1

Vector spaces

In this chapter we fix our terminology and notation, mostly related to (real and complex) linear algebra. We will consider only algebraic properties. Infinite-dimensional vector spaces will not be equipped with any topology.

Let us stress that using precise terminology and notation concerning linear algebra is very useful in describing various aspects of quantization and quantum fields. Even though the material of this chapter is elementary, the terminology and notation introduced in this chapter will play an important role throughout our work. In particular we should draw the reader's attention to the notion of the complex conjugate space (Subsect. 1.2.3), and of the holomorphic and anti-holomorphic subspaces (Subsect. 1.3.6).

Throughout the book $\mathbb{K}$ will denote either the field $\mathbb{R}$ or $\mathbb{C}$, all vector spaces being either real or complex, unless specified otherwise.

1.1 Elementary linear algebra

The material of this section is well known and elementary. Among other things, we discuss four basic kinds of structures, which will serve as the starting point for quantization:

(1) Symplectic spaces – classical phase spaces of neutral bosons,
(2) Euclidean spaces – classical phase spaces of neutral fermions,
(3) Charged symplectic spaces – classical phase spaces of charged bosons,
(4) Unitary spaces – classical phase spaces of charged fermions.

Throughout the section, $\mathcal{Y}, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{W}$ are vector spaces over $\mathbb{K}$.

1.1.1 Vector spaces and linear operators

Definition 1.1 *If $\mathcal{U} \subset \mathcal{Y}$, then* Span $\mathcal{U}$ *denotes the space of finite linear combinations of elements of $\mathcal{U}$.*

Definition 1.2 $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ *denotes the* external direct sum of $\mathcal{Y}_1$ and $\mathcal{Y}_2$, *that is, the Cartesian product $\mathcal{Y}_1 \times \mathcal{Y}_2$ equipped with its vector space structure. If $\mathcal{Y}_1$, $\mathcal{Y}_2$ are subspaces of a vector space $\mathcal{Y}$ and $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \{0\}$, then the same notation $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ stands for the* internal direct sum of $\mathcal{Y}_1$ and $\mathcal{Y}_2$, *that is, $\mathcal{Y}_1 + \mathcal{Y}_2$ (which is a subspace of $\mathcal{Y}$).*

Definition 1.3 *$L(\mathcal{Y},\mathcal{W})$ denotes the space of linear maps from $\mathcal{Y}$ to $\mathcal{W}$. We set $L(\mathcal{Y}) := L(\mathcal{Y},\mathcal{Y})$.*

Definition 1.4 *$L^{\mathrm{fd}}(\mathcal{Y},\mathcal{W})$, resp. $L^{\mathrm{fd}}(\mathcal{Y})$ denote the space of* finite-dimensional *(or* finite rank) linear operators *in $L(\mathcal{Y},\mathcal{W})$, resp. $L(\mathcal{Y})$.*

Definition 1.5 *Let $a_i \in L(\mathcal{Y}_i,\mathcal{W})$, $i = 1,2$. We say that $a_1 \subset a_2$ if $\mathcal{Y}_1 \subset \mathcal{Y}_2$ and a_1 is the restriction of a_2 to $\mathcal{Y}_1$, that is, $a_2\big|_{\mathcal{Y}_1} = a_1$.*

Definition 1.6 *If $a \in L(\mathcal{Y},\mathcal{W})$, then* Ker a *denotes the* kernel *(or* null space) of a *and* Ran a *denotes its* range.

Definition 1.7 $\mathbb{1}_{\mathcal{Y}}$ *stands for the* identity on $\mathcal{Y}$.

1.1.2 2×2 *block matrices*

If $\mathcal{Y} = \mathcal{Y}_+ \oplus \mathcal{Y}_-$, every $r \in L(\mathcal{Y})$ can be written as a 2×2 block matrix. The following decomposition, possible if a is invertible, is often useful:

$$r = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \mathbb{1} & 0 \\ ca^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d - ca^{-1}b \end{bmatrix} \begin{bmatrix} \mathbb{1} & a^{-1}b \\ 0 & \mathbb{1} \end{bmatrix}. \tag{1.1}$$

Here are some expressions for the inverse of r:

$$r^{-1} = \begin{bmatrix} \mathbb{1} & -a^{-1}b \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & (d - ca^{-1}b)^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ -ca^{-1} & \mathbb{1} \end{bmatrix} \tag{1.2}$$

$$= \begin{bmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{bmatrix}. \tag{1.3}$$

If $\mathcal{Y}$ is finite-dimensional, then, using the decomposition (1.1), we obtain the following formulas for the determinant:

$$\begin{aligned} \det r &= \det a \det(d - ca^{-1}b) \\ &= \det c \det b \det(ac^{-1}db^{-1} - \mathbb{1}). \end{aligned} \tag{1.4}$$

1.1.3 Duality

Definition 1.8 *The* dual *of $\mathcal{Y}$, denoted by $\mathcal{Y}^{\#}$, is the space of* linear functionals on $\mathcal{Y}$. *Three kinds of notation for the action of $v \in \mathcal{Y}^{\#}$ on $y \in \mathcal{Y}$ will be used:*

(1) *the* bra–ket notation $\langle v|y\rangle = \langle y|v\rangle$,
(2) *the* simplified notation $v \cdot y = y \cdot v$,
(3) *the* functional notation $v(y)$.

There is a canonical injection $\mathcal{Y} \to \mathcal{Y}^{\#\#}$. We have $\mathcal{Y} = \mathcal{Y}^{\#\#}$ iff $\dim\mathcal{Y} < \infty$.

Definition 1.9 *If $y \in \mathcal{Y}$, we will sometimes write $|y\rangle$ for the operator*

$$\mathbb{K} \ni \lambda \mapsto |y\rangle\lambda := \lambda y \in \mathcal{Y}.$$

If $v \in \mathcal{Y}^{\#}$, we will sometimes write $\langle v|$ instead of v.

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As an example of this notation, suppose that $y \in \mathcal{Y}$ and $v \in \mathcal{Y}^{\#}$ satisfy $\langle v|y\rangle = 1$. Then $|y\rangle\langle v|$ is the projection onto the space spanned by y along the kernel of v.

Definition 1.10 *Let $(e_1, \dots, e_n)$ be a basis of a finite-dimensional space $\mathcal{Y}$. Then there exists a unique basis of $\mathcal{Y}^{\#}$, $(e^1, \dots, e^n)$, called the* dual basis*, such that $\langle e^i|e_j\rangle = \delta^i_j$.*

1.1.4 Annihilator

Definition 1.11 *The* annihilator of $\mathcal{X} \subset \mathcal{Y}$ *is defined as*

$$\mathcal{X}^{\mathrm{an}} := \big\{ v \in \mathcal{Y}^{\#} \ : \ \langle v|y\rangle = 0, \ y \in \mathcal{X} \big\}.$$

The pre-annihilator of $\mathcal{V} \subset \mathcal{Y}^{\#}$ *is defined as*

$$\mathcal{V}_{\mathrm{an}} := \big\{ y \in \mathcal{Y} \ : \ \langle v|y\rangle = 0, \ v \in \mathcal{V} \big\}.$$

Note that

$$(\mathcal{X}^{\mathrm{an}})_{\mathrm{an}} = \mathrm{Span}\mathcal{X}, \quad (\mathcal{V}_{\mathrm{an}})^{\mathrm{an}} = \mathrm{Span}\mathcal{V}.$$

1.1.5 Transpose of an operator

Definition 1.12 *If $a \in L(\mathcal{Y}_1, \mathcal{Y}_2)$, then $a^{\#}$ will denote the* transpose of a*, that is, the operator in $L(\mathcal{Y}_2^{\#}, \mathcal{Y}_1^{\#})$ defined by*

$$\langle a^{\#} v|y\rangle := \langle v|ay\rangle, \quad v \in \mathcal{Y}_2^{\#}, \quad y \in \mathcal{Y}_1. \tag{1.5}$$

Note that a is bijective iff $a^{\#}$ is. We have $a^{\#\#} \in L(\mathcal{Y}_1^{\#\#}, \mathcal{Y}_2^{\#\#})$ and $a \subset a^{\#\#}$.

1.1.6 Dual pairs

Definition 1.13 *A* dual pair *is a pair $(\mathcal{V}, \mathcal{Y})$ of vector spaces equipped with a bilinear form*

$$(\mathcal{V}, \mathcal{Y}) \ni (v, y) \mapsto \langle v|y\rangle \in \mathbb{K}$$

such that

$$\langle v|y\rangle = 0, \ v \in \mathcal{V} \ \Rightarrow \ y = 0, \tag{1.6}$$

$$\langle v|y\rangle = 0, \ y \in \mathcal{Y} \ \Rightarrow \ v = 0. \tag{1.7}$$

Clearly, if $(\mathcal{V}, \mathcal{Y})$ is a dual pair, then so is $(\mathcal{Y}, \mathcal{V})$. If $\mathcal{Y}$ is finite-dimensional and $(\mathcal{V}, \mathcal{Y})$ is a dual pair, then $\mathcal{V}$ is naturally isomorphic to $\mathcal{Y}^{\#}$.

In general, $(\mathcal{V}, \mathcal{Y})$ is a dual pair iff $\mathcal{V}$ can be identified with a subspace of $\mathcal{Y}^{\#}$ (this automatically guarantees (1.7)) satisfying $\mathcal{V}_{\mathrm{an}} = \{0\}$ (this implies (1.6)).

1.1.7 Bilinear forms

Definition 1.14 *Elements of* $L(\mathcal{Y}, \mathcal{Y}^{\#})$ *will be called* bilinear forms.

