) = \frac{1}{\rho(b^{n-j})} = \frac{\omega_2^j}{v_{j-1}(b, b)}. \tag{4.10}$$

Notice that (4.10) coincides with (4.9) when $j = 1$. Now, for $j = \lfloor (n + 1)/2 \rfloor$,

$$\rho(b^{\lfloor (n+1)/2 \rfloor}) = \frac{1}{\rho(b^{\lfloor n/2 \rfloor})} = \frac{v_{\lfloor n/2 \rfloor - 1}(b, b)}{\omega_2^{\lfloor n/2 \rfloor}} \stackrel{(4.9)}{=} \frac{\omega_2^{\lfloor (n+1)/2 \rfloor}}{v_{\lfloor (n-1)/2 \rfloor}(b, b)}.$$

We also want

$$v_j(a, b^l) = \frac{\rho^j(a) \rho(b^l)}{\rho(a^j b^l)} \tag{4.11}$$

in the following cases:

- $1 \leq j \leq m$ and $1 \leq l \leq \lfloor (n - 1)/2 \rfloor$ (if $n \geq 3$);
- $1 \leq j \leq \lfloor (m + 1)/2 \rfloor$ and $l = n/2$ (if n is even).

Thus, we set

$$\rho(a) = \frac{1}{\rho(a^{m-1})} = \rho(1) = 1$$

and

$$\rho(a^j b^l) = \frac{1}{\rho(a^{m-j} b^{n-l})} \stackrel{(4.11)}{=} \frac{\rho^j(a) \rho(b^l)}{v_j(a, b^l)} \stackrel{(4.10)}{=} \frac{\omega_2^l}{v_j(a, b^l) v_{l-1}(b, b)} \tag{4.12}$$

in these cases.

Finally, in order to get $v(a^i, b) = \rho(a^i) \rho(b) \rho(a^i b)^{-1}$ when $1 \leq i \leq \lfloor m/2 \rfloor$, define

$$\rho(a^i) = \frac{1}{\rho(a^{m-i})} = \frac{v(a^i, b) \rho(a^i b)}{\rho(b)} \stackrel{(4.12)}{=} \frac{v(a^i, b) \omega_2}{\omega_2 v_i(a, b) v_0(b, b)} = \frac{v(a^i, b)}{v_i(a, b)}.$$

The map ρ just defined satisfies the following conditions.

- If $1 \leq k \leq m - 1$ and $1 \leq l \leq \lfloor (n - 1)/2 \rfloor$, or if n is even, $l = n/2$ and $1 \leq k \leq \lfloor (m - 1)/2 \rfloor$, then

$$\frac{\rho(a) \rho(a^k b^l)}{\rho(a^{k+1} b^l)} = \frac{\omega_2^l}{v_k(a, b^l) v_{l-1}(b, b)} \frac{v_{k+1}(a, b^l) v_{l-1}(b, b)}{\omega_2^l} = v(a, a^k b^l).$$

- For $1 \leq i \leq \lfloor m/2 \rfloor$,

$$\frac{\rho(a^i)\rho(b)}{\rho(a^i b)} = \frac{\nu(a^i, b)}{\nu_i(a, b)} \omega_2 \frac{\nu_i(a, b)\nu_0(b, b)}{\omega_2} = \nu(a^i, b).$$

- If $n \geq 3$ and $1 \leq l \leq \lfloor (n - 1)/2 \rfloor$, then

$$\frac{\rho(b)\rho(b^l)}{\rho(b^{l+1})} = \omega_2 \frac{\omega_2^l}{\nu_{l-1}(b, b)} \frac{\nu_l(b, b)}{\omega_2^{l+1}} = \nu(b, b^l).$$

This completes the proof. □

Let $c_{m,n} = |S_{C_m \times C_n}|$ be the number of effective S_3 -orbits of the group $C_m \times C_n$. By Example 3.8, $c_{m,n} = |S_{C_m}| + |B_1| + |B_2|$, where B_1 and B_2 are as in the proof of Proposition 4.1. Thus,

$$|B_1| = \begin{cases} \left\lfloor \frac{m-1}{2} \right\rfloor m(n-1) + \frac{m(n-1)}{2} & \text{if } m \equiv 0 \pmod{2}, \\ \left\lfloor \frac{m-1}{2} \right\rfloor m(n-1) & \text{if } m \not\equiv 0 \pmod{2}, \end{cases}$$

$$|B_2| = \begin{cases} m^2(|S_{C_n}| - 1) + m + 2|S_{C_m}| & \text{if } n \geq 3 \text{ and } n \equiv 0 \pmod{3}, \\ m^2|S_{C_n}| & \text{if } n \geq 3 \text{ and } n \not\equiv 0 \pmod{3}, \\ 0 & \text{if } n = 2 \end{cases}$$

and, using (3.1),

$$|S_{C_n}| = \begin{cases} \frac{(n-1)(n-2)}{6} & \text{if } n \not\equiv 0 \pmod{3}, \\ \frac{(n-1)(n-2) + 4}{6} & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

Now we give the principal result of this section.

THEOREM 4.2. *If $G = C_m \times C_n$, then $pM_{G \times G}(G) \simeq (K^*)^{c_{m,n} - |Q_1| - |Q_2|}$, where $Q_1 \subseteq S_{C_m \times C_n}$ is the set of pairs given by (4.1)–(4.3) and $Q_2 \subseteq S_{C_m \times C_n}$ is given by (4.4).*

PROOF. Considering $S_{C_m \times C_n}$ as in Example 3.8, write

$$(K^*)^{c_{m,n}} = \{ \{x(u, v)\}_{(u,v) \in S_{C_m \times C_n}} \}.$$

By Proposition 4.1, the subgroup $L = L_{G \times G} = \{x \in (K^*)^{c_{m,n}} \mid \sigma_x \sim 1\}$ of $(K^*)^{c_{m,n}}$ is

$$L = \left\{ x \in (K^*)^{c_{m,n}} \mid \begin{aligned} &x(a^i, a^j) = \frac{x(a, a^i) \dots x(a, a^{i+j-1})}{x(a, a) \dots x(a, a^{j-1})}, \text{ for } (a^i, a^j) \in J_1; \\ &x(a, a^j) = x(a, a^{m-j-1}), \text{ for } j \in J_2; \quad x(a, a^j) = \frac{x(a, b)x(a^j, ab)}{x(a^{1+j}, b)}, \text{ for } j \in J_3; \\ &x(a^i, a^k b^l) = \frac{x(a, a^i b^l) \dots x(a, a^{i+k-1} b^l)x(a^i, b^l)}{x(a, b^l) \dots x(a, a^{k-1} b^l)}, \text{ for } (a^i, a^k b^l) \in J_4; \\ &x(a, a^k b^l) = x(a, a^{m-1-k} b^{n-l}), \text{ for } (k, l) \in J_5; \\ &x(a, a^k b^{n/2}) = x(a, a^{m-1-k} b^{n/2}), \text{ for } k \in J_6; \\ &x(a^i, b^l) = x(a^i, b) \frac{x(a, b^l) \dots x(a, a^{i-1} b^l)}{x(a, b) \dots x(a, a^{i-1} b)}, \text{ for } (i, l) \in J_7; \\ &x(a^i b^j, a^k b^l) = x(b^j, b^l) \frac{x(a^i, a^k b^l)x(a^{i+k}, b^{j+l})}{x(a^i, b^j)x(a^{i+k}, b^l)}, \text{ for } (i, l) \in J_8; \\ &x(b^j, b^l) = \frac{x(b, b^j) \dots x(b, b^{j+l-1})}{x(b, b) \dots x(b, b^{l-1})}, \text{ for } (b^j, b^l) \in J_9; \\ &x(b, b^l) = x(b, b^{n-l-1}), \text{ for } l \in J_{10} \end{aligned} \right\},$$

