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ON COABSOLUTE PARACOMPACT SPACES

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Abstract

We are interested in determining whether two spaces are coabsolute by comparing their Boolean algebras of regular closed sets. It is known that when the spaces are compact Hausdorff they are coabsolute precisely when the Boolean algebras of regular closed sets are isomorphic; but in general this condition is not strong enough to insure that the spaces be coabsolute. In this paper we show that for paracompact Hausdorff spaces, the spaces are coabsolute when the Boolean algebra isomorphism and its inverse 'preserve' local finiteness, and for locally compact paracompact Hausdorff spaces, the spaces are coabsolute when the collections of compact regular closed subsets are 'isomorphic'.

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The concept of the absolute of a space has been developed using different approaches by Gleason (1958), Iliadis and Fomin (1966) and Ponomarev (1962) to mention a few. It is well known that every regular space has an absolute which is unique (up to homeomorphism); but in general this is a many-to-one relation. Two spaces are said to be coabsolute if their absolutes are homeomorphic. If two spaces are coabsolute, then their families of regular closed subsets are isomorphic Boolean algebras. In the case that the spaces are compact, this is also a sufficient condition for the spaces to be coabsolute since then the absolute is the Stone space of the Boolean algebra of regular closed sets. In general, however, this is not a sufficient condition for spaces to be coabsolute; for example Q and P are not coabsolute where Q is the rationals and P is the irrationals although the regular closed subsets of Q and P are isomorphic Boolean algebras. We are interested in determining necessary and sufficient conditions for spaces to be coabsolute.

In the first section of this paper we give necessary definitions and background material. In the second section we develop a condition which is equivalent to the condition that the Boolean algebras of the regular closed subsets of two spaces be isomorphic. In the third section we determine restrictions on the Boolean algebra isomorphism between the regular closed subsets which yield a necessary and sufficient condition for spaces to be coabsolute when the spaces in question are (1) paracompact and (2) locally compact paracompact.

1. Preliminaries

The notation and terminology of Gillman and Jerison (1960) and Walker (1974) will be used. In particular, βX will denote the Stone–Čech compactification of a completely regular space X, and $S(U)$ will denote the Stone space of U where U is a complete Boolean algebra. All spaces are regular Hausdorff and all maps are continuous.

Recall that a continuous map f from X onto Y is *closed irreducible* if the image under f of each proper closed subset of X is a proper closed subset of Y. A space is *extremally disconnected* if the closure of every open set is open. For each space X there is a unique (up to homeomorphism) extremally disconnected space, EX, that can be mapped onto X by a closed irreducible perfect continuous map. EX is called the *absolute of* X. If there is a closed irreducible perfect map of X onto Y, then X and Y are coabsolute.

For a space X, $R(X)$ will denote the set of regular closed subsets of X where $R \subseteq X$ is regular closed if $\mathrm{cl}_X \mathrm{int}_X R = R$. $\mathrm{R}(X)$ is a complete Boolean algebra under the following operations: Let $A, B, A_\alpha \in R(X)$.

(i) $A \leqslant B$ if and only if $A \subseteq B$;
(ii) $\bigvee_\alpha A_\alpha = \mathrm{cl}_X \cup_\alpha \mathrm{int}_X A_\alpha = \mathrm{cl}_X \cup_\alpha A_\alpha$;
(iii) $\bigwedge_\alpha A_\alpha = \mathrm{cl}_X \mathrm{int}_X \cap_\alpha A_\alpha$;
(iv) $A' = \mathrm{cl}_X(X \backslash A)$.