Let $\nu \in L(\mathcal{Y}, \mathcal{Y}^{\#})$. Then ν determines a bilinear map on $\mathcal{Y}$:

$$\mathcal{Y} \times \mathcal{Y} \ni (y_1, y_2) \mapsto y_1 \cdot \nu y_2 = \langle y_1 | \nu y_2 \rangle \in \mathbb{K}. \tag{1.8}$$

Definition 1.15 *We say that* ν *is* non-degenerate *if* $\operatorname{Ker} \nu = 0$.

Definition 1.16 *We say that* $r \in L(\mathcal{Y})$ preserves the form ν *if*

$$r^{\#} \nu r = \nu, \text{ i.e.} \quad (r y_1) \cdot \nu r y_2 = y_1 \cdot \nu y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

We say that $a \in L(\mathcal{Y})$ infinitesimally preserves the form ν *if*

$$a^{\#} \nu + \nu a = 0, \text{ i.e.} \quad (a y_1) \cdot \nu y_2 = -y_1 \cdot \nu a y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

Remark 1.17 *We will use three kinds of notation for bilinear forms:*

(1) *the* bra–ket notation $\langle y_1 | \nu y_2 \rangle$*, going back to Dirac,*
(2) *the* simplified notation $y_1 \cdot \nu y_2$*,*
(3) *the* functional notation $\nu(y_1, y_2)$*.*

Usually, we prefer the first two kinds of notation (both appear in (1.8)).

1.1.8 Symmetric forms

Definition 1.18 *We will say that* $\nu \in L(\mathcal{Y}, \mathcal{Y}^{\#})$ *is* symmetric *if*

$$\nu \subset \nu^{\#}, \text{ i.e.} \quad y_1 \cdot \nu y_2 = y_2 \cdot \nu y_1, \quad y_1, y_2 \in \mathcal{Y}.$$

The space of all symmetric elements of $L(\mathcal{Y}, \mathcal{Y}^{\#})$ *will be denoted by* $L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}^{\#})$.

Let $\nu \in L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}^{\#})$.

Definition 1.19 *A subspace* $\mathcal{X} \subset \mathcal{Y}$ *is called* isotropic *if*

$$y_1 \cdot \nu y_2 = 0, \qquad y_1, y_2 \in \mathcal{X}.$$

Definition 1.20 *Let* $\mathcal{Y}$ *be a real vector space.* ν *is called* positive semi-definite *if* $y \cdot \nu y \geq 0$ *for* $y \in \mathcal{Y}$*. It is called* positive definite *if* $y \cdot \nu y > 0$ *for* $y \neq 0$.

A positive definite form is always non-degenerate.

Assume that ν is non-degenerate. Using that ν is symmetric and non-degenerate we see that $\langle v | y \rangle = 0$ for all $v \in \nu\mathcal{Y}$ implies $y = 0$. Thus $(\nu\mathcal{Y}, \mathcal{Y})$ is a dual pair and $\mathcal{Y}$ can be treated as a subspace of $(\nu\mathcal{Y})^{\#}$. Hence, ν^{-1}, a priori defined as a map from $\nu\mathcal{Y}$ to $\mathcal{Y}$, can be understood as a map from $\nu\mathcal{Y}$ to $(\nu\mathcal{Y})^{\#}$. We easily check that ν^{-1} is symmetric and non-degenerate. If ν is positive definite, then so is ν^{-1}.

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Proposition 1.21 *Let $\mathcal{Y}$ be finite-dimensional. Then,*

(1) $\nu \in L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}^{\#})$ *iff* $\nu^{\#} = \nu$.
(2) *If ν is non-degenerate, then* $\nu\mathcal{Y} = \mathcal{Y}^{\#}$, *so that* $\nu^{-1} \in L_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$ *is a non-degenerate symmetric form.*

1.1.9 (Pseudo-)Euclidean spaces

Definition 1.22 *A couple $(\mathcal{Y}, \nu)$, where $\nu \in L_{\mathrm{s}}(\mathcal{Y}, \mathcal{Y}^{\#})$ is non-degenerate, is called a* pseudo-Euclidean space. *If $\mathcal{Y}$ is real and ν is positive definite, then $(\mathcal{Y}, \nu)$ is called a* Euclidean space. *In such a case we can define the norm of $y \in \mathcal{Y}$, denoted by $\|y\| := \sqrt{y \cdot \nu y}$. If $\mathcal{Y}$ is complete for this norm, it is called a* real Hilbert space.

Let $(\mathcal{Y}, \nu)$ be a pseudo-Euclidean space.

Definition 1.23 *If $\mathcal{X} \subset \mathcal{Y}$, then $\mathcal{X}^{\nu\perp}$ denotes the ν-orthogonal complement of $\mathcal{X}$:*

$$\mathcal{X}^{\nu\perp} := \{y \in \mathcal{Y} \ : \ y \cdot \nu x = 0, \ x \in \mathcal{X}\}.$$

Definition 1.24 *A symmetric form on a real space, especially if it is positive definite, is often called a* scalar product *and denoted $\langle y_1|y_2\rangle$ or $y_1 \cdot y_2$. In such a case, the orthogonal complement of $\mathcal{X}$ is denoted $\mathcal{X}^{\perp}$. For $x \in \mathcal{Y}$, $\langle x|$ will denote the following operator:*

$$\mathcal{Y} \ni y \mapsto \langle x|y := \langle x|y\rangle \in \mathbb{K}.$$

If $\langle x|x\rangle = 1$, then $|x\rangle\langle x|$ is the orthogonal projection onto x.

Most Euclidean spaces considered in our work will be real Hilbert spaces. Real Hilbert spaces will be further discussed in Subsect. 2.2.2.

1.1.10 Inertia of a symmetric form

Let $\mathcal{Y}$ be a finite-dimensional space equipped with a symmetric form ν. In the real case we can find a basis

$$(e_{1,+}, \dots, e_{p,+}, e_{1,-}, \dots, e_{q,-}, e_1, \dots, e_r)$$

such that if

$$(e^{1,+}, \dots, e^{p,+}, e^{1,-}, \dots, e^{q,-}, e^1, \dots, e^r)$$

is the dual basis in $\mathcal{Y}^{\#}$, then

$$\nu e_{j,+} = e^{j,+}, \qquad \nu e_{j,-} = -e^{j,-}, \qquad \nu e_j = 0.$$

The numbers (p, q) do not depend on the choice of the basis. ν is positive definite iff $q = r = 0$.

Definition 1.25 *We set* inert $\nu := p - q$.

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In the complex case, we can find a basis

$$(e_{1,+}, \dots, e_{p,+}, e_1, \dots, e_r)$$

such that if

$$(e^{1,+}, \dots, e^{p,+}, e^1, \dots, e^r)$$

is the dual basis in $\mathcal{Y}^{\#}$, then

$$\nu e_{j,+} = e^{j,+}, \qquad \nu e_j = 0.$$

The number p does not depend on the choice of the basis.

Definition 1.26 *We set* inert $\nu := p$.

1.1.11 Group $O(\mathcal{Y})$ and Lie algebra $o(\mathcal{Y})$

Let $(\mathcal{Y}, \nu)$ be a Euclidean space and $a \in L(\mathcal{Y})$.

Definition 1.27 *We say that*

$$\begin{array}{rl} a \text{ is isometric} & \text{if } a^{\#}\nu a = \nu, \\ a \text{ is orthogonal} & \text{if } a \text{ is isometric and bijective}, \\ a \text{ is anti-self-adjoint} & \text{if } a^{\#}\nu = -\nu a, \\ a \text{ is self-adjoint} & \text{if } a^{\#}\nu = \nu a. \end{array}$$

The set of orthogonal elements in $L(\mathcal{Y})$ is a group for the operator composition, denoted by $O(\mathcal{Y})$. The set of anti-self-adjoint elements in $L(\mathcal{Y})$, denoted by $o(\mathcal{Y})$, is a Lie algebra, equipped with the commutator $[a, b]$.

Definition 1.28 *If $(\mathcal{Y}, \nu)$ is pseudo-Euclidean, we keep the same definitions, except we replace isometric, orthogonal, anti-self-adjoint and self-adjoint with* pseudo-isometric, pseudo-orthogonal, anti-pseudo-self-adjoint *and* pseudo-self-adjoint.

1.1.12 Anti-symmetric forms

Definition 1.29 *We will say that $\omega \in L(\mathcal{Y}, \mathcal{Y}^{\#})$ is* anti-symmetric *if*

$$-\omega \subset \omega^{\#}, \text{ i.e.} \quad y_1 \cdot \omega y_2 = -y_2 \cdot \omega y_1, \quad y_1, y_2 \in \mathcal{Y}.$$

The space of all anti-symmetric elements of $L(\mathcal{Y}, \mathcal{Y}^{\#})$ will be denoted by $L_{\mathrm{a}}(\mathcal{Y}, \mathcal{Y}^{\#})$.

Let $\omega \in L_{\mathrm{a}}(\mathcal{Y}, \mathcal{Y}^{\#})$.

Definition 1.30 *A subspace $\mathcal{X} \subset \mathcal{Y}$ is called* isotropic *if*

$$y_1 \cdot \omega y_2 = 0, \qquad y_1, y_2 \in \mathcal{X}.$$

A maximal isotropic subspace is called Lagrangian.

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Definition 1.31 *A non-degenerate anti-symmetric bilinear form is called* symplectic.

If ω is symplectic, then $(\omega\mathcal{Y}, \mathcal{Y})$ is a dual pair and we can treat $\mathcal{Y}$ as a subspace of $(\omega\mathcal{Y})^{\#}$. We can also define a symplectic form $\omega^{-1} \in L_{\mathrm{a}}(\omega\mathcal{Y}, \mathcal{Y}) \subset L_{\mathrm{a}}(\omega\mathcal{Y}, (\omega\mathcal{Y})^{\#})$.