where

$$\begin{aligned} J_1 &= \{(a^i, a^j) \in S_{C_m} \mid i, j \geq 2\}, \\ J_2 &= \{j \in \mathbb{N} \mid \lfloor (m+1)/2 \rfloor \leq j \leq m-3\}, \\ J_3 &= \{j \in \mathbb{N} \mid 1 \leq j \leq \lfloor (m-1)/2 \rfloor\}, \\ J_4 &= \{(a^i, a^k b^l) \in B_1 \mid 2 \leq i \text{ and } 1 \leq k\}, \\ J_5 &= \{(k, l) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq k \leq m-1 \text{ and } \lfloor n/2 \rfloor + 1 \leq l \leq n-1\}, \\ J_6 &= \{k \in \mathbb{N} \mid \lfloor m/2 \rfloor \leq k \leq m-1\}, \\ J_7 &= \{(i, l) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq m/2 \text{ and } 2 \leq l \leq n/2\}, \\ J_8 &= \{(a^i b^j, a^k b^l) \in B_2 \mid (i, k) \neq (0, 0)\}, \\ J_9 &= \{(b^j, b^l) \in S_{C_n} \mid j, l \geq 2\}, \\ J_{10} &= \{l \in \mathbb{N} \mid \lfloor (n+1)/2 \rfloor \leq l \leq n-3\}. \end{aligned}$$

Notice that

$$|Q_1| = \begin{cases} m \lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor - 1 + \lfloor (m+1)/2 \rfloor & \text{if } n \equiv 0 \pmod{2}, \\ m \lfloor (n-1)/2 \rfloor + \lfloor m/2 \rfloor - 1 & \text{if } n \not\equiv 0 \pmod{2} \end{cases}$$

and that

$$|Q_2| = \begin{cases} \lfloor (n-1)/2 \rfloor & \text{if } n \geq 3, \\ 0 & \text{if } n = 2. \end{cases}$$

Furthermore, $|B_1 \setminus Q_1| = |J_4| + |J_5| + |J_6| + |J_7|$ and $|B_2 \setminus Q_2| = |J_8| + |J_9| + |J_{10}|$. In addition, $c_{m,n} - |Q_1| - |Q_2| = |S_{C_m \times C_n}| + |B_1 \setminus Q_1| + |B_2 \setminus Q_2|$.

Inspired by the relations defining L , we consider the homomorphism $\Lambda : (K^*)^{c_{m,n}} \rightarrow (K^*)^{c_{m,n} - |Q_1| - |Q_2|}$ which maps any $\{x(u, v)\}_{(u,v) \in S_{C_m \times C_n}}$ to

$$\begin{aligned} & \left(\frac{x(a, a^i) \dots x(a, a^{i+j-1})}{x(a^i, a^j)x(a, a) \dots x(a, a^{j-1})} \right)_{(a^i, a^j) \in J_1}, \left(\frac{x(a, a^{m-j-1})}{x(a, a^j)} \right)_{j \in J_2}, \\ & \left(\frac{x(a, b)x(a^j, ab)}{x(a, a^j)x(a^{1+j}, b)} \right)_{j \in J_3}, \\ & \left(\frac{x(a, a^i b^l) \dots x(a, a^{i+k-1} b^l)x(a^i, b^l)}{x(a^i, a^k b^l)x(a, b^l) \dots x(a, a^{k-1} b^l)} \right)_{(a^i, a^k b^l) \in J_4}, \\ & \left(\frac{x(a, a^{m-1-k} b^{n-l})}{x(a, a^k b^l)} \right)_{(k,l) \in J_5}, \left(\frac{x(a, a^{m-1-k} b^{n/2})}{x(a, a^k b^{n/2})} \right)_{k \in J_6}, \\ & \left(\frac{x(a^i, b)x(a, b^l) \dots x(a, a^{i-1} b^l)}{x(a^i, b^l)x(a, b) \dots x(a, a^{i-1} b)} \right)_{(i,l) \in J_7}, \\ & \left(\frac{x(b^j, b^l)x(a^i, a^k b^l)x(a^{i+k}, b^{j+l})}{x(a^i b^j, a^k b^l)x(a^i, b^j)x(a^{i+k}, b^l)} \right)_{(i,l) \in J_8}, \\ & \left(\frac{x(b, b^j) \dots x(b, b^{j+l-1})}{x(b^j, b^l)x(b, b) \dots x(b, b^{l-1})} \right)_{(b^j, b^l) \in J_9}, \left(\frac{x(b, b^{n-l-1})}{x(b, b^l)} \right)_{l \in J_{10}}. \end{aligned}$$

It follows by construction that Λ is an epimorphism of groups whose kernel is L . Consequently, by Theorem 2.16,

$$pM_{G \times G}(G) \simeq \frac{(K^*)^{c_{m,n}}}{L} \simeq (K^*)^{c_{m,n} - |Q_1| - |Q_2|}. \quad \square$$

5. Dihedral groups

In this section we calculate $pM_{G \times G}(G)$ for any dihedral group G .

5.1. Finite dihedral group. Consider the dihedral group $D_{2m} = \langle a, b \mid a^m = b^2 = (ab)^2 = 1 \rangle, m \in \mathbb{N}$. We denote by $d_m = s(D_{2m}, D_{2m} \times D_{2m})$ the number of effective orbits of D_{2m} , which is given by (3.1).

In the proof of Theorem 5.2, we will not determine explicitly the kernel of the map ψ ; instead, we shall obtain a chain of subgroups $W \leq \ker \psi \leq R$: then

$$\frac{(K^*)^{d_m}}{\ker(\psi)} \simeq \frac{(K^*)^{d_m}/W}{\ker(\psi)/W}.$$

As to the quotients $(K^*)^{d_m}/W$ and $\ker(\psi)/W$, we construct a map Λ with domain $(K^*)^{d_m}$ and whose kernel will be W ; then Λ induces a group isomorphism $\bar{\Lambda} : (K^*)^{d_m}/W \rightarrow \text{im}(\Lambda)$. Finally, to obtain the desired quotient $(K^*)^{d_m}/\ker(\psi)$, we apply to $\bar{\Lambda}$ the following result.