The points of $S(R(X))$ will be identified with the regular closed Boolean algebra ultrafilters and the clopen subsets of $S(R(X))$ will be denoted by

$$\lambda_X(R) = \{\alpha \in S(R(X)) \colon R \in \alpha\},$$

where $R \in R(X)$. If X is completely regular then $k_{\beta X} \colon S(R(X)) \to \beta X$ defined by

$$k_{\beta X}(\alpha) = \bigcap_{R \in \alpha} \mathrm{cl}_{\beta X} R$$

is a well-defined, closed irreducible, continuous map with $k_{\beta X}^{\leftarrow}(X)$ dense in the extremally disconnected space $S(R(X))$. Thus *we will identify* EX *with* $k_{\beta X}^{\leftarrow}(X)$ and so $\beta EX = S(R(X))$. If X is compact, then $EX = S(R(X))$. If $x \in X$, let

$\omega_X(x) = k_{\beta X}^{\angle}(x) = \{\alpha \in S(R(X)): R \in \alpha \text{ whenever } x \in \text{int}_X R\}$. The preceding material may be found in Gates (1977), Woods (1971) and Walker (1974).

If X and Y are coabsolute completely regular spaces, then there is a homeomorphism $f: EX \to EY$; the Stone extension $f^\beta: S(R(X)) \to S(R(Y))$ is a homeomorphism and induces a Boolean algebra isomorphism $h: R(X) \to R(Y)$ by $f^\beta(\lambda_X(R)) = \lambda_Y(h(R))$ for $R \in R(X)$.

2. Condition equivalent to $R(X) \approx R(Y)$

In this section it is not necessary that the spaces be regular. A π-basis of a space X is a collection $\mathscr{B}$ of non-empty open sets such that for every non-empty open set $U \subseteq X$, there exists $B \in \mathscr{B}$ such that $B \subseteq U$. We will need the following property of a π-basis.

LEMMA 2.1. *Let X be a space and let $\mathscr{B}$ be a π-basis of X. Then if $R \subseteq X$, $\text{cl}_X \cup \{B \in \mathscr{B}: B \subseteq R\} = \text{cl}_X \text{int}_X R$. So if $R \in R(X)$, $\text{cl}_X \cup \{B \in \mathscr{B}: B \subseteq R\} = R$.*

The proof of Lemma 2.1 is straightforward.

Let X and Y be spaces with π-bases $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively and suppose k is a mapping from $\mathscr{B}_1$ onto $\mathscr{B}_2$ with the property (*) whenever $A, B \in \mathscr{B}_1$, then $A \cap B = \varnothing$ if and only if $k(A) \cap k(B) = \varnothing$. Some properties of k are given in the next lemma.

LEMMA 2.2. *Let $A, B, C, D \in \mathscr{B}_1$.*

(i) $A \subseteq B$ implies $k(A) \subseteq \text{cl}_Y k(B)$ and $k(C) \subseteq k(D)$ implies $C \subseteq \text{cl}_X D$.

(ii) Let $S \in R(Y)$ and let $R = \text{cl}_X \cup \{B \in \mathscr{B}_1: k(B) \subseteq S\}$. Then if $B \in \mathscr{B}_1$, $B \subseteq R$ if and only if $k(B) \subseteq S$.

PROOF. (i) Let $A \subseteq B$ and suppose $k(A) \cap Y \backslash \text{cl}_Y k(B) \neq \varnothing$. Then there exists $B_0 \in \mathscr{B}_1$ such that $k(B_0) \subseteq k(A) \cap Y \backslash \text{cl}_Y k(B)$. However, by property (*) $B_0 \cap A \neq \varnothing$ but $B_0 \cap B = \varnothing$ which contradicts $A \subseteq B$. The second part of the statement is proved similarly.

(ii) Let $B \in \mathscr{B}_1$. It is clear that if $k(B) \subseteq S$, then $B \subseteq R$. On the other hand, let $B \subseteq R$ and suppose $k(B) \cap Y \backslash S \neq \varnothing$. There is $B_1 \in \mathscr{B}_1$ such that $k(B_1) \subseteq k(B) \cap Y \backslash S$ and so by (i), $B_1 \subseteq \text{cl}_X B \subseteq R$. Thus $B_1 \cap \bigcup_{k(B) \subseteq S} B \neq \varnothing$; so there is $B_0 \in \mathscr{B}_1$ such that $B_0 \subseteq B_1 \cap B_2$ for some $B_2 \in \mathscr{B}_1$ with $k(B_2) \subseteq S$. Now $k(B_0) \subseteq \text{cl}_Y k(B_1) \subseteq Y \backslash \text{int}_Y S$ and so $k(B_0) \cap S = \varnothing$. But $k(B_0) \subseteq \text{cl}_Y k(B_2) \subseteq S$. This is a contradiction.