Proposition 1.32 *Let $\mathcal{Y}$ be finite-dimensional.*

(1) *ω is anti-symmetric iff $\omega^{\#} = -\omega$.*
(2) *An isotropic subspace $\mathcal{X}$ is Lagrangian iff $\dim \mathcal{X} = \frac{1}{2}\dim \mathcal{Y}$.*
(3) *If ω is symplectic, then $\omega\mathcal{Y} = \mathcal{Y}^{\#}$, so that $\omega^{-1} \in L_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y})$ is a symplectic form.*

1.1.13 Symplectic spaces

Definition 1.33 *The pair $(\mathcal{Y}, \omega)$, where ω is a symplectic form on $\mathcal{Y}$, is called a* symplectic space.

Let $(\mathcal{Y}, \omega)$ be a symplectic space.

Definition 1.34 *The symplectic complement of $\mathcal{X} \subset \mathcal{Y}$ is defined as*

$$\mathcal{X}^{\omega\perp} := \{y \in \mathcal{Y} \ : \ y \cdot \omega x = 0, \ x \in \mathcal{X}\}.$$

Let $\mathcal{X}$ be a subspace of $\mathcal{Y}$. Note that $\mathcal{X}$ is isotropic iff $\mathcal{X}^{\omega\perp} \supset \mathcal{X}$ and it is Lagrangian iff $\mathcal{X}^{\omega\perp} = \mathcal{X}$.

Definition 1.35 *We say that $\mathcal{X}$ is* co-isotropic *if $\mathcal{X}^{\omega\perp} \subset \mathcal{X}$.*

If $\mathcal{X}$ is co-isotropic, then $\mathcal{X}/\mathcal{X}^{\omega\perp}$ is naturally a symplectic space.

Note that $\mathcal{X}$ is isotropic in $\mathcal{Y}$ iff $\mathcal{X}^{\mathrm{an}}$ is co-isotropic in $\mathcal{Y}^{\#}$.

1.1.14 Group $Sp(\mathcal{Y})$ and Lie algebra $sp(\mathcal{Y})$

Let $(\mathcal{Y}, \omega)$ be a symplectic space and $a \in L(\mathcal{Y})$.

Definition 1.36 *We say that*

$$\begin{aligned} a \textit{ is } \text{symplectic} \quad &\textit{if} \quad a \textit{ is bijective and } a^{\#}\omega a = \omega, \\ a \textit{ is } \text{anti-symplectic} \quad &\textit{if} \quad a \textit{ is bijective and } a^{\#}\omega a = -\omega, \\ a \textit{ is } \text{infinitesimally symplectic} \quad &\textit{if} \quad a^{\#}\omega = -\omega a. \end{aligned}$$

The set of symplectic elements in $L(\mathcal{Y})$ is a group for the operator composition denoted by $Sp(\mathcal{Y})$. The set of infinitesimally symplectic elements, denoted by $sp(\mathcal{Y})$, is a Lie algebra equipped with the commutator.

Proposition 1.37 *Assume that $\mathcal{Y}$ is finite-dimensional and $r \in L(\mathcal{Y})$. Then*

(1) *$r \in Sp(\mathcal{Y})$ iff $r^{\#}\omega r = \omega$.*
(2) *$r \in Sp(\mathcal{Y}, \omega)$ iff $r^{\#} \in Sp(\mathcal{Y}^{\#}, \omega^{-1})$.*
(3) *$r \in Sp(\mathcal{Y})$ implies $r^{-1} = \omega^{-1}r^{\#}\omega$.*

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1.1.15 Involutions and super-spaces

Definition 1.38 *$\epsilon \in L(\mathcal{Y})$ is called an* involution *if $\epsilon^2 = \mathbb{1}$.*

Definition 1.39 *If $\epsilon \in L(\mathcal{Y})$ is an involution, we set $\mathcal{Y}^{\pm\epsilon} := \mathrm{Ker}(\mathbb{1} \mp \epsilon)$.*

Every involution determines a decomposition $\mathcal{Y} = \mathcal{Y}^{\epsilon} \oplus \mathcal{Y}^{-\epsilon}$, the operators $\frac{1}{2}(\mathbb{1} \pm \epsilon)$ being the projections onto $\mathcal{Y}^{\pm\epsilon}$ along $\mathcal{Y}^{\mp\epsilon}$.

Conversely, a decomposition $\mathcal{Y} = \mathcal{Y}_0 \oplus \mathcal{Y}_1$ determines an involution given by the matrix $\epsilon = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$.

Operators $a \in L(\mathcal{Y})$ commuting with ϵ are of the form $a = \begin{bmatrix} a_{00} & 0 \\ 0 & a_{11} \end{bmatrix}$.

Definition 1.40 *We say that $(\mathcal{Y}, \epsilon)$ is a $\mathbb{Z}_2$-*graded space *or a* super-space *if ϵ is an involution on $\mathcal{Y}$. ϵ is often called the $\mathbb{Z}_2$-*grading.

Definition 1.41 *In the context of super-spaces one often writes $\mathcal{Y}_0$ for $\mathcal{Y}^{\epsilon}$ and its elements are called* even*. One writes $\mathcal{Y}_1$ for $\mathcal{Y}^{-\epsilon}$ and its elements are called* odd*. Elements of $\mathcal{Y}_0 \cup \mathcal{Y}_1$ will be called* homogeneous *or* pure*. The operator $p = 0 \oplus \mathbb{1}$ is called the* parity*, so that $\epsilon = (-\mathbb{1})^p$. Sometimes, the parity of a homogeneous element $y \in \mathcal{Y}$ is denoted $|y|$.*

Remark 1.42 *The name "super-space" came into use under the influence of super-symmetric quantum field theory. The prefix "super" is often attached to mean "$\mathbb{Z}_2$-graded" in various contexts; see e.g. Subsects. 3.3.9 and 6.1.4.*

If $\mathcal{Y}$ has an additional structure, we will often assume that it is preserved by ϵ. For instance, we have the following terminology (see Subsect. 1.3.8):

Definition 1.43 *$(\mathcal{Y}, \epsilon)$ is a* super-Hilbert space *if $\mathcal{Y}$ is a Hilbert space and ϵ is a unitary involution; it is a* super-Kähler space *if $\mathcal{Y}$ is a Kähler space and ϵ is a symplectic and orthogonal (and hence complex linear) involution.*

Let $(\mathcal{Y}, \epsilon)$, $(\mathcal{W}, \varepsilon)$ be two super-spaces. The space of linear transformations from $\mathcal{Y}$ to $\mathcal{W}$, that is, $L(\mathcal{Y}, \mathcal{W})$, is itself naturally a super-space, with the grading given by

$$L(\mathcal{Y}, \mathcal{W}) \ni r \mapsto \varepsilon r \epsilon \in L(\mathcal{Y}, \mathcal{W}).$$

Written in the matrix notation, the decomposition of an element of $L(\mathcal{Y}, \mathcal{W})$ into its even and odd parts is

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} = \begin{bmatrix} a_{00} & 0 \\ 0 & a_{11} \end{bmatrix} + \begin{bmatrix} 0 & a_{01} \\ a_{10} & 0 \end{bmatrix}.$$

We can form other super-spaces in an obvious way, for example, $(\mathcal{Y} \oplus \mathcal{W}, \epsilon \oplus \varepsilon)$, $(\mathcal{Y} \otimes \mathcal{W}, \epsilon \otimes \varepsilon)$.

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1.1.16 Conjugations on a symplectic space

Let $(\mathcal{Y}, \omega)$ be a symplectic space.

Definition 1.44 *A map $\tau \in L(\mathcal{Y})$ is called a* conjugation *if it is an anti-symplectic involution.*

Let $(\mathcal{V}, \mathcal{X})$ be a dual pair of vector spaces. Define $\omega \in L(\mathcal{V} \oplus \mathcal{X}, \mathcal{V}^{\#} \oplus \mathcal{X}^{\#})$ and $\tau \in L(\mathcal{V} \oplus \mathcal{X})$ by

$$\omega = \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}. \tag{1.9}$$

In other words, for $(\eta_1, q_1), (\eta_2, q_2) \in \mathcal{V} \oplus \mathcal{X}$ we have

$$(\eta_1, q_1) \cdot \omega(\eta_2, q_2) = \eta_1 \cdot q_2 - \eta_2 \cdot q_1, \quad \tau(\eta_1, q_1) = (\eta_1, -q_1). \tag{1.10}$$

Then ω is a symplectic form on $\mathcal{V} \oplus \mathcal{X}$ and τ is a conjugation.

We can also define ω^{-1} and $\tau^{\#}$ on $\mathcal{V}^{\#} \oplus \mathcal{X}^{\#}$. We obtain a symplectic form and a conjugation:

$$\omega^{-1} = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}, \quad \tau^{\#} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \tag{1.11}$$

or equivalently

$$(x_1, \xi_1) \cdot \omega^{-1}(x_2, \xi_2) = \xi_1 \cdot x_2 - \xi_2 \cdot x_1, \quad \tau^{\#}(x_1, \xi_1) = (x_1, -\xi_1). \tag{1.12}$$

We will see below that the above construction describes a general form of a symplectic space equipped with a conjugation.