LEMMA 5.1. *Let $f : H \rightarrow A$ be an isomorphism of abelian groups and $H_1 \leq H$; then the map $H/H_1 \ni hH_1 \rightarrow f(h)f(H_1) \in A/f(H_1)$ is an isomorphism.*

This argument will be used not only for $pM_{D_{2m} \times D_{2m}}(D_{2m})$ but in the calculation of other total components.

THEOREM 5.2. *We have $pM_{D_{2m} \times D_{2m}}(D_{2m}) \simeq (K^*)^{d_m - \lfloor m-1/2 \rfloor}$.*

PROOF. Let $\psi : (K^*)^{d_m} \rightarrow pM_{D_{2m} \times D_{2m}}(D_{2m})$ be the group epimorphism given by (2.11) and $x \in \ker(\psi) = L_{D_{2m} \times D_{2m}}$, where σ_x is as in Corollary 2.14(3). Then $\sigma = \sigma_x \in N_{D_{2m} \times D_{2m}}$. By Corollary 2.14(2), there is a map $\rho : D_{2m} \rightarrow K^*$ such that

$$\sigma(a^i, a^j) = \frac{\rho(a^i)\rho(a^j)}{\rho(a^{i+j})} \quad \text{and} \quad \sigma(a^k, a^l b) = \frac{\rho(a^k)\rho(a^l b)}{\rho(a^{k+l} b)},$$

for all $i, j, k, l \in \mathbb{N}$. Moreover, ρ satisfies

$$\rho(1) = \rho(a^i)\rho(a^{-i}) = \rho(a^i b)^2 = 1,$$

for all $i \in \mathbb{N}$. From the equalities above, it is easily verified that $\sigma^2(a^k, a^l b) = \rho^2(a^k)$, for every k and l . In particular, if m is even, then $\sigma^2(a^{m/2}, a^l b) = 1$, for any l . It follows from (2.15) that

$$\sigma(a, a^j) = \frac{\sigma(a, b)\sigma(a^j, ab)}{\sigma(a^{j+1}, b)},$$

for any $j = 1, \dots, \lfloor (m-1)/2 \rfloor$. Consequently, considering σ_{ij} and τ_{kl} as in (3.5) and, using Lemmas 3.3 and 3.4,

$$\ker(\psi) \subseteq R = \left\{ (\sigma_{ij}, \tau_{kl}) \in (K^*)^{d_m} \mid \begin{aligned} &\sigma_{ij} = \frac{\sigma_{1i} \cdots \sigma_{1,i+j-1}}{\sigma_{11} \cdots \sigma_{1,j-1}}, \text{ for } (i, j) \in J_1; \\ &\sigma_{1j} = \sigma_{1,m-j-1}, \text{ for } j \in J_2; \sigma_{1j} = \frac{\tau_{10}\tau_{j1}}{\tau_{1+j,0}}, \text{ for } j \in J_3; \\ &\tau_{kl}^2 = \tau_{k0}^2, \text{ for } (k, l) \in J_4; \tau_{m/2,l}^2 = 1, \text{ for } l \in J_5 \end{aligned} \right\},$$

where

$$\begin{aligned} J_1 &= \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i, j \geq 2 \text{ and } (a^i, a^j) \in S_{C_m}\}, \\ J_2 &= \{j \in \mathbb{N} \mid \lfloor (m+1)/2 \rfloor \leq j \leq m-3\}, \\ J_3 &= \{j \in \mathbb{N} \mid 1 \leq j \leq \lfloor (m-1)/2 \rfloor\}, \\ J_4 &= \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq \lfloor (m-1)/2 \rfloor; 0 \leq j \leq m-1\}, \\ J_5 &= \begin{cases} \{l \in \mathbb{N} \mid 0 \leq l \leq (m/2) - 1\} & \text{if } m \equiv 0 \pmod{2}, \\ \emptyset & \text{if } m \not\equiv 0 \pmod{2}. \end{cases} \end{aligned}$$

Notice that $\sum_{i=1}^5 |J_i| = d_m$. Now we consider the group

$$\begin{aligned} W &= \{(\sigma_{ij}, \tau_{kl}) \in R \mid \tau_{kl} = \mu_k \in K^*, \text{ for } (k, l) \in J_4, \\ &\quad \tau_{m/2,l} = \mu_{m/2}, \text{ for } l \in J_5, \text{ where } \mu_{m/2} = 1 \text{ if } m \equiv 0 \pmod{2}\}, \end{aligned}$$

which is obtained from R by removing the squares in the relations corresponding to J_4 and J_5 .

We shall show that $W \leq \ker(\psi)$. Indeed, given $(\sigma_{ij}, \tau_{kl}) \in W$, let $\lambda : G \rightarrow K^*$ be the map defined by

$$\begin{aligned} \lambda(a^i b) &= \lambda(1) = 1 \quad \text{for each } 0 \leq i \leq m - 1, \\ \lambda(a^i) &= \frac{1}{\lambda(a^{m-i})} = \mu_i \quad \text{for each } 1 \leq i \leq \lfloor (m - 1)/2 \rfloor \end{aligned}$$

and $\lambda(a^{m/2}) = 1$ if $m \equiv 0 \pmod 2$. Then, by the construction of W , it follows that

$$\sigma_{1j} = \frac{\tau_{10}\tau_{j1}}{\tau_{1+j,0}} = \frac{\mu_1\mu_j}{\mu_{1+j}} = \frac{\lambda(a)\lambda(a^j)}{\lambda(a^{1+j})}, \quad \text{for all } j \in J_3$$

and

$$\sigma_{1j} = \sigma_{1,m-j-1} = \frac{\lambda(a)\lambda(a^{m-j-1})}{\lambda(a^{m-j})} = \frac{\lambda(a)\lambda(a^j)}{\lambda(a^{j+1})}, \quad \text{for any } j \in J_2.$$

These imply that

$$\sigma_{ij} = \frac{\sigma_{1i} \cdots \sigma_{1,i+j-1}}{\sigma_{11} \cdots \sigma_{1,j-1}} = \frac{\lambda^i(a)\lambda(a^i) \lambda(a^j)}{\lambda(a^{i+j}) \lambda^j(a)} = \frac{\lambda(a^i)\lambda(a^j)}{\lambda(a^{i+j})},$$

for every $(i, j) \in J_1$, and

$$\tau_{kl} = \mu_k = \frac{\lambda(a^k)\lambda(a^l b)}{\lambda(a^{k+l} b)}, \quad \text{for each } (k, l) \in J_4 \cup (\{m/2\} \times J_5).$$

Consequently, any partial factor set $\sigma_x, x \in W$, is a coboundary, that is, $W \triangleleft \ker(\psi)$.