The mapping $k: \mathscr{B}_1 \to \mathscr{B}_2$ induces a mapping $\tilde{k}: R(X) \to R(Y)$ by defining $\tilde{k}(R) = \text{cl}_Y \bigcup \{k(B): B \in \mathscr{B}_1, B \subseteq R\}$ for $R \in R(X)$. Note that $\tilde{k}(\phi) = \varnothing$. Some properties of $\tilde{k}$ are given in the following lemma.

LEMMA 2.3. (i) *$\bar{k}$: $R(X) \to R(Y)$ is onto.*
(ii) *$\bar{k}(R \wedge S) = \bar{k}(R) \wedge \bar{k}(S)$ for $R, S \in R(X)$.*
(iii) *$\bar{k}(R') = (\bar{k}(R))'$ for $R \in R(X)$.*
(iv) *$\bar{k}(R) = \varnothing$ implies $R = \varnothing$ where $R \in R(X)$.*

PROOF. (i) Let $S \in R(Y)$. By Lemma 2.1, $S = \mathrm{cl}_Y \cup \{B \in \mathscr{B}_2 : B \subseteq S\}$. Let $R = \mathrm{cl}_X \bigcup \{B \in \mathscr{B}_1 : k(B) \subseteq S\}$. Then $\bar{k}(R) = \mathrm{cl}_Y \bigcup \{k(B) : B \in \mathscr{B}_1, B \subseteq R\}$. By Lemma 2.2, $B \subseteq R$ if and only if $k(B) \subseteq S$; so

$$\mathrm{cl}_Y \bigcup \{k(B) : B \in \mathscr{B}_1, B \subseteq R\} = \mathrm{cl}_Y \bigcup \{k(B) \subseteq S : B \in \mathscr{B}_1\}$$
$$= \mathrm{cl}_Y \bigcup \{B \in \mathscr{B}_2 : B \subseteq S\} = S.$$

The proofs of (ii), (iii) and (iv) are straightforward using the definition of a π-basis and the definition of $\bar{k}$.

A subset U of a space X is *regular open* if $\mathrm{int}_X \mathrm{cl}_X U = U$.

THEOREM 2.4. *Let X and Y be spaces such that the regular open subsets form a π-basis. Then $R(X)$ and $R(Y)$ are isomorphic Boolean algebras if and only if there exist π-bases $\mathscr{B}_1$ and $\mathscr{B}_2$ of X and Y respectively and a mapping $k: \mathscr{B}_1 \to \mathscr{B}_2$ which is onto and has the property* (*) *if $A, B \in \mathscr{B}_1$ then $A \cap B = \varnothing$ if and only if $k(A) \cap k(B) = \varnothing$.*

PROOF. Suppose $\mathscr{B}_1$ and $\mathscr{B}_2$ are π-bases of X and Y respectively and $k: \mathscr{B}_1 \to \mathscr{B}_2$ is onto with property (*). Define $\bar{k}: R(X) \to R(Y)$ as it was defined following Lemma 2.2. By Lemma 2.3, $\bar{k}$ is a Boolean algebra isomorphism.

Conversely, suppose $h: R(X) \to R(Y)$ is a Boolean algebra isomorphism. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be the regular open subsets of X and Y respectively; each set B in $\mathscr{B}_1$ (respectively $\mathscr{B}_2$) has a unique representation as $\mathrm{int}_X R$ (respectively $\mathrm{int}_Y S$) for some $R \in R(X)$ (respectively $S \in R(Y)$). Define $k: \mathscr{B}_1 \to \mathscr{B}_2$ by $k(\mathrm{int}_X R) = \mathrm{int}_Y h(R)$. Then k is well defined and maps $\mathscr{B}_1$ onto $\mathscr{B}_2$. Let $U, V \in \mathscr{B}_1$ with $U = \mathrm{int}_X R$, $V = \mathrm{int}_X T$ for $R, T \in R(X)$. Suppose $U \cap V = \varnothing$; then

$$k(U) \cap k(V) = \mathrm{int}_Y h(R) \cap \mathrm{int}_Y h(T) \subseteq h(R) \wedge h(T).$$