Proposition 1.45 *Let τ be a conjugation on a symplectic space $\mathcal{Y}$. Then the spaces $\mathcal{Y}^{\pm\tau}$ are Lagrangian.*

Proof The spaces $\mathcal{Y}^{\pm\tau}$ are clearly isotropic. Since $\mathcal{Y} \simeq \mathcal{Y}^{\tau} \oplus \mathcal{Y}^{-\tau}$ we have $\mathcal{Y}^{\#} \simeq (\mathcal{Y}^{\tau})^{\#} \oplus (\mathcal{Y}^{-\tau})^{\#}$, and we can write ω as the matrix

$$\begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix},$$

where $a : \mathcal{Y}^{-\tau} \to (\mathcal{Y}^{\tau})^{\#}$ and $b : \mathcal{Y}^{\tau} \to (\mathcal{Y}^{-\tau})^{\#}$ are injective and

$$a^{\#}\big|_{\mathcal{Y}^{\tau}} = b, \qquad b^{\#}\big|_{\mathcal{Y}^{-\tau}} = a.$$

If $\mathcal{Y}^{\tau} \subsetneq \mathcal{X}$, where $\mathcal{X}$ is isotropic, then there exists $e \notin \mathcal{Y}^{\tau}$ such that $y \cdot \omega e = 0$ for all $y \in \mathcal{Y}^{\tau}$. Then $(\mathbb{1} - \tau)e \neq 0$ and $y \cdot \omega(\mathbb{1} - \tau)e = y \cdot a(\mathbb{1} - \tau)e = 0$ for all $y \in \mathcal{Y}^{\tau}$, which contradicts the fact that a is injective. Hence $\mathcal{Y}^{\pm\tau}$ are Lagrangian. □

Proposition 1.46 *Let $\mathcal{Y}$ be a symplectic space $\mathcal{Y}$ with a conjugation τ. We use the notation of the proof of Prop. 1.45. Set*

$$\mathcal{X} := \mathcal{Y}^{-\tau}, \quad \mathcal{V} := b\mathcal{Y}^{\tau}.$$

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Then $(\mathcal{V}, \mathcal{X})$ is a dual pair and $b \oplus \mathbb{1}$ sends bijectively $\mathcal{Y} = \mathcal{Y}^{\tau} \oplus \mathcal{Y}^{-\tau}$ onto $\mathcal{V} \oplus \mathcal{X}$. With this identification, ω and τ are given by (1.10).

If in addition the dimension of $\mathcal{Y}$ is finite, then $\mathcal{V} = \mathcal{X}^{\#}$ and we obtain a bijection of $\mathcal{Y}$ onto $\mathcal{X}^{\#} \oplus \mathcal{X}$ and of $\mathcal{Y}^{\#}$ onto $\mathcal{X} \oplus \mathcal{X}^{\#}$.

Proof Clearly, $\mathcal{V} \subset \mathcal{X}^{\#}$. We need to show that $\mathcal{V}_{\mathrm{an}} = \{0\}$. Let $x \in \mathcal{V}_{\mathrm{an}}$. For any $y \in \mathcal{Y}^{\tau}$, we have

$$0 = \langle by|x\rangle = \langle y|b^{\#}x\rangle = \langle y|ax\rangle,$$

since $b^{\#}\big|_{\mathcal{Y}^{-\tau}} = a$. This implies that $ax = 0$, and hence $x = 0$, since a is injective. Therefore, $(\mathcal{V}, \mathcal{X})$ is a dual pair. □

Theorem 1.47 *Let $\mathcal{Y}$ be a finite-dimensional symplectic space. There exists a conjugation in $L(\mathcal{Y})$. Consequently, there exists a vector space $\mathcal{X}$ such that $\mathcal{Y}$ is isomorphic to $\mathcal{X}^{\#} \oplus \mathcal{X}$.*

Proof Let f_1 be an arbitrary non-zero vector in $\mathcal{Y}$. Since ω is non-degenerate, we can find a vector e_1 such that $f_1 \cdot \omega e_1 = 1$. f_1 is not proportional to e_1, because ω is anti-symmetric. Let $\mathcal{Y}_1 = \{y \in \mathcal{Y} \ : \ y \cdot \omega f_1 = y \cdot \omega e_1 = 0\}$. Then $\dim \mathcal{Y}_1 = \dim \mathcal{Y} - 2$. We continue our construction in $\mathcal{Y}_1$, finding vectors f_2, e_2 etc.

In the end we set $\tau = \mathbb{1}$ on $\mathrm{Span}\{f_1, \dots, f_d\}$ and $\tau = -\mathbb{1}$ on $\mathrm{Span}\{e_1, \dots, e_d\}$. □

1.2 Complex vector spaces

Throughout the section, $\mathcal{Z}, \mathcal{W}$ are complex vector spaces.

1.2.1 Anti-linear operators

Definition 1.48 *Let a be a map from $\mathcal{Z}$ to $\mathcal{W}$. We say that it is* anti-linear *if it is linear over $\mathbb{R}$ and* $\mathrm{i}a = -a\mathrm{i}$.

Definition 1.49 *Let a be anti-linear from $\mathcal{Z}$ to $\mathcal{W}$. The* transpose of a *is the operator in $L(\mathcal{W}^{\#}, \mathcal{Z}^{\#})$ defined by*

$$\langle a^{\#} v|y\rangle := \langle v|ay\rangle, \quad v \in \mathcal{Y}_2^{\#}, \quad y \in \mathcal{Y}_1. \tag{1.13}$$

Note that the transpose of an anti-linear operator is also anti-linear.

1.2.2 Internal conjugations

Definition 1.50 *An anti-linear map χ on $\mathcal{Z}$ such that $\chi^2 = \mathbb{1}$ is called an* (internal) conjugation. *The subspace $\mathcal{Z}^{\chi} := \{z \in \mathcal{Z} \ : \ \chi z = z\}$ is sometimes called a* real form *of $\mathcal{Z}$. According to an alternative terminology, $\mathcal{Z}^{\chi}$ is called the* real subspace *and $\mathcal{Z}^{-\chi} := \{z \in \mathcal{Z} \ : \ \chi z = -z\}$ the* imaginary subspace (for χ).

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Definition 1.51 *Operators $a \in L(\mathcal{Z}, \mathcal{W})$ satisfying $a = \chi a \chi$ will be sometimes called* real *(for χ).*

Clearly, the space of real operators can be identified with $L(\mathcal{Z}^\chi, \mathcal{W}^\chi)$.

Sometimes, an internal conjugation will be denoted by $\overline{z}$ instead of χz. In such a case, if $a \in L(\mathcal{Z})$, we will write $\overline{a}$ for $\chi a \chi$.

1.2.3 Complex conjugate spaces

In this subsection we discuss the external approach to the complex conjugation. This is a very simple and elementary subject, which, however, can be a little confusing.

Definition 1.52 *$\overline{\mathcal{Z}}$ will denote a complex space equipped with an anti-linear isomorphism*

$$\mathcal{Z} \ni z \mapsto \overline{z} \in \overline{\mathcal{Z}}. \tag{1.14}$$

We will call $\overline{\mathcal{Z}}$ the space complex conjugate to $\mathcal{Z}$. *We will use the convention that the inverse of (1.14) is denoted by the same symbol, so that $\overline{\overline{z}} = z$, $z \in \mathcal{Z}$ and $\overline{\overline{\mathcal{Z}}} = \mathcal{Z}$.*

In practice, one often uses one of the following two concrete realizations of the complex conjugate space.

The first approach is the most canonical (it does not introduce additional structure). We set $\overline{\mathcal{Z}}$ to be equal to $\mathcal{Z}$ as a real vector space. The map $\mathcal{Z} \ni z \mapsto \overline{z} \in \overline{\mathcal{Z}}$ is just the identity. One defines the multiplication by $\lambda \in \mathbb{C}$ on $\overline{\mathcal{Z}}$ as

$$\overline{\lambda}\overline{z} := \overline{\lambda z}, \quad z \in \mathcal{Z}, \ \lambda \in \mathbb{C}.$$

In the second approach, we choose $\overline{\mathcal{Z}} = \mathcal{Z}$ as complex vector spaces and we fix an internal conjugation χ. Then we set $\overline{z} := \chi z$. Thus we are back in the framework of Subsect. 1.2.2.

Definition 1.53 *If $a \in L(\mathcal{Z}, \mathcal{W})$, then one defines $\overline{a} \in L(\overline{\mathcal{Z}}, \overline{\mathcal{W}})$ by*

$$\overline{a}\,\overline{z} := \overline{az}. \tag{1.15}$$

The map $L(\mathcal{Z}, \mathcal{W}) \ni a \mapsto \overline{a} \in L(\overline{\mathcal{Z}}, \overline{\mathcal{W}})$ is an anti-linear isomorphism which allows us to identify $L(\overline{\mathcal{Z}}, \overline{\mathcal{W}})$ and $\overline{L(\mathcal{Z}, \mathcal{W})}$ as complex vector spaces.

Sometimes the notation $z \mapsto \overline{z}$ is inconvenient for typographical reasons, and we will denote the complex conjugation by a letter, e.g. χ. Thus $\chi : \mathcal{Z} \to \overline{\mathcal{Z}}$ is a fixed anti-linear map and we write χz for $\overline{z}$.

In particular, if $a \in L(\mathcal{Z}_1, \mathcal{Z}_2)$, and the conjugations $\mathcal{Z}_i \to \overline{\mathcal{Z}}_i$ are denoted by χ_i, then $\overline{a} = \chi_2 a \chi_1^{-1}$.

A typical situation when this alternative notation is more convenient is the following. Suppose that b is an anti-linear map from $\mathcal{Z}_1$ to $\mathcal{Z}_2$. Then, instead of b, it may be more convenient to use one of the following two *linear* maps:

$$b\chi_1^{-1} \in L(\overline{\mathcal{Z}}_1, \mathcal{Z}_2), \quad \text{or} \quad \chi_2 b \in L(\mathcal{Z}_1, \overline{\mathcal{Z}}_2). \tag{1.16}$$

Note that b is a conjugation on $\mathcal{Z}$ iff the linear map $a := b\chi^{-1} \in L(\overline{\mathcal{Z}}, \mathcal{Z})$ satisfies

$$\overline{a}a = 1\!\mathrm{l}.$$

1.2.4 Anti-linear functionals

If $\overline{w} \in \overline{\mathcal{Z}^{\#}}$, we let it act on $\overline{\mathcal{Z}}$ as

$$\langle \overline{w} | \overline{z} \rangle := \overline{\langle w | z \rangle},\ z \in \mathcal{Z}.$$

This identifies $\overline{\mathcal{Z}^{\#}}$ with $\overline{\mathcal{Z}}^{\#}$. (This is a special case of (1.15) for $\mathcal{W} = \mathbb{C}$).