We will construct an epimorphism whose kernel is W . Since $\sum_{i=1}^5 |J_i| = d_m$ and $|(k, l) \in J_4, l \neq 0| = |J_4| - \lfloor (m - 1)/2 \rfloor$, we may define a homomorphism $\Lambda : (K^*)^{d_m} \rightarrow (K^*)^{d_m - \lfloor (m-1)/2 \rfloor}$ sending (σ_{ij}, τ_{kl}) to $((\sigma_{1j}\tau_{j+1,0}/\tau_{10}\tau_{j1})_{j \in J_3}, (\sigma_{1j}/\sigma_{1,m-j-1})_{j \in J_2}, (\sigma_{ij}\sigma_{11} \cdots \sigma_{1,j-1}/\sigma_{1i} \cdots \sigma_{1,i+j-1})_{(i,j) \in J_1}, (\tau_{kl}/\tau_{k0})_{(k,l) \in J_4, l \neq 0}, (\tau_{m/2,l})_{l \in J_5}$.

Then, by construction, $W = \ker(\Lambda)$, and Λ is an epimorphism, because for any $z = ((u_i)_{i \in J_3}, (v_i)_{i \in J_2}, (w_{i,j})_{(i,j) \in J_1}, (x_{i,j})_{(i,j) \in J_4, j \neq 0}, (y_i)_{i \in J_5})$ belonging to $(K^*)^{d_m - \lfloor (m-1)/2 \rfloor}$, one has $\Lambda(\sigma_{ij}, \tau_{kl}) = z$, where

$$\begin{aligned} \tau_{k0} &= 1, \quad \text{for } (k, 0) \in J_4, \\ \tau_{kl} &= x_{kl}, \quad \text{for } (k, l) \in J_4, l \neq 0, \\ \tau_{(m/2)l} &= y_l, \quad \text{for } l \in J_5, \\ \sigma_{1j} &= \frac{u_j \tau_{10} \tau_{j1}}{\tau_{j+1,0}}, \quad \text{for } j \in J_3, \\ \sigma_{1j} &= v_j \sigma_{1,m-j-1}, \quad \text{for } j \in J_2 \end{aligned}$$

and

$$\sigma_{ij} = \frac{w_{ij} \sigma_{1i} \cdots \sigma_{1,i+j-1}}{\sigma_{11} \cdots \sigma_{1,j-1}}, \quad \text{for } (i, j) \in J_1.$$

On the other hand,

$$\Lambda(x) = (1, \dots, 1, \varepsilon_1, \dots, \varepsilon_t), \quad \text{for any } x \in \ker(\psi) \subseteq R, \tag{5.1}$$

where the number of ones in (5.1) is $|J_3| + |J_2| + |J_1|$ and $\varepsilon_i^2 = 1$, for every $1 \leq i \leq t = |J_4| - \lfloor (m-1)/2 \rfloor + |J_5|$. In other words, $\Lambda(\ker(\psi)) \triangleleft C_2^{d_m - \lfloor (m-1)/2 \rfloor}$. We get that $pM_{D_{2m} \times D_{2m}}(D_{2m})$ is isomorphic to

$$\frac{(K^*)^{d_m}}{\ker(\psi)} \simeq \frac{\frac{(K^*)^{d_m}}{\ker(\Lambda)}}{\frac{\ker(\psi)}{\ker(\Lambda)}} \simeq \frac{(K^*)^{d_m - \lfloor (m-1)/2 \rfloor}}{\Lambda(\ker(\psi))} \simeq (K^*)^{d_m - \lfloor (m-1)/2 \rfloor},$$

where the last isomorphism follows from (3.2) and (5.1). □

5.2. Infinite dihedral group. Now we proceed with the calculation of the total component of the partial Schur multiplier, for $D_\infty = \langle a, b \mid b^2 = (ab)^2 = 1 \rangle$. First we give a result that can be proved similarly to Lemma 3.3.

LEMMA 5.3. *Any element of $pm'_{D_\infty \times D_\infty}(D_\infty)$ is uniquely determined by its values on pairs in*

$$\{(a^i, a^j) \mid (i, j) \in \mathbb{N} \times \mathbb{N}\} \cup \{(a^k, a^l b) \mid (k, l) \in \mathbb{N} \times \mathbb{Z}\}.$$

Moreover, these values can be chosen arbitrarily in K^* .

THEOREM 5.4. *We have $pm_{D_\infty \times D_\infty}(D_\infty) \simeq (K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z}^*)}$.*

PROOF. By Lemma 5.3, the effective orbits of D_∞ can be indexed by the set $(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z})$. So, we may consider $\psi : (K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z})} \rightarrow pm_{D_\infty \times D_\infty}(D_\infty)$ as the epimorphism of groups given by (2.11) and take $x \in \ker(\psi) = L_{D_\infty \times D_\infty}$. Then $\sigma = \sigma_x \in N_{D_\infty \times D_\infty}$ and there is a map $\rho : D_\infty \rightarrow K^*$ such that (2.9) and (2.10) are satisfied. As above, it is easily verified that $\sigma^2(a^k, a^l b) = \rho^2(a^k)$, for every k and l .

For $i, j \geq 1$, we get from (2.15) that

$$\sigma(a^i, a^j) = \frac{\sigma(a^i, b)\sigma(a^j, a^i b)}{\sigma(a^{i+j}, b)}.$$

Given $j \in \mathbb{Z}$ and $x, y \in D_\infty$, set

$$\sigma_j(x, y) = \begin{cases} \sigma(x, y)\sigma(x, xy) \dots \sigma(x, x^{j-1}y) & \text{if } j > 0, \\ 1 & \text{if } j = 0, \\ \sigma(x, x^{-1}y)\sigma(x, x^{-2}y) \dots \sigma(x, x^jy) & \text{if } j < 0. \end{cases} \tag{5.2}$$

In particular, for $x = a$ and $y = b$,

$$\sigma_j(a, b) = \frac{\rho^{|j|}(a)\rho(b)}{\rho(a^j b)},$$

for any $j \in \mathbb{Z}$.

Using (5.2), it is also seen that for any $i, j \in \mathbb{Z}$,

$$\frac{\sigma(a^i, a^j b)}{\sigma_{i+j}(a, b)\sigma(a^i, b)} = \frac{\rho(a^i b)\rho(a^j b)}{\rho^{i+j}(a)} = \begin{cases} \sigma_i^{-1}(a, b)\sigma_j^{-1}(a, b) & \text{if } i, j > 0, \\ \sigma_i^{-1}(a, b)\sigma_j(a, b) & \text{if } 0 < -j \leq i, \\ \sigma_i(a, b)\sigma_j^{-1}(a, b) & \text{if } 0 < i \leq -j. \end{cases}$$

Consequently, the values of σ are determined by its values on

$$\{(a^k, b) \mid k \in \mathbb{N}\} \cup \{(a, a^l b) \mid l \in \mathbb{Z}\}.$$