But since $R \wedge T = \varnothing$, $h(R) \wedge h(T) = \varnothing$ so $k(U) \cap k(V) = \varnothing$. On the other hand, if $k(U) \cap k(V) = \varnothing$ then $\mathrm{cl}_Y(\mathrm{int}_Y h(R) \cap \mathrm{int}_Y h(T)) = h(R) \wedge h(T) = \varnothing$; so

$$R \wedge T = \mathrm{cl}_X(U \cap V) = \varnothing$$

and thus $U \cap V = \varnothing$. So k has the required properties.

NOTE 2.5. The condition required on the π-bases in Theorem 2.4 is stronger than the requirement that the two spaces have the same π-weight, even in the case where the spaces are compact. For example, consider the compact spaces βN and $\beta\mathbf{R}$ where N is the countable discrete space and $\mathbf{R}$ is the real numbers. Both of these spaces have π-weight $\aleph_0$. Since βN is extremally disconnected $\beta N = E\beta N$, but $E\beta N$ cannot be homeomorphic to the absolute of $\beta\mathbf{R}$ because closed irreducible maps preserve isolated points. So $R(\beta N)$ and $R(\beta\mathbf{R})$ are not isomorphic Boolean algebras.

3. Necessary and sufficient conditions for spaces to be coabsolute

In the first theorem of this section we determine a restriction on the isomorphism between $R(X)$ and $R(Y)$ which gives a necessary and sufficient condition for X and Y to be coabsolute when X and Y are paracompact spaces.

For a space Z, let C_Z be the collection of all regular closed, locally finite covers of Z for which the interiors of the sets are pairwise disjoint, and partially order C_Z by refinement. Let J_Z be an indexing set for C_Z with the induced partial ordering. If $C_a \in C_Z$, let Z_a be the discrete space whose points are the sets in C_a. If $a, b \in J_Z$ and $a \geqslant b$ (in other words C_a refines C_b), then define $f_Z ab$: $Z_a \to Z_b$ by $f_Z ab(R) = S$ if $R \in C_a$ and S is the unique element in C_b for which $R \subseteq S$. It is known (Ponomarev (1962), p. 143) that EZ is homeomorphic to the inverse limit induced by the system $\{Z_a, f_Z ab, J_Z\}$ when Z is a paracompact space. Now suppose X and Y are paracompact spaces with h: $R(X) \to R(Y)$ a Boolean algebra isomorphism which induces a mapping $\tilde{h}$: $C_X \to C_Y$ by $\tilde{h}(C) = \{h(R): R \in C\}$ for $C \in C_X$ and suppose $\tilde{h}$ has the property that $\{\tilde{h}(C): C \in C_X\}$ is cofinal in C_Y. Then $\tilde{h}$ induces a family of homeomorphisms $\{h_a: a \in J_X\}$ between $\{X_a: a \in J_X\}$ and a cofinal subset of $\{Y_b: b \in J_Y\}$ such that $h_b . f_X ab = f_Y a'b' . h_a$, where $\tilde{h}(C_a) = D_{a'} \in C_Y$ and $\tilde{h}(C_b) = D_{b'} \in C_Y$. Thus $\lim\operatorname{inv}\{X_a, f_X ab, J_X\}$ and $\lim\operatorname{inv}\{Y_a, f_Y ab, J_Y\}$ are homeomorphic so X and Y are coabsolute. Next we show that this condition on the Boolean algebra isomorphism is also necessary. So let X and Y be coabsolute paracompact spaces and let f: $EX \to EY$ be a homeomorphism. Then f^β: $S(R(X)) \to S(R(Y))$ induces the Boolean algebra isomorphism h between $R(X)$ and $R(Y)$ by $f^\beta(\lambda_X(R)) = \lambda_Y(h(R))$ for $R \in R(X)$. For $C \in C_X$, define $\tilde{h}(C) = \{h(R): R \in C\}$. The following lemmas show that $\tilde{h}$ maps C_X into C_Y.