Definition 1.54 *The* anti-dual of $\mathcal{Z}$ *is defined as*

$$\mathcal{Z}^* := \overline{\mathcal{Z}}^{\#}.$$

Thus $\mathcal{Z}^*$ is the space $\overline{L(\mathcal{Z}, \mathbb{C})}$ of anti-linear functionals *on $\mathcal{Z}$. Several kinds of notation for the action of $w \in \mathcal{Z}^*$ on $z \in \mathcal{Z}$ will be used:*

(1) *the* bra–ket notation $(z|w) = \langle \overline{z} | w \rangle = \langle w | \overline{z} \rangle$,
(2) *the* simplified notation $\overline{z} \cdot w = w \cdot \overline{z}$,
(3) *the* functional notation $w(z)$.

Since $\overline{\mathcal{Z}^{\#}} = \overline{\mathcal{Z}}^{\#}$, we see that $\mathcal{Z}^{**} = \mathcal{Z}^{\#\#}$, so that $\mathcal{Z} \subset \mathcal{Z}^{**}$ and in the finite-dimensional case $\mathcal{Z} = \mathcal{Z}^{**}$.

Remark 1.55 *We will consistently use the following convention. The round brackets in a pairing of two vectors will indicate that the expression depends anti-linearly on the first argument and linearly on the second argument. In the case of the angular brackets the dependence on both arguments will always be linear, in both the real and the complex case.*

1.2.5 Adjoint of an operator

Let $a \in L(\mathcal{Z}_1, \mathcal{Z}_2)$.

Definition 1.56 *We define the* adjoint of a, *denoted by $a^* \in L(\mathcal{Z}_2^*, \mathcal{Z}_1^*)$, by*

$$(a^* w_2 | z_1) := (w_2 | a z_1),\quad w_2 \in \mathcal{Z}_2^*,\quad z_1 \in \mathcal{Z}_1. \tag{1.17}$$

We see that

$$a^* = \overline{a}^{\#} = \overline{a^{\#}},\quad a \subset a^{**}. \tag{1.18}$$

Definition 1.57 *Let a be an anti-linear map from $\mathcal{Z}_1$ to $\mathcal{Z}_2$. The* adjoint of a, *instead of by (1.17), is defined by*

$$(z_1 | a^* w_2) = (w_2 | a z_1),$$
$$\text{or, equivalently,}\quad \overline{(a^* w_2 | z_1)} = (w_2 | a z_1),\quad w_2 \in \mathcal{Z}_2^*,\quad z_1 \in \mathcal{Z}_1. \tag{1.19}$$

It is an anti-linear operator from $\mathcal{Z}_2^*$ to $\mathcal{Z}_1^*$ satisfying (1.18).

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1.2.6 Anti-dual pairs

Definition 1.58 *An* anti-dual pair *is a pair* $(\mathcal{W}, \mathcal{Z})$ *of complex vector spaces equipped with a form*

$$(\mathcal{W}, \mathcal{Z}) \ni (w, z) \mapsto (w|z) \in \mathbb{C}$$

anti-linear in $\mathcal{W}$ *and linear in* $\mathcal{Z}$ *such that*

$$\begin{aligned} (w|z) = 0,\ w \in \mathcal{V} \ &\Rightarrow\ z = 0, \\ (w|z) = 0,\ z \in \mathcal{Z} \ &\Rightarrow\ w = 0. \end{aligned}$$

Properties of anti-dual pairs are obvious analogs of the properties of dual pairs. For instance, if $\mathcal{Z}$ is finite-dimensional and $(\mathcal{W}, \mathcal{Z})$ is a dual pair, then $\mathcal{W}$ is naturally isomorphic to $\mathcal{Z}^*$.

1.2.7 Sesquilinear forms

Definition 1.59 *Elements of* $L(\mathcal{Z}, \mathcal{Z}^*)$ *will be called* sesquilinear forms.

Let $\beta \in L(\mathcal{Z}, \mathcal{Z}^*)$. β determines a map

$$\mathcal{Z} \times \mathcal{Z} \ni (z_1, z_2) \mapsto (z_1|\beta z_2) = \overline{z_1} \cdot \beta z_2 \in \mathbb{C} \tag{1.20}$$

anti-linear in the first argument and linear in the second argument.

Definition 1.60 *We say that* β *is* non-degenerate *if* $\operatorname{Ker} \beta = \{0\}$.

Definition 1.61 *An operator* $r \in L(\mathcal{Z})$ preserves β *if*

$$r^* \beta r = \beta, \text{ i.e.} \quad (r z_1|\beta r z_2) = (z_1|\beta z_2),\ z_1, z_2 \in \mathcal{Z}.$$

An operator $a \in L(\mathcal{Z})$ infinitesimally preserves β *if*

$$a^* \beta + \beta a = 0, \text{ i.e.} \quad (a z_1|\beta z_2) = -(z_1|\beta a z_2),\ z_1, z_2 \in \mathcal{Z}.$$

Remark 1.62 *Note that we adopt the so-called* physicist's convention *for sesquilinear forms. A part of the mathematical community adopts the reverse convention: they assume sesquilinear forms to be linear in the first and anti-linear in the second argument.*

Remark 1.63 *We will use three kinds of notation for sesquilinear forms:*

(1) *the* bra–ket notation $(z_1|\beta z_2)$, *going back to Dirac,*
(2) *the* simplified notation $\overline{z}_1 \cdot \beta z_2$,
(3) *the* functional notation $\beta(\overline{z}_1, z_2)$.

Note that in all cases the notation indicates that the form is sesquilinear and not bilinear: by the use of round instead of angular brackets in the first case, and by the use of the bar in the remaining cases. Usually, we will prefer the first two notations, both given in (1.20).

1.2.8 Hermitian forms

Let $\beta \in L(\mathcal{Z}, \mathcal{Z}^*)$.

Definition 1.64 *We will say that*

$$\begin{aligned} \beta \text{ is Hermitian} \quad & \text{if } \beta \subset \beta^*, \quad \text{i.e. } (z_2|\beta z_1) = \overline{(z_1|\beta z_2)},\ z_1, z_2 \in \mathcal{Z}, \\ & \text{or equivalently } (z|\beta z) \in \mathbb{R},\ z \in \mathcal{Z}; \\ \beta \text{ is anti-Hermitian} \quad & \text{if } \beta \subset -\beta^*, \quad \text{i.e. } (z_2|\beta z_1) = -\overline{(z_1|\beta z_2)},\ z_1, z_2 \in \mathcal{Z}, \\ & \text{or equivalently } (z|\beta z) \in \mathrm{i}\mathbb{R},\ z \in \mathcal{Z}. \end{aligned}$$

Clearly, β is Hermitian iff $\mathrm{i}\beta$ is anti-Hermitian.

Definition 1.65 *The space of all Hermitian elements of $L(\mathcal{Z}, \mathcal{Z}^*)$ will be denoted $L_{\mathrm{h}}(\mathcal{Z}, \mathcal{Z}^*)$. Such operators are also called* Hermitian forms.

If $\mathcal{Z}$ is finite-dimensional then $\beta \in L_{\mathrm{h}}(\mathcal{Z}, \mathcal{Z}^*)$ iff $\beta^* = \beta$.

Definition 1.66 *A Hermitian form β is called* positive semi-definite *if $(z|\beta z) \geq 0$ for $z \in \mathcal{Z}$. It is called* positive definite *if $(z|\beta z) > 0$ for $z \neq 0$. A positive definite form is also often called a* scalar product.

Positive definite forms are always non-degenerate.

If $\beta \in L_{\mathrm{h}}(\mathcal{Z}, \mathcal{Z}^*)$ is non-degenerate, then $(\beta\mathcal{Z}, \mathcal{Z})$ is an anti-dual pair. Hence, we can define $\beta^{-1} \in L_{\mathrm{h}}(\beta\mathcal{Z}, \mathcal{Z}) \subset L_{\mathrm{h}}(\beta\mathcal{Z}, (\beta\mathcal{Z})^*)$. (Note that $\mathcal{Z} \subset (\beta\mathcal{Z})^*$.) The form β^{-1} is non-degenerate and is positive definite iff β is positive definite.

1.2.9 (Pseudo-)unitary spaces

Definition 1.67 *A couple $(\mathcal{Z}, \beta)$, where $\beta \in L_{\mathrm{h}}(\mathcal{Z}, \mathcal{Z}^*)$ is non-degenerate, is called a* pseudo-unitary space. *If β is positive definite, then $(\mathcal{Z}, \beta)$ is called a* unitary space. *In such a case we can define the norm of $z \in \mathcal{Z}$ denoted by $\|z\| := \sqrt{(y|\beta y)}$. If $\mathcal{Z}$ is complete for this norm, it is called a* Hilbert space.

Note that the notion of a pseudo-unitary space is closely related to that of a *charged symplectic space*, which is defined later, in Subsect. 1.2.11.

Let $(\mathcal{Z}, \beta)$ be a pseudo-unitary space.

Definition 1.68 *If $\mathcal{U} \subset \mathcal{Z}$, then $\mathcal{U}^{\beta\perp}$ denotes the β-orthogonal complement of $\mathcal{U}$:*

$$\mathcal{U}^{\beta\perp} := \big\{ z \in \mathcal{Z} \ :\ (u|\beta z) = 0,\ u \in \mathcal{U} \big\}.$$

Definition 1.69 *Let $(\mathcal{Z}, \beta)$ be a unitary, pseudo-unitary, resp. charged symplectic space. Then $\overline{\mathcal{Z}}$ has a natural unitary, pseudo-unitary, resp. charged symplectic structure:*

$$(\overline{z}_1|\overline{\beta}\overline{z}_2) := \overline{(z_1|\beta z_2)}.$$

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Definition 1.70 *A non-degenerate Hermitian form, especially if it is positive definite, is often called a* scalar product *and denoted* $(z_1|z_2)$ *or* $\overline{z}_1 \cdot z_2$. *In such a case, the orthogonal complement of* $\mathcal{U}$ *is denoted* $\mathcal{U}^\perp$. *For* $w \in \mathcal{Z}$, $(w|$ *will denote the following operator:*

$$\mathcal{Z} \ni z \mapsto (w|z := (w|z) \in \mathbb{C}.$$

For example, if $(w|w) = 1$, then $|w)(w|$ is the orthogonal projection onto w.