Therefore, using Lemma 5.3 and considering σ_{ij} and τ_{kl} as in (3.5), and π_i as in (2.12), we conclude that

$$\begin{aligned} \ker(\psi) \subseteq R = & \left\{ (\sigma_{ij}, \tau_{kl}) \in (K^*)^{\mathbb{N} \times \mathbb{N}} \times (K^*)^{\mathbb{N} \times \mathbb{Z}} \mid \sigma_{ij} = \frac{\pi_{i+j}}{\pi_i \pi_j}, \text{ for } i, j \geq 2; \right. \\ & \sigma_{i1} = \sigma_{1i}, \text{ for } i \geq 2; \sigma_{1j} = \frac{\tau_{10} \tau_{j1}}{\tau_{1+j,0}}, \text{ for } j \in \mathbb{N}; \\ & \left. \tau_{kl}^2 = \tau_{k0}^2, \text{ for } (k, l) \in \mathbb{N} \times \mathbb{Z} \right\}. \end{aligned}$$

Consider the group

$$W = \{(\sigma_{ij}, \tau_{kl}) \in R \mid \tau_{kl} = \mu_k, \text{ for } (k, l) \in \mathbb{N} \times \mathbb{Z} \text{ and some } \mu_k \in K^*\}.$$

Given $(\sigma_{ij}, \tau_{kl}) \in W$, let $\lambda : G \rightarrow K^*$ be the map defined by

$$\begin{aligned} \lambda(a^i b) &= \lambda(1) = 1, \quad \text{for } i \in \mathbb{Z}, \\ \lambda(a^i) &= \frac{1}{\lambda(a^{m-i})} = \mu_i, \quad \text{for } i \in \mathbb{N}. \end{aligned}$$

Then it follows from the definition of R and W that $\sigma_{1j} = \tau_{10} \tau_{j1} / \tau_{1+j,0} = \mu_1 \mu_j / \mu_{1+j} = \lambda(a)\lambda(a^j) / \lambda(a^{1+j})$, for $j \in \mathbb{N}$, and $\sigma_{i1} = \sigma_{1i} = \lambda(a)\lambda(a^i) / \lambda(a^{1+i})$, for $i \geq 2$. The latter equality implies that

$$\sigma_{ij} \frac{\sigma_{1i} \dots \sigma_{1,i+j-1}}{\sigma_{11} \dots \sigma_{1,j-1}} = \frac{\lambda^j(a)\lambda(a^i)\lambda(a^j)}{\lambda(a^{i+j})\lambda^j(a)} = \frac{\lambda(a^i)\lambda(a^j)}{\lambda(a^{i+j})},$$

for every $i, j \geq 2$ and

$$\tau_{kl} = \mu_k = \frac{\lambda(a^k)\lambda(a^l b)}{\lambda(a^{k+l b})}, \quad \text{for each } (k, l) \in \mathbb{N} \times \mathbb{Z}.$$

We conclude that for every $x \in W$, the partial factor set σ_x is a coboundary. Consequently, $W \triangleleft \ker(\psi)$.

As in the proof of Theorem 5.2, we will construct an epimorphism of groups whose kernel is W . Let $\Lambda : (K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z})} \rightarrow (K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z}^*)}$ be the homomorphism of groups defined by mapping (σ_{ij}, τ_{kl}) to

$$\begin{aligned} & \left(\left(\frac{\sigma_{i1}}{\sigma_{1i}} \right)_{i \in \mathbb{N} \setminus \{1\}}, \left(\frac{\sigma_{ij}\sigma_{11} \dots \sigma_{1,j-1}}{\sigma_{1i} \dots \sigma_{1,i+j-1}} \right)_{i, j \in \mathbb{N} \setminus \{1\}}, \right. \\ & \left. \left(\frac{\sigma_{1j}\tau_{j+1,0}}{\tau_{10}\tau_{j1}} \right)_{j \in \mathbb{N}}, \left(\frac{\tau_{kl}}{\tau_{k0}} \right)_{(k,l) \in \mathbb{N} \times \mathbb{Z}^*} \right). \end{aligned}$$

Taking any $z = ((u_{i1})_{i \in \mathbb{N} \setminus \{1\}}, (v_{ij})_{i,j \in \mathbb{N} \setminus \{1\}}, (w_{1j})_{j \in \mathbb{N}}, (x_{k,l})_{(k,l) \in \mathbb{N} \times \mathbb{Z}^*})$ in the group $(K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z}^*)}$, one gets $\Lambda(\sigma_{ij}, \tau_{kl}) = z$, where

$$\begin{aligned} \tau_{k0} &= 1, & \text{for } (k, 0) \in \mathbb{N} \times \{0\}, \\ \tau_{kl} &= x_{kl}, & \text{for } (k, l) \in \mathbb{N} \times \mathbb{Z}^*, \\ \sigma_{1j} &= \frac{w_{1j} \tau_{10} \tau_{j1}}{\tau_{j+1,0}}, & \text{for } (1, j) \in \{1\} \times \mathbb{N}, \\ \sigma_{i1} &= \sigma_{1,i} u_{i1}, & \text{for } (i, 1) \in \mathbb{N} \times \{1\} \end{aligned}$$

and

$$\sigma_{ij} = \frac{v_{ij} \sigma_{1i} \cdots \sigma_{1,i+j-1}}{\sigma_{11} \cdots \sigma_{1,j-1}}, \quad \text{for } (i, j) \in (\mathbb{N} \setminus \{1\}) \times (\mathbb{N} \setminus \{1\}).$$

Then Λ is an epimorphism. Moreover, it is immediate that $W = \ker(\Lambda)$ and, given $\mu \in \ker(\psi)$, it follows that

$$\Lambda(\mu) = ((1_{ij})_{(i,j) \in \mathbb{N} \times \mathbb{N}}, (\varepsilon_{kl})_{(k,l) \in \mathbb{N} \times \mathbb{Z}^*}),$$

where $1_{ij} = 1 = \varepsilon_{kl}^2$, for every $(i, j) \in \mathbb{N} \times \mathbb{N}$ and $(k, l) \in \mathbb{N} \times \mathbb{Z}^*$. Hence, $\Lambda(\ker(\psi)) \triangleleft C_2^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z}^*)}$.

Therefore,

$$\begin{aligned} pM_{D_\infty \times D_\infty}(D_\infty) &= \frac{(K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z})}}{\ker(\psi)} \simeq \frac{(K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z})}}{\frac{\ker(\Lambda)}{\ker(\Lambda)}} \\ &\simeq \frac{(K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z}^*)}}{\Lambda(\ker(\psi))} \simeq (K^*)^{(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{Z}^*)}, \end{aligned}$$

where the last isomorphism follows from (3.2). □

6. Dicyclic groups

Take $m \in \mathbb{N}$, $m > 0$ and let $\text{Dic}_m = \langle a, b \mid a^{2m} = 1, b^2 = a^m, b^{-1}ab = a^{-1} \rangle$. Our aim in this section is to determine $pM_{\text{Dic}_m \times \text{Dic}_m}(\text{Dic}_m)$.

Notice that every element of Dic_m can be written in the form $a^i b^l$, where $0 \leq i \leq 2m - 1$ and $l \in \{0, 1\}$. Moreover,

$$(a^k b)^{-1} = a^{k+m} b \quad \text{and} \quad (a^k)^{-1} = a^{2m-k}. \tag{6.1}$$

In the next result, $S_{C_{2m}}$ denotes the full set of representatives of the effective orbits of C_{2m} defined in (3.3).