We will say that a mapping k: $R(X) \to R(Y)$ *preserves local finiteness* if whenever $\mathscr{H} \subseteq R(X)$ is a locally finite collection of X, then $\{k(R): R \in \mathscr{H}\}$ is a locally finite collection of Y.

LEMMA 3.1. *Let h: $R(X) \to R(Y)$ be a Boolean algebra isomorphism induced as above from a homeomorphism f between EX and EY. Then h preserves local finiteness.*

PROOF. Let $\mathscr{H} \subseteq R(X)$ be a locally finite subfamily of $R(X)$ and let $y \in Y$. $f^{\angle}(\omega_Y(y)) \subseteq EX$ and is compact. Let $y_X = \{x \in X: \omega_X(x) \cap f^{\angle}(\omega_Y(y)) \neq \emptyset\}$ and for each $x \in y_X$, let N_x be an open neighborhood of x meeting only finitely many members of $\mathscr{H}$. Then $f^{\angle}(\omega_Y(y)) \subseteq \bigcup_{x \in y_X} \lambda_X(\mathrm{cl}_X N_x)$ and the compactness of $f^{\angle}(\omega_Y(y))$ implies that $f^{\angle}(\omega_Y(y)) \subseteq \lambda_X(\mathrm{cl}_X N)$ where $N = N_{x_1} \cup \ldots \cup N_{x_k}$ for some $\{x_1, \ldots, x_k\} \subseteq y_X$. Thus $\omega_Y(y) = f^{\beta}(f^{\angle}(\omega_Y(y)) \subseteq f^{\beta}(\lambda_X(\mathrm{cl}_X N)) = \lambda_Y(h(\mathrm{cl}_X N))$ and so $y \in \mathrm{int}_Y h(\mathrm{cl}_X N)$. Since N misses all but finitely many members of $\mathscr{H}$, and h is a Boolean algebra isomorphism, $\mathrm{int}_Y h(\mathrm{cl}_X N)$ misses all but finitely many members of $\{h(R): R \in \mathscr{H}\}$.

LEMMA 3.2. *Let $h: R(X) \to R(Y)$ be a Boolean algebra isomorphism which preserves local finiteness. Then if $\mathscr{H} \subseteq R(X)$ is a locally finite cover of X, $\{h(R): R \in \mathscr{H}\}$ covers Y.*

PROOF. Since $\{h(R): R \in \mathscr{H}\}$ is a locally finite collection of Y, then $\bigvee_{\mathscr{H}} h(R) = \bigcup_{\mathscr{H}} h(R)$. A Boolean algebra isomorphism preserves existing suprema and so $h(\bigvee_{\mathscr{H}} R) = \bigvee_{\mathscr{H}} h(R)$. But $\bigvee_{\mathscr{H}} R = X$, and $h(X) = Y$ and so $Y = \bigcup_{\mathscr{H}} h(R)$.

If $C \in C_X$, it is clear that the sets in $\tilde{h}(C)$ have pairwise disjoint interiors and so $\tilde{h}$ maps C_X into C_Y. The above arguments can also be applied to $h^{\angle}$ and $(\tilde{h}^{\angle})$. Thus $\tilde{h}$ is onto. So we have proved the following theorem.

THEOREM 3.3. *Let X and Y be paracompact spaces. X and Y are coabsolute if and only if there is a Boolean algebra isomorphism h between $R(X)$ and $R(Y)$ which induces a mapping $\tilde{h}: C_X \to C_Y$ by $\tilde{h}(C) = \{h(R): R \in C\}$ for $C \in C_X$ and such that $\tilde{h}(C_X)$ is cofinal in C_Y.*

Using Theorem 3.3 and the lemmas used in its proof, we can reformulate the necessary and sufficient condition for paracompact spaces to be coabsolute in a more descriptive form as follows:

COROLLARY 3.4. *Let X and Y be paracompact spaces. X and Y are coabsolute if and only if there exists a Boolean algebra isomorphism between $R(X)$ and $R(Y)$ such that the isomorphism and its inverse preserve local finiteness.*

PROOF. By Lemma 3.1, if X and Y are coabsolute, then the isomorphism between $R(X)$ and $R(Y)$ induced by the homeomorphism between EX and EY preserves local finiteness. Conversely, if $h: R(X) \to R(Y)$ is a Boolean algebra isomorphism such that h and $h^{\angle}$ preserve local finiteness, then by Lemma 3.2, h induces a mapping $\tilde{h}: C_X \to C_Y$ and this will be onto. So by Theorem 3.3, X and Y are coabsolute.