Most unitary spaces considered in our work will be (complex) Hilbert spaces. Hilbert spaces will be further discussed in Subsect. 2.2.2.

1.2.10 Group $U(\mathcal{Z})$ *and Lie algebra* $u(\mathcal{Z})$

Let $(\mathcal{Z}, \beta)$ be an unitary space and $a \in L(\mathcal{Z})$.

Definition 1.71 *We say that*

$$\begin{array}{rl} a \textit{ is } \text{isometric} & \textit{if } a^*\beta a = \beta, \\ a \textit{ is } \text{unitary} & \textit{if } a \textit{ is isometric and bijective}, \\ a \textit{ is } \text{self-adjoint} & \textit{if } a^*\beta = \beta a, \\ a \textit{ is } \text{anti-self-adjoint} & \textit{if } a^*\beta = -\beta a. \end{array}$$

The set of unitary operators on $\mathcal{Z}$ *is a group for the operator composition denoted by* $U(\mathcal{Z})$. *The space of anti-self-adjoint operators on* $\mathcal{Z}$, *denoted by* $u(\mathcal{Z})$, *is a Lie algebra equipped with the usual commutator.*

Let b be an anti-linear operator on $\mathcal{Z}$.

Definition 1.72 *We say that*

$$\begin{array}{rl} b \textit{ is } \text{anti-unitary} & \textit{if } b^*\beta b = \beta \textit{ and } a \textit{ is bijective}, \\ b \textit{ is a } \text{conjugation} & \textit{if it is an anti-unitary involution}. \end{array}$$

Recall from Subsect. 1.2.3 that we sometimes use two alternative symbols for the complex conjugation: χ and the “bar”.

Clearly, b is anti-unitary iff $\chi b : \mathcal{Z} \to \overline{\mathcal{Z}}$ is unitary.

If $\mathcal{Z}$ is a pseudo-unitary space, we can repeat Subsect. 1.2.10, replacing the terms isometric, unitary, anti-self-adjoint and self-adjoint with *pseudo-isometric, pseudo-unitary, anti-pseudo-self-adjoint* and *pseudo-self-adjoint.*

1.2.11 Charged symplectic spaces

Definition 1.73 *If* ω *is anti-Hermitian and non-degenerate, then* $(\mathcal{Z}, \omega)$ *is called a* charged symplectic space.

Note that the difference between a pseudo-unitary and charged symplectic space is minor (passing from β to $\omega = \mathrm{i}\beta$ changes a pseudo-unitary space into a charged symplectic space). We will, however, more often use the framework of a charged symplectic space. The terminology in this case is somewhat different.

Let $(\mathcal{Z}, \omega)$ be a charged symplectic space and $a \in L(\mathcal{Z})$.

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Definition 1.74 *We say that*

$$\begin{array}{rl} a \text{ preserves } \omega & \textit{if } a^*\omega a = \omega, \\ a \text{ anti-preserves } \omega & \textit{if } a^*\omega a = -\omega, \\ a \textit{ is } \text{charged symplectic} & \textit{if } a \textit{ preserves } \omega \textit{ and is bijective}, \\ a \textit{ is } \text{charged anti-symplectic} & \textit{if } a \textit{ anti-preserves } \omega \textit{ and is bijective}, \\ a \textit{ is } \text{infinitesimally charged symplectic} & \textit{if } a^*\omega = -\omega a. \end{array}$$

The set of charged symplectic operators on $\mathcal{Z}$ is a group for the operator composition denoted by $ChSp(\mathcal{Z})$. The space of infinitesimally charged symplectic operators on $\mathcal{Z}$, denoted by $chsp(\mathcal{Z})$, is a Lie algebra equipped with the usual commutator.

Let a be an anti-linear operator on $\mathcal{Z}$.

Definition 1.75 *We say that*

$$\begin{array}{rl} a \text{ preserves } \omega & \textit{if } a^*\omega a = \omega, \textit{ or } (z_1|\omega z_2) = \overline{(az_1|\omega az_2)}, \\ a \text{ anti-preserves } \omega & \textit{if } a^*\omega a = -\omega, \textit{ or } (z_1|\omega z_2) = -\overline{(az_1|\omega az_2)}, \\ a \textit{ is } \text{anti-charged symplectic} & \textit{if } a \textit{ preserves } \omega \textit{ and is bijective}, \\ a \textit{ is } \text{anti-charged anti-symplectic} & \textit{if } a \textit{ anti-preserves } \omega \textit{ and is bijective}. \end{array}$$

Remark 1.76 *The terminology "charged symplectic space" is motivated by applications in quantum field theory: such spaces describe charged bosons.*

1.3 Complex structures

When we quantize a classical system, the phase space is often naturally equipped with more than one complex structure. Therefore, it is useful to develop this concept in more detail.

Besides complex structures, in this section we discuss the so-called (pseudo-) Kähler spaces, which can be described as (pseudo-)unitary spaces treated as real spaces.

1.3.1 Anti-involutions

Let $\mathcal{Y}$ be a vector space.

Definition 1.77 *We say that* $\mathrm{j} \in L(\mathcal{Y})$ *is an* anti-involution *if* $\mathrm{j}^2 = -\mathbb{1}$.

If $\mathcal{Y}$ is a real vector space with an anti-involution j, then $\mathcal{Y}$ can be naturally endowed with the structure of a complex space:

$$(\lambda + \mathrm{i}\mu)y := \lambda y + \mu \mathrm{j} y, \quad y \in \mathcal{Y}, \quad \lambda, \mu \in \mathbb{R}. \tag{1.21}$$

Therefore, anti-involutions on real spaces are often called *complex structures.*

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Definition 1.78 *$\mathcal{Y}$ converted into a vector space over $\mathbb{C}$ with the multiplication (1.21) will be denoted $\mathcal{Y}^{\mathbb{C}}$, or by $(\mathcal{Y}^{\mathbb{C}}, \mathrm{j})$ if we need to specify the complex structure that we use. It will be called a* complex form of $\mathcal{Y}$.

Definition 1.79 *Conversely, any complex space $\mathcal{W}$ can be considered as a real vector space, called the* realification of $\mathcal{W}$ *and denoted $\mathcal{W}_{\mathbb{R}}$. It is equipped with an anti-involution $\mathrm{j} \in L(\mathcal{W}_{\mathbb{R}})$ (the multiplication by the complex number i).*

Let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be real spaces with anti-involutions j_1, j_2. Then

$$L(\mathcal{Y}_1^{\mathbb{C}}, \mathcal{Y}_2^{\mathbb{C}}) = \big\{ a \in L(\mathcal{Y}_1, \mathcal{Y}_2) \;:\; a\mathrm{j}_1 = \mathrm{j}_2 a \big\}.$$

1.3.2 Conjugations on a space with an anti-involution

Let $\mathcal{Y}$ be a vector space equipped with an anti-involution $\mathrm{j} \in L(\mathcal{Y})$.

Definition 1.80 *We say that $\chi \in L(\mathcal{Y})$ is a* conjugation *if it is an involution and $\mathrm{j}\chi = -\chi\mathrm{j}$.*

Recall that χ determines a decomposition $\mathcal{Y} = \mathcal{Y}^{\chi} \oplus \mathcal{Y}^{-\chi}$ (see Def. 1.39). Let us write $\mathcal{X} := \mathcal{Y}^{-\chi}$. Then $\mathrm{j}\mathcal{X} = \mathcal{Y}^{\chi}$. The map

$$\mathcal{Y} \ni y \mapsto \left(\mathrm{j}\frac{\mathbb{1} + \chi}{2} y, \frac{\mathbb{1} - \chi}{2} y \right) \in \mathcal{X} \oplus \mathcal{X} \tag{1.22}$$

is bijective. Thus $\mathcal{Y}$ can be identified with $\mathcal{X} \oplus \mathcal{X}$, so that

$$\mathrm{j} = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}, \quad \chi = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}.$$

$r \in L(\mathcal{X} \oplus \mathcal{X})$ commutes with j iff it is of the form

$$r = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \tag{1.23}$$

for $a, b \in L(\mathcal{X})$.

r commutes with both j and χ iff

$$r = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \tag{1.24}$$

for $a \in L(\mathcal{X})$.

1.3.3 Complexification

Let $\mathcal{X}$ be a real vector space

Definition 1.81 *The* complexification of $\mathcal{X}$, *denoted by $\mathbb{C}\mathcal{X}$, is the complex vector space $(\mathcal{X} \oplus \mathcal{X})^{\mathbb{C}}$, equipped with the anti-involution given by $\begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}$,*

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which will be denoted simply by i. $\mathbb{C}\mathcal{X}$ *is also equipped with the conjugation* χ *given by* $\begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$. *According to the convention in Subsect. 1.2.3, we will usually write* $\overline{z} := \chi z$, $z \in \mathbb{C}\mathcal{X}$.

Note that $L(\mathbb{C}\mathcal{X})$, in the representation $\mathcal{X} \oplus \mathcal{X}$, consists of matrices of the form (1.23).

Let $a \in L(\mathcal{X})$.