LEMMA 6.1. *The set S_{Dic_m} defined by*

$$\begin{aligned} S_{C_{2m}} \cup \{ &(a^i, a^k b) \mid 1 \leq i \leq m - 1, 0 \leq k \leq 2m - 1 \} \\ &\cup \{ (a^m, a^k b) \mid 0 \leq k \leq m - 1 \} \end{aligned}$$

contains exactly one representative of each effective orbit of S_3 on $\text{Dic}_m \times \text{Dic}_m$.

PROOF. By (2.4), we have $S_3(x, a^k) = S_3(a^{2m-k}, x^{-1})$; then every orbit $S_3(x, a^k)$, where $x \in \text{Dic}_m$, is represented by a pair whose first coordinate is a power of a . Moreover, since $ba = a^{-1}b$,

$$a^i ba^k b = a^{i-1} ba^{k-1} b = a^{i-2} ba^{k-2} b = \dots = a^{i-k} b^2 = a^{m+i-k}$$

and, consequently, $S_3(a^i b, a^k b) = S_3(a^i ba^k b, (a^k b)^{-1}) = S_3(a^{m+i-k}, a^{k+m} b)$, thanks to (2.4).

Since the set $S_{C_{2m}}$ from (3.3) contains precisely one element of each effective S_3 -orbit in $C_{2m} \times C_{2m}$, then, to find the effective orbits, it remains to consider the pairs $(a^i, a^k b)$. But

$$S_3(a^i, a^k b) = S_3((a^i)^{-1}, a^i a^k b) = S_3(a^{2m-i}, a^{i+k} b),$$

and we can assume that $i \leq 2m - i$, that is, $i \leq m$.

Finally, for $i = m$, one gets $S_3(a^m, a^k b) = S_3(a^m, a^{m+k} b)$, and we can choose $k \leq m - 1$. Indeed, $a^{m+k} b = a^{m+k-2m} b = a^{k-m} b$ and, if $m \leq k \leq 2m - 1$, then $0 \leq m - k \leq m - 1$. This completes the proof. \square

PROPOSITION 6.2. *Let $G = \text{Dic}_m$, for some natural number $m \geq 2$. If $\sigma \in pm'_{G \times G}(G)$ and $\sigma \sim 1$, then σ is uniquely determined by its values on the pairs*

$$(a, a^k b), \quad \text{where } 0 \leq k \leq m \text{ and} \tag{6.2}$$

$$(a^i, b), \quad \text{where } 2 \leq i \leq m - 1, \tag{6.3}$$

which can be chosen in K^* arbitrarily, and also by its value on (a^m, b) , which must satisfy

$$\sigma^2(a^m, b) = \frac{(\sigma_m(a, b))^2}{(\sigma(a, b)\sigma(a, a^m b))^m}. \tag{6.4}$$

PROOF. Let $\sigma \in pm'_{G \times G}(G)$ and suppose that $\sigma \sim 1$. By (2.15),

$$\sigma(a^i, a^k) = \frac{\sigma(a^i, b)\sigma(a^k, a^i b)}{\sigma(a^{i+k}, b)},$$

for any i, k . Thus, using Lemma 6.1, it is clear that each value of σ on the pairs (a^i, a^k) from $S_{C_{2m}}$ is determined by its value on the set $B = S_{\text{Dic}_m} \setminus S_{C_{2m}}$ given by

$$\{(a^i, a^k b) \mid 1 \leq i \leq m - 1, 0 \leq k \leq 2m - 1\} \cup \{(a^m, a^k b) \mid 0 \leq k \leq m - 1\}. \tag{6.5}$$

By (2.14), the values of σ on those pairs for which $i \geq 2$ and $k \geq 1$ are determined by the values $\sigma(a, a^k b)$ and $\sigma(a^i, b)$, where $0 \leq k \leq 2m - 1$ and $2 \leq i \leq m$. Indeed,

$$\sigma(a^i, a^k b) = \frac{\sigma_{i+k}(a, b)\sigma(a^i, b)}{\sigma_i(a, b)\sigma_k(a, b)} = \frac{\sigma(a, a^i b) \dots \sigma(a, a^{i+k-1} b)\sigma(a^i, b)}{\sigma(a, b) \dots \sigma(a, a^{k-1} b)}.$$

Using (6.1) and the fact that σ is a coboundary,

$$\sigma(a, a^k b) = \frac{\sigma(a, b)\sigma(a, a^m b)}{\sigma(a, a^{k-m} b)}, \quad \text{where } 1 \leq k - m \leq m - 1$$

or, equivalently, for $m + 1 \leq k \leq 2m - 1$. Hence, the values of $\sigma(a, a^k b)$, for $k = m + 1, \dots, 2m - 1$, are determined by the values $\sigma(a, a^k b)$ for which $1 \leq k \leq m$. Consequently, the values of σ on elements of B are determined by those on pairs from (6.2) and (6.3).

Moreover, let $\sigma = \partial\rho$, for some $\rho : G \rightarrow K^*$ verifying (2.9) and (2.10). Then

$$\sigma(a, a^k b)\sigma(a, a^{k+m} b) = \frac{\rho(a)\rho(a^k b)}{\rho(a^{k+1} b)} \frac{\rho(a)\rho(a^{k+m} b)}{\rho(a^{k+m+1} b)} = \rho^2(a). \tag{6.6}$$

Further, (6.4) follows from

$$\begin{aligned} \sigma^2(a^m, b)(\sigma(a, b)\sigma(a, a^m b))^m &\stackrel{(6.6)}{=} \left(\frac{\rho(a^m)\rho(b)}{\rho(a^m b)}\right)^2 (\rho^2(a))^m \\ &= \left(\frac{\rho^m(a)\rho(b)}{\rho(a^m b)}\right)^2 = (\sigma_m(a, b))^2. \end{aligned}$$

Now fix a family $(v(x, y))_{(x,y) \in I}$ of elements of K^* , indexed by the list of pairs given by (6.2) and (6.3). Write

$$v_j(a, b) = v(a, b)v(a, ab) \dots v(a, a^{j-1} b), \tag{6.7}$$

for each $1 \leq j \leq m + 1$. Suppose that these values satisfy a condition analogous to (6.4), that is,

$$v^2(a^m, b) = \frac{(v_m(a, b))^2}{(v(a, b)v(a, a^m b))^m}. \tag{6.8}$$

Our intent is to show that these values determine a partial factor set $\sigma \sim 1$. By Corollary 2.14(2), if we construct a map $\rho : G \rightarrow K^*$ satisfying (2.9) and (2.10) such that $\sigma(x, y) = \rho(x)\rho(y)\rho(xy)^{-1}$, for any of the pairs (x, y) listed above, then this equality will define a partial factor set $\sigma \in pm'_{G \times G}(G)$ such that $\sigma \sim 1$.