If we add the hypothesis that the spaces in question be locally compact as well as paracompact, the necessary and sufficient condition for the spaces to be coabsolute simplifies. For any space Z, let $K(Z)$ be the collection of regular closed compact subsets of Z. $K(Z)$ plays a role for locally compact paracompact spaces Z analogous to that of $R(Z)$ for compact spaces Z—namely, we have the following theorem.

THEOREM 3.5. *Let X and Y be locally compact paracompact spaces. Then X and Y are coabsolute if and only if there is a one-to-one onto mapping of $K(X)$ onto $K(Y)$ which preserves $\vee$ and $\wedge$.*

PROOF. Suppose X and Y are coabsolute. It suffices to show that the Boolean algebra isomorphism h induced by the Stone extension of a homeomorphism $f: EX \to EY$ (in other words $f^{\beta}(\lambda_X(R)) = \lambda_Y(h(R))$), when restricted to $K(X)$, maps onto $K(Y)$. For any regular space Z and $R \in R(Z)$, $R \in K(Z)$ if and only if $\lambda_Z(R) \subseteq EZ$. So if $R \in K(X)$, $\lambda_X(R) \subseteq EX$ and

$$\lambda_Y(h(R)) = f^{\beta}(\lambda_X(R)) = f(\lambda_X(R)) \subseteq EY;$$

thus $h(R) \in K(Y)$. Similarly, if $R \in K(Y)$, $R = h(S)$ for $S \in R(X)$ and so $f^{\beta}(\lambda_X(S)) = \lambda_Y(R) \subseteq EY$ and since f^{β} is a homeomorphism,

$$f^{\beta\angle}(\lambda_Y(R)) = \lambda_X(S) \subseteq f^{\beta\angle}(EY) = EX.$$

Thus $S \in K(X)$.

Conversely, let $h: K(X) \to K(Y)$ be one-to-one, onto and preserve $\vee$ and $\wedge$. Note the following facts:

(i) If $S \in K(X)$ then $h|R(S): R(S) \to R(h(S))$ is a Boolean algebra isomorphism and so S and $h(S)$ are coabsolute.

(ii) If $C = \{R_a: a \in A\} \in C_X$ and $j: \sum_A R_a \to X$ is defined by $j(x) = x$ for every $x \in \sum_A R_a$, then j is closed irreducible, perfect and continuous; so X and $\sum_A R_a$ are coabsolute. Furthermore, the absolute of $\sum_A R_a$ is homeomorphic to $\sum_A ER_a$.

Thus it suffices to find $C \in C_X$ such that $C \subseteq K(X)$ and for which $\{h(R): R \in C\} \in C_Y$.

Since X is locally compact and paracompact, X has a locally finite cover F of compact regular closed subsets. By Dugundji (1968), p. 178, Exercise 1.7, there is a family of pairwise disjoint open sets whose closures form a cover C of X which is a refinement of F. So $C \in C_X$ and $C \subseteq K(X)$. The sets in $\{h(R): R \in C\}$ have pairwise disjoint interiors.

(iii) $\{h(R): R \in C\}$ is a locally finite collection of Y. Let $y \in Y$ and let $\mathrm{cl}_Y M$ be a compact neighborhood of y. Then $\mathrm{cl}_Y M = h(S)$ for some $S \in K(X)$. For each $x \in S$, there is a compact neighborhood $\mathrm{cl}_X N_x$ of x which meets only finitely many

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members of C, and since $S \subseteq \bigcup_{x \in S} N_x$ and S is compact, $S \subseteq N = N_{x_1} \cup \ldots \cup N_{x_k}$ for some $\{x_1, \ldots, x_k\} \subseteq S$. Since $\mathrm{cl}_X N$ meets only finitely many members of C, and h preserves $\wedge$, $\mathrm{int}_Y h(S)$ meets only finitely many members of $\{h(R) \colon R \in C\}$.