Definition 1.82 *We set*

$$a_{\mathbb{C}} := \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \qquad a_{\overline{\mathbb{C}}} := \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}. \tag{1.25}$$

$a_{\mathbb{C}}$, resp. $a_{\overline{\mathbb{C}}}$, is the unique (complex) linear, resp. anti-linear extension of a to an operator on $\mathbb{C}\mathcal{X}$. Often, we simply write a instead of $a_{\mathbb{C}}$.

1.3.4 Complexification of a Euclidean space

Let $(\mathcal{X}, \nu)$ be a Euclidean space. Then the scalar product in $\mathcal{X}$ has two natural extensions to $\mathbb{C}\mathcal{X}$: if $w_i = (x_i + \mathrm{i}y_i) \in \mathbb{C}\mathcal{X}$, $i = 1, 2$, we can define the *bilinear form*

$$w_1 \cdot \nu_{\mathbb{C}} w_2 := x_1 \cdot \nu x_2 - y_1 \cdot \nu y_2 + \mathrm{i}x_1 \cdot \nu y_2 + \mathrm{i}y_1 \cdot \nu x_2$$

and the *sesquilinear form*

$$(w_1|w_2) = \overline{w_1} \cdot \nu_{\mathbb{C}} w_2 := x_1 \cdot \nu x_2 + y_1 \cdot \nu y_2 + \mathrm{i}x_1 \cdot \nu y_2 - \mathrm{i}y_1 \cdot \nu x_2.$$

We will more often use the latter. It makes $\mathbb{C}\mathcal{X}$ into a unitary space. The canonical conjugation χ defined in Subsect. 1.3.3 is anti-unitary. We also see that if $r \in O(\mathcal{X})$, resp. $r \in o(\mathcal{X})$, then $r_{\mathbb{C}} \in U(\mathbb{C}\mathcal{X})$, resp. $r_{\mathbb{C}} \in u(\mathbb{C}\mathcal{X})$.

Assume now that $(\mathcal{W}, (\cdot|\cdot))$ is a unitary space and that χ is a conjugation on $\mathcal{X}$ in the sense of Subsect. 1.2.8. Let $\mathcal{X} := \mathcal{W}^{\chi}$ as in Subsect. 1.3.2. Then $\mathcal{X}$ equipped with $y_1 \cdot \nu y_2 := (y_1|y_2)$ is a Euclidean space. The identification of $\mathcal{X} \oplus \mathcal{X} \simeq \mathbb{C}\mathcal{X}$ with $\mathcal{W}$ as complex spaces defined in Subsect. 1.3.2 is unitary from $(\mathbb{C}\mathcal{X}, (\cdot|\cdot))$ to $(\mathcal{W}, (\cdot|\cdot))$.

1.3.5 Complexification of a symplectic space

Let $(\mathcal{X}, \omega)$ be a symplectic space. Then $\mathbb{C}\mathcal{X}$ can be equipped with the non-degenerate anti-symmetric form ω defined for $w_i = (x_i + \mathrm{i}y_i) \in \mathbb{C}\mathcal{X}$, $i = 1, 2$, by

$$w_1 \cdot \omega_{\mathbb{C}} w_2 := x_1 \cdot \omega x_2 - y_1 \cdot \omega y_2 + \mathrm{i}x_1 \cdot \omega y_2 + \mathrm{i}y_1 \cdot \omega x_2,$$

as well as a charged symplectic form

$$\overline{w}_1 \cdot \omega_{\mathbb{C}} w_2 := x_1 \cdot \omega x_2 + y_1 \cdot \omega y_2 + \mathrm{i}x_1 \cdot \omega y_2 - \mathrm{i}y_1 \cdot \omega x_2.$$

where $w_i = (x_i + \mathrm{i}y_i)$, $i = 1, 2$.

1.3.6 Holomorphic and anti-holomorphic subspaces

Assume that a real space $\mathcal{Y}$ is equipped with an anti-involution $\mathrm{j} \in L(\mathcal{Y})$. Thus $(\mathbb{C}\mathcal{Y})_{\mathbb{R}}$ has two distinguished anti-involutions: the usual i, and also $\mathrm{j}_{\mathbb{C}}$.

Definition 1.83 *Set*

$$\mathcal{Z} := \{y - \mathrm{ij}y \ : \ y \in \mathcal{Y}\}.$$

$\mathcal{Z}$ *will be called the* holomorphic subspace of $\mathbb{C}\mathcal{Y}$.

$$\overline{\mathcal{Z}} := \{y + \mathrm{ij}y \ : \ y \in \mathcal{Y}\}$$

will be called the anti-holomorphic subspace of $\mathbb{C}\mathcal{Y}$.

The corresponding projections are $\mathbb{1}_{\mathcal{Z}} := \frac{1}{2}(\mathbb{1} - \mathrm{ij}_{\mathbb{C}})$ and $\mathbb{1}_{\overline{\mathcal{Z}}} := \frac{1}{2}(\mathbb{1} + \mathrm{ij}_{\mathbb{C}})$. Clearly, $\mathbb{1} = \mathbb{1}_{\mathcal{Z}} + \mathbb{1}_{\overline{\mathcal{Z}}}$, and $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. We have $\mathcal{Z} = \mathrm{Ker}(\mathrm{j}_{\mathbb{C}} - \mathrm{i})$, $\overline{\mathcal{Z}} = \mathrm{Ker}(\mathrm{j}_{\mathbb{C}} + \mathrm{i})$, thus on $\mathcal{Z}$ the complex structures i and $\mathrm{j}_{\mathbb{C}}$ coincide, whereas on $\overline{\mathcal{Z}}$ they are opposite.

The canonical conjugation on $\mathbb{C}\mathcal{Y}$ is bijective from $\mathcal{Z}$ to $\overline{\mathcal{Z}}$, which shows that we can treat $(\overline{\mathcal{Z}}, \mathrm{i})$ as the conjugate vector space $\overline{(\mathcal{Z}, \mathrm{i})}$.

Using the decomposition

$$\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}, \tag{1.26}$$

we can write

$$\mathrm{i} = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}, \quad \mathrm{j}_{\mathbb{C}} = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}.$$

The converse construction is as follows: Let $(\mathcal{Z}, \mathrm{i})$ be a complex vector space. Set

$$\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) := \big\{(z, \overline{z}) \in \mathcal{Z} \oplus \overline{\mathcal{Z}} \ : \ z \in \mathcal{Z}\big\}.$$

Clearly, $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ is a real vector space. It can be equipped with the anti-involution

$$\mathrm{j}(z, \overline{z}) := (\mathrm{i}z, \overline{\mathrm{i}z}) = (\mathrm{i}z, -\mathrm{i}\overline{z}).$$

We identify $\mathbb{C}\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ with $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ as follows: if $y_i = (z_i, \overline{z_i}) \in \mathcal{Y}$ for $i = 1, 2$, then

$$\mathbb{C}\mathcal{Y} \ni y_1 + \mathrm{i}y_2 \mapsto (z_1 + \mathrm{i}z_2, \overline{z}_1 + \mathrm{i}\overline{z}_2) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}. \tag{1.27}$$

With this identification we have

$$\mathrm{j}_{\mathbb{C}} \simeq \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix},$$

which shows that this is the converse construction.

$\mathcal{Z} \oplus \overline{\mathcal{Z}}$ is equipped with a conjugation

$$\epsilon(z_1, \overline{z}_2) := (\overline{z}_2, z_1).$$

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Note that $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ is the real subspace of $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ for the conjugation ϵ. Clearly, under the identification (1.27), ϵ coincides with the usual complex conjugation on $\mathbb{C}\mathcal{Y}$.

Often it is convenient to identify the space $\mathcal{Z}$ with $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) = \mathcal{Y}$.

Definition 1.84 *For any $\lambda \neq 0$, we introduce an identification between a space with an anti-involution and the corresponding holomorphic space:*

$$\mathcal{Y} \ni y \mapsto T_\lambda y = \lambda \frac{\mathbb{1} - \mathrm{ij}}{2} y \in \mathcal{Z}. \tag{1.28}$$

The inverse map is

$$\mathcal{Z} \ni z \mapsto T_\lambda^{-1} z := \frac{1}{\lambda}(z + \overline{z}) \in \mathcal{Y}. \tag{1.29}$$

In the literature one can find at least two special cases of these identifications: for $\lambda = 1$ and for $\lambda = \sqrt{2}$. Each one has its own advantages. Note that in the bosonic case, we will typically use the identification $T_{\sqrt{2}}$, and in the fermionic case, the identification T_1. The arguments in favor of $T_{\sqrt{2}}$ will be given in Subsect. 1.3.9.

Let us discuss an argument in favor of T_1. Consider the natural projection from $\mathbb{C}\mathcal{Y}$ onto $\mathcal{Y}$:

$$\mathbb{C}\mathcal{Y} \ni w \mapsto \frac{w + \overline{w}}{2} + \mathrm{j}\frac{w - \overline{w}}{2\mathrm{i}} \in \mathcal{Y}. \tag{1.30}$$

Then

$$\mathcal{Z} \ni z \mapsto T_1^{-1} z = z + \overline{z} \in \mathcal{Y} \tag{1.31}$$

is the restriction of (1.30) to $\mathcal{Z}$.

T_1 appears naturally in the following context. Suppose that we have a function $\mathcal{Z} \ni z \mapsto F(z) \in \mathbb{C}$. One often prefers to move its domain onto $\mathcal{Y}$ by considering

$$\mathcal{Y} \ni (z, \overline{z}) \mapsto F(T_1(z, \overline{z})) = F(z). \tag{1.32}$$

Abusing notation, one can denote (1.32) by $F(z, \overline{z})$. This notation is especially common in the literature if F is not holomorphic.