Motivated by (6.6), we start by defining

$$\rho(a) = \frac{1}{\rho(a^{2m-1})} = \omega_1,$$

where ω_1 satisfies $\omega_1^2 = v(a, b)v(a, a^m b)$. Define also

$$\rho(1) = 1 \quad \text{and} \quad \rho(a^m) = \frac{\omega_1^m v(a^m, b)}{v_m(a, b)};$$

by (6.8), this implies that $\rho^2(a^m) = (\omega_1^{2m} v(a^m, b)^2 / v_m(a, b)^2) = 1$. Since we want the equality $v(a^m, b) = (\rho(a^m)\rho(b) / \rho(a^m b)) = \rho(a^m)\rho^2(b)$, define

$$\rho(b) = \frac{1}{\rho(a^m b)} = \omega_2,$$

where ω_2 satisfies

$$\omega_2^2 = \frac{v(a^m, b)}{\rho(a^m)} = \frac{v(a^m, b)v_m(a, b)}{\omega_1^m v(a^m, b)} = \frac{v_m(a, b)}{\omega_1^m}.$$

Furthermore, set

$$\rho(a^j b) \stackrel{(6.1)}{=} \frac{1}{\rho(a^{m+j}b)} = \frac{\rho^j(a)\rho(b)}{\nu_j(a,b)} = \frac{\omega_1^j \omega_2}{\nu_j(a,b)},$$

for each $j = 1, \dots, m - 1$, and

$$\rho(a^j) \stackrel{(6.1)}{=} \frac{1}{\rho(a^{2m-j})} = \frac{\nu(a^j, b)\rho(a^j b)}{\rho(b)} = \frac{\nu(a^j, b)\omega_1^j \omega_2}{\omega_2 \nu_j(a,b)} = \frac{\nu(a^j, b)\omega_1^j}{\nu_j(a,b)},$$

for each $j = 1, \dots, m - 1$ (notice that the same formula is also valid for $j = m$).

With these definitions, it follows that

$$\begin{aligned} \frac{\rho(a)\rho(b)}{\rho(ab)} &= \omega_1 \omega_2 \frac{\nu(a,b)}{\omega_1 \omega_2} = \nu(a,b), \\ \frac{\rho(a)\rho(a^m b)}{\rho(a^{m+1}b)} &= \frac{\omega_1}{\omega_2} \frac{\omega_1 \omega_2}{\nu(a,b)} = \frac{\omega_1^2}{\nu(a,b)} = \nu(a, a^m b) \end{aligned}$$

and, for any $k = 1, \dots, m - 1$,

$$\frac{\rho(a)\rho(a^k b)}{\rho(a^{k+1}b)} = \omega_1 \frac{\omega_1^k \omega_2}{\nu_k(a,b)} \frac{\nu_{k+1}(a,b)}{\omega_1^{k+1} \omega_2} = \frac{\nu_{k+1}(a,b)}{\nu_k(a,b)} \stackrel{(6.7)}{=} \nu(a, a^k b).$$

In addition,

$$\frac{\rho(a^m)\rho(b)}{\rho(a^m b)} = \frac{\omega_1^m \nu(a^m, b)}{\nu_m(a,b)} \omega_2^2 = \frac{\omega_1^m \nu(a^m, b)}{\nu_m(a,b)} \frac{\nu_m(a,b)}{\omega_1^m} = \nu(a^m, b)$$

and, finally, for any $2 \leq i \leq m - 1$,

$$\frac{\rho(a^i)\rho(b)}{\rho(a^i b)} = \frac{\nu(a^i, b)\omega_1^i}{\nu_i(a,b)} \omega_2 \frac{\nu_i(a,b)}{\omega_1^i \omega_2} = \nu(a^i, b).$$

Thus, we have obtained a map $\rho : G \rightarrow K^*$ with the desired properties. □

Let $dc_m = |S_{\text{Dic}_m}|$ be the number of effective S_3 -orbits of Dic_m , which, by Lemma 6.1, is given by the formula

$$dc_m = \begin{cases} \frac{(4m-1)(4m-2)+4}{6} & \text{if } m \equiv 0 \pmod{3}, \\ \frac{(4m-1)(4m-2)}{6} & \text{if } m \not\equiv 0 \pmod{3}. \end{cases}$$

Notice that $dc_m = |S_{C_{2m}}| + |B|$, where $S_{C_{2m}}$ is given by (3.3) and B is as in (6.5).

THEOREM 6.3. *If $G = \text{Dic}_m$, then $pM_{G \times G}(G) \simeq (K^*)^{dc_m - 2m + 1}$.*

PROOF. Considering σ_{ij} and τ_{kl} as in (3.5), write $(K^*)^{dc_m} = (K^*)^{|S_{C_{2m}}|} \times (K^*)^{|B|}$. By Proposition 6.2, the subgroup $L = L_{G \times G} = \{x \in (K^*)^{dc_m} \mid \sigma_x \sim 1\}$ of $(K^*)^{dc_m}$ is given by

$$L = \left\{ (\sigma_{ij}, \tau_{kl}) \in (K^*)^{|S_{C_{2m}}|} \times (K^*)^{|B|} \mid \begin{aligned} &\sigma_{ij} = \frac{\sigma_{1i} \cdots \sigma_{1,i+j-1}}{\sigma_{11} \cdots \sigma_{1,j-1}}, \text{ for } (i, j) \in J_1; \\ &\sigma_{1j} = \sigma_{1,2m-j-1}, \text{ for } j \in J_2; \sigma_{1j} = \frac{\tau_{10}\tau_{j1}}{\tau_{1+j,0}}, \text{ for } j \in J_3; \\ &\tau_{i,k} = \frac{(\tau_{1,i} \cdots \tau_{1,i+k-1})\tau_{i,0}}{\tau_{1,0} \cdots \tau_{1,k-1}} \text{ for } (i, k) \in J_4; \\ &\tau_{1,k}\tau_{1,m+k} = \tau_{1,0}\tau_{1,m}, \text{ for } k \in J_5; \\ &\tau_{m,0}^2 = \frac{(\tau_{1,0}\tau_{1,1} \cdots \tau_{1,m-1})^2}{(\tau_{1,0}\tau_{1,m})^m} \end{aligned} \right\},$$

where

$$\begin{aligned} J_1 &= \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i, j \geq 2 \text{ and } (a^i, a^j) \in S_{C_{2m}}\}, \\ J_2 &= \{j \in \mathbb{N} \mid m \leq j \leq 2m - 3\}, \\ J_3 &= \{j \in \mathbb{N} \mid 1 \leq j \leq \lfloor (2m - 1)/2 \rfloor = m - 1\}, \\ J_4 &= \{(i, k) \in \mathbb{N} \times \mathbb{N} \mid (a^i, a^k b) \in B, 2 \leq i \text{ and } 1 \leq k\}, \\ J_5 &= \{k \in \mathbb{N} \mid 1 \leq k \leq m - 1\}. \end{aligned}$$