(iv) $\{h(R) \colon R \in C\}$ is a cover of Y. By (iii), $\bigcup_{R \in C} h(R)$ is closed. If $Y \setminus \bigcup_{R \in C} h(R) \neq \varnothing$, then there is a non-empty $S \in K(Y)$ such that $S \subseteq Y \setminus \bigcup_{R \in C} h(R)$ and $S = h(T)$ for some non-empty $T \in K(X)$. Then $T \wedge R = \varnothing$ since $h(T) \wedge h(R) = \varnothing$ for every $R \in C$ and thus $\mathrm{int}_X T \subseteq X \setminus R$ for all $R \in C$. But then

$$\mathrm{int}_X T \subseteq \bigcap_C X \setminus R = X \setminus \bigcup_C R = \varnothing.$$

So (iii) and (iv) show that $\{h(R) \colon R \in C\} \in C_Y$.

REMARKS. (1) The condition given in Theorem 3.5 for locally compact paracompact spaces is strictly stronger than the condition that the Boolean algebras of regular closed subsets be isomorphic as the following example shows. This example was suggested by the referee. Let $(A_n)_{n \in N}$ be a partition of N into $\aleph_0$ infinite sets. Let $X = \bigcup_{n \in N} \mathrm{cl}_{\beta N} A_n$. Since N is extremally disconnected, so is βN and also any dense subset of βN; thus $EN = N$ and $EX = X$. Since N is dense in X, $R(N)$ and $R(X)$ are isomorphic Boolean algebras. Obviously N and X are not homeomorphic, even though they are both locally compact and σ-compact and have the same density and cellularity. Since N and X are not coabsolute, Theorem 3.5 assures us that although $R(N)$ and $R(X)$ are isomorphic, there is no mapping between $K(N)$ and $K(X)$ which is one-to-one, onto, and preserves $\vee$ and $\wedge$.

(2) We will now see that the condition given in Theorem 3.5 *is* the same as the condition that the Boolean algebra of regular closed sets be isomorphic if the spaces are locally compact, non-compact metric spaces without isolated points. Let X and Y be two such spaces with $R(X)$ and $R(Y)$ isomorphic. Then the following argument shows that X and Y have the same density: EX and EY are dense open subsets of $S(R(X)) \approx S(R(Y))$ and so have the same density; since a space and its absolute have the same density, the conclusion follows. It is known (for example, Gates (1977)) that two locally compact, non-compact metric spaces without isolated points are coabsolute if they have the same density. This, along with Theorem 3.5, yields the statement that $R(X)$ and $R(Y)$ are isomorphic Boolean algebras if and only if there is a one-to-one, onto mapping between $K(X)$ and $K(Y)$ which preserves $\vee$ and $\wedge$ when X and Y are locally compact, non-compact metric spaces without isolated points. Note that the restriction of a particular isomorphism between $R(X)$ and $R(Y)$ may not be the required mapping between $K(X)$ and $K(Y)$; for example, if $h \colon R([0,1)) \to R((0,1))$ is defined by $h(S) = S \cap (0,1)$ where $S \in R([0,1))$, h is a Boolean algebra isomorphism, but the restriction of h to $K([0,1))$ does not map into $K((0,1))$

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(3) If $\{\lambda_X(R) \cap EX : R \in \mathscr{H}\}$ is a point-finite collection of clopen sets in EX where $\mathscr{H} \subseteq R(X)$, then $\{\text{int}_X R : R \in \mathscr{H}\}$ is a point-finite collection of open sets in X. This raises an interesting question suggested by the referee: Is there a result analogous to Corollary 3.4 for metacompact spaces with point-finiteness being the property to be 'preserved' in some sense?

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