Let us assume for a moment that $\mathcal{Y}$ is a complex space. We can realify $\mathcal{Y}$, and then complexify it, obtaining $\mathbb{C}\mathcal{Y}_\mathbb{R}$. Denote the original imaginary unit of $\mathcal{Y}$ by j. Introducing $\mathcal{Z}$ and identifying it with $\mathcal{Y}$ with help of T_1 we can write

$$\mathbb{C}\mathcal{Y}_\mathbb{R} \simeq \mathcal{Y} \oplus \overline{\mathcal{Y}}. \tag{1.33}$$

1.3.7 Operators on a space with an anti-involution

Let $\mathcal{Y}$ be a real space with an anti-involution j. Let $\mathcal{Z}$, $\overline{\mathcal{Z}}$ be the holomorphic and anti-holomorphic spaces defined in Subsect. 1.3.6. Let us collect the form of various operators on $\mathbb{C}\mathcal{Y}$ after the identification of $\mathbb{C}\mathcal{Y}$ with $\mathcal{Z} \oplus \overline{\mathcal{Z}}$.

We have

$$\epsilon = \begin{bmatrix} 0 & \chi \\ \overline{\chi} & 0 \end{bmatrix}, \quad \mathrm{j}_{\mathbb{C}} = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \quad \mathrm{i} = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}.$$

where $\mathcal{Z} \ni z \mapsto \epsilon z := \overline{z} \in \overline{\mathcal{Z}}$.

An operator in $L(\mathbb{C}\mathcal{Y})$ is of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a \in L(\mathcal{Z})$, $b \in L(\overline{\mathcal{Z}}, \mathcal{Z})$, $c \in L(\mathcal{Z}, \overline{\mathcal{Z}})$, $d \in L(\overline{\mathcal{Z}})$.

An operator in $L(\mathbb{C}\mathcal{Y})$ equal to $r_{\mathbb{C}}$ for some $r \in L(\mathcal{Y})$ is of the form

$$\begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix},$$

where $p \in L(\mathcal{Z})$, $q \in L(\overline{\mathcal{Z}}, \mathcal{Z})$.

Finally an operator $L(\mathbb{C}\mathcal{Y})$ equal to $r_{\mathbb{C}}$ for $r \in L(\mathcal{Y}^{\mathbb{C}})$ (which means that $[r, \mathrm{j}] = 0$) is of the form

$$\begin{bmatrix} p & 0 \\ 0 & \overline{p} \end{bmatrix},$$

for $p \in L(\mathcal{Z})$.

1.3.8 (Pseudo-)Kähler spaces

Let $(\mathcal{Y}, (\cdot|\cdot))$ be a (pseudo-)unitary space. Then $\mathcal{Y}_{\mathbb{R}}$ is a (pseudo-)Euclidean space for the scalar product

$$y_2 \cdot \nu y_1 := \mathrm{Re}(y_2|y_1), \tag{1.34}$$

a symplectic space for the symplectic form

$$y_2 \cdot \omega y_1 := \mathrm{Im}(y_2|y_1), \tag{1.35}$$

and has an anti-involution

$$\mathrm{j}y := \mathrm{i}y. \tag{1.36}$$

The name "*(pseudo-)Kähler space*" is used for a unitary space treated as a real space with the three structures (1.34), (1.35) and (1.36). Below we give a more precise definition:

Definition 1.85 *We say that a quadruple* $(\mathcal{Y}, \nu, \omega, \mathrm{j})$ *is a* pseudo-Kähler space *if*

(1) $\mathcal{Y}$ *is a real vector space,*
(2) ν *is a non-degenerate symmetric form,*
(3) ω *is a symplectic form,*
(4) j *is an anti-involution,*
(5) $\omega\mathrm{j} = \nu$.

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If in addition ν is positive definite, then we say that $(\mathcal{Y}, \nu, \omega, \mathrm{j})$ is a Kähler space*.*

Definition 1.86 *If $(\mathcal{Y}, \nu, \omega, \mathrm{j})$ is a (pseudo-)Kähler space, we set*

$$(y_1|y_2) := y_1 \cdot \nu y_2 + \mathrm{i} y_1 \cdot \omega y_2. \tag{1.37}$$

Then $(\mathcal{Y}^{\mathbb{C}}, (\cdot|\cdot))$ is a (pseudo-)unitary space.

Definition 1.87 *If a Kähler space $\mathcal{Y}$ is complete for the norm $(y \cdot \nu y)^{\frac{1}{2}}$, we say that $\mathcal{Y}$ is a* complete Kähler space*. In other words $\mathcal{Y}^{\mathbb{C}}$ equipped with $(\cdot|\cdot)$ is a Hilbert space.*

Two structures out of ν, ω, j determine the other. This is used in the following three definitions. In all of them $\mathcal{Y}$ is a real vector space, ω is a symplectic form and ν is a non-degenerate symmetric form.

Definition 1.88 (1) *We say that a pair (ω, j) is* pseudo-Kähler *if $\omega\mathrm{j}$ is symmetric. If in addition $\omega\mathrm{j}$ is positive definite, then we say that (ω, j) is* Kähler*.*
(2) *We say that a pair (ν, j) is* pseudo-Kähler *if $-\nu\mathrm{j}$ is a symplectic form. If in addition ν is positive definite, then we say that (ν, j) is* Kähler*.*
(3) *We say that a pair (ν, ω) is* pseudo-Kähler *if $\mathrm{Ran}\,\omega = \mathrm{Ran}\,\nu$ and $\omega^{-1}\nu$ is an anti-involution. If in addition ν is positive definite, we say that (ν, ω) is* Kähler*.*

The definitions (1) and (2) have other equivalent versions, as seen from the following theorem:

Theorem 1.89 (1) *Let $(\mathcal{Y}, \omega)$ be a symplectic space. Consider the following conditions:*

(i) $\mathrm{j}^{\#}\omega\mathrm{j} = \omega$ *(j preserves ω),*
(ii) $\mathrm{j}^{\#}\omega + \omega\mathrm{j} = 0$ *($\mathrm{j} \in sp(\mathcal{Y})$, or equivalently $\omega\mathrm{j}$ is symmetric),*
(iii) $\mathrm{j}^2 = -1\!\!1$ *(j is an anti-involution).*

Then any pair of the conditions (i), (ii), (iii) implies the third condition and that the pair (ω, j) is pseudo-Kähler.

(2) *Let $(\mathcal{Y}, \nu)$ be a (pseudo-)Euclidean space. Consider the following conditions:*

(i) $\mathrm{j}^{\#}\nu\mathrm{j} = \nu$ *(j is (pseudo-)isometric),*
(ii) $\mathrm{j}^{\#}\nu + \nu\mathrm{j} = 0$ *($\mathrm{j} \in o(\mathcal{Y})$, or equivalently $\nu\mathrm{j}$ is anti-symmetric),*
(iii) $\mathrm{j}^2 = -1\!\!1$ *(j is an anti-involution).*

Then any pair of the conditions (i), (ii), (iii) implies the third condition and that the pair (ν, j) is (pseudo-)Kähler.

1.3.9 Complexification of a (pseudo-)Kähler space

Let $(\mathcal{Y}, \nu, \omega, \mathrm{j})$ be a (pseudo-)Kähler space. We have seen that the space $\mathbb{C}\mathcal{Y}$ is equipped with

(1) the symmetric form $w_1 \cdot \nu_{\mathbb{C}} w_2$,
(2) the Hermitian form $(w_1|w_2) := \overline{w_1} \cdot \nu_{\mathbb{C}} w_2$,
(3) the symplectic form $w_1 \cdot \omega_{\mathbb{C}} w_2$, and
(4) the charged symplectic form $\overline{w_1} \cdot \omega_{\mathbb{C}} w_2$,

where $w_1, w_2 \in \mathbb{C}\mathcal{Y}$.

The spaces $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ introduced in Subsect. 1.3.6 are isotropic for both bilinear forms $\nu_{\mathbb{C}}$ and $\omega_{\mathbb{C}}$ and are mutually orthogonal for both sesquilinear forms.

Let us concentrate on the (pseudo-)unitary structure on $\mathbb{C}\mathcal{Y}$ given by the form $(\cdot|\cdot)$. Using the fact that j is anti-self-adjoint for ν on $\mathcal{Y}$ we see that $\mathrm{j}_{\mathbb{C}}$ is anti-self-adjoint for $(\cdot|\cdot)$ on $\mathbb{C}\mathcal{Y}$. Therefore, the projections $\mathbb{1}_{\mathcal{Z}}$ and $\mathbb{1}_{\overline{\mathcal{Z}}}$ are orthogonal projections and hence the spaces $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are orthogonal for $(\cdot|\cdot)$. The map $T_{\sqrt{2}}$, introduced in (1.29) is (pseudo-)unitary, if we interpret $\mathcal{Y}$ as a (pseudo-)unitary space $\mathcal{Y}^{\mathbb{C}}$ equipped with the scalar product (1.37). This is the main reason why the identification $T_{\sqrt{2}}$ is often used, at least for bosonic systems.

The converse construction is as follows. Let $\mathcal{Z}$ be a (pseudo-)unitary space. Set $\mathcal{Y} := \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. Recall from Subsect. 1.3.6 that $\mathcal{Z}$ is naturally isomorphic to the holomorphic space for $(\mathcal{Y}, \mathrm{j})$, where the anti-involution j is given by

$$\mathrm{j}(z, \overline{z}) = (\mathrm{i}z, \overline{\mathrm{i}z}) = (\mathrm{i}z, -\mathrm{i}\overline{z}).$$

$\mathcal{Y}$ is equipped with the symmetric form

$$(z_1, \overline{z}_1) \cdot \nu(z_2, \overline{z}_2) := 2\mathrm{Re}(z_1|z_2),$$

and the symplectic form

$$(z_1, \overline{z}_1) \cdot \omega(z_2, \overline{z}_2) = 2\mathrm{Im}(z_1|z_2).$$

Then $(\mathcal{Y}, \nu, \omega, \mathrm{j})$ is a (pseudo-