Let Q be the set of pairs given by (6.2) and (6.3). Then $|Q| = 2m - 1$, $Q \subseteq B$ and $|B \setminus Q| = |J_4| + |J_5| + 1$. Write $(K^*)^{dc_m - 2m + 1} = (K^*)^{|S_{C_{2m}}|} \times (K^*)^{|B \setminus Q|}$ and consider the group homomorphism $\Lambda : (K^*)^{dc_m} \rightarrow (K^*)^{dc_m - 2m + 1}$ which maps (σ_{ij}, τ_{kl}) to

$$\left(\left(\frac{\sigma_{ij}\sigma_{11} \cdots \sigma_{1,j-1}}{\sigma_{1i} \cdots \sigma_{1,i+j-1}} \right)_{(i,j) \in J_1}, \left(\frac{\sigma_{1j}}{\sigma_{1,m-j-1}} \right)_{j \in J_2}, \left(\frac{\sigma_{1j}\tau_{j+1,0}}{\tau_{10}\tau_{j1}} \right)_{j \in J_3}, \right. \\ \left. \left(\frac{(\tau_{1,i} \cdots \tau_{1,i+k-1})\tau_{i,0}}{(\tau_{1,0} \cdots \tau_{1,k-1})\tau_{i,k}} \right)_{(i,k) \in J_4}, \left(\frac{\tau_{1,0}\tau_{1,m}}{\tau_{1,k}\tau_{1,m+k}} \right)_{k \in J_5}, \left(\frac{(\tau_{1,0}\tau_{1,1} \cdots \tau_{1,m-1})^2}{\tau_{m,0}^2(\tau_{1,0}\tau_{1,m})^m} \right) \right).$$

Then Λ is an epimorphism whose kernel is L . Consequently, $pM_{G \times G}(G) \simeq (K^*)^{dc_m} / L \simeq (K^*)^{dc_m - 2m + 1}$. □

7. The total component $pM_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{Z})$

For maps $\sigma : \mathbb{Z} \times \mathbb{Z} \rightarrow K$ and $\rho : \mathbb{Z} \rightarrow K^*$, denote $\sigma_{ij} = \sigma(i, j)$ and $\rho_i = \rho(i)$, for every $i, j \in \mathbb{Z}$. To determine $pM_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{Z})$, we need a couple of facts.

LEMMA 7.1. *Any element of $pm'_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{Z})$ is uniquely determined by its values on pairs $(i, j) \in \mathbb{N} \times \mathbb{N}$. Moreover, these values can be chosen arbitrarily in K^* .*

PROOF. For any $i, j \in \mathbb{Z}$,

$$S_3(i, j) = \{(i, j), (-i - j, i), (j, -i - j), (-j, -i), (-i, i + j), (i + j, -j)\}.$$

Let $S_3(i, j)$ be an effective orbit, that is, $0 \notin \{i, j, i + j\}$. If $(i, j) \in (\mathbb{Z} \times \mathbb{Z}) \setminus (\mathbb{N} \times \mathbb{N})$, we shall prove that there is exactly one $(m, n) \in (\mathbb{N} \times \mathbb{N}) \cap S_3(i, j)$. There are a few cases to consider.

- If $i, j < 0$, then $(-j, -i) \in \mathbb{N} \times \mathbb{N}$.
- If $i < 0 < j$ and $i + j > 0$, then $(-i, i + j) \in \mathbb{N} \times \mathbb{N}$. On the other hand, if $i + j < 0$, then $(j, -i - j) \in \mathbb{N} \times \mathbb{N}$.
- If $j < 0 < i$ and $i + j > 0$, then $(i + j, -j) \in \mathbb{N} \times \mathbb{N}$. Finally, if $i + j < 0$, then $(-i - j, i) \in \mathbb{N} \times \mathbb{N}$.

When $(i, j) \in \mathbb{N} \times \mathbb{N}$, it is clear that the other elements of its S_3 -orbit are not in $\mathbb{N} \times \mathbb{N}$. □

PROPOSITION 7.2. *If $\sigma \in pm'_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{Z})$ and $\sigma \sim 1$, then σ is uniquely determined by its values on $\{1\} \times \mathbb{N}$, which can be chosen arbitrarily in K^* .*

PROOF. Let $\sigma \in pm'_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{Z})$ and suppose that $\sigma \sim 1$. This means that there is a map $\rho : \mathbb{Z} \rightarrow K^*$ such that $\sigma_{ij} = \rho_i \rho_j / \rho_{i+j}$, for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, and $\rho_0 = \rho_i \rho_{-i} = 1$, for all $i \in \mathbb{Z}$. Observe that $\sigma_{ij} = \sigma_{ji}$ and, setting $\pi_j = \sigma_{11} \sigma_{12} \dots \sigma_{1,j-1}$, for each $j \geq 2$,

$$\pi_j = \frac{\rho_1^j}{\rho_j} \tag{7.1}$$

and, in addition, $\sigma_{ij} = \pi_{i+j} / \pi_i \pi_j$, for every $i, j \geq 2$. Therefore, σ is completely determined by its values $\sigma_{1n}, n \in \mathbb{N}$, which, by Lemma 7.1, correspond to pairs belonging to distinct S_3 -orbits.

On the other hand, fixing $(v_{1j})_{j \in \mathbb{N}}$ arbitrarily in $(K^*)^{\mathbb{N}}$, let us show that these values determine a partial factor set $\sigma \sim 1$. By Corollary 2.14(2), it is enough to construct a map $\rho : \mathbb{Z} \rightarrow K^*$ satisfying (2.9) and such that

$$v_{1j} = \frac{\rho_1 \rho_j}{\rho_{j+1}}, \tag{7.2}$$

for any $j \in \mathbb{N}$. For $j \geq 2$, write $\pi_j = v_{11} \dots v_{1,j-1}$. In order to get (7.1), for all $j \geq 2$, define $\rho_1 = 1$ and $\rho_j = 1 / (\rho_{-j}) = 1 / \pi_j$. Hence, we have constructed a map $\rho : \mathbb{Z} \rightarrow K^*$ satisfying (2.9) and (7.2). Indeed, $\sigma_{11} = \pi_2 = \rho_1^2 / \rho_2$ and, for every $j \geq 2$,

$$\sigma_{1j} = \frac{\pi_{j+1}}{\pi_j} = \frac{\frac{1}{\rho_{j+1}}}{\frac{1}{\rho_j}} = \frac{\rho_j}{\rho_{j+1}} = \frac{\rho_1 \rho_j}{\rho_{j+1}}. \tag{7.2} \quad \square$$

THEOREM 7.3. *We have $pm_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{Z}) \simeq (K^*)^{\mathbb{N}}$.*

PROOF. Using Lemma 7.1 and Proposition 7.2, we get $pm_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{Z}) \simeq (K^*)^{\mathbb{N} \times \mathbb{N}} / (K^*)^{\{1\} \times \mathbb{N}} \simeq (K^*)^{\mathbb{N}}$. □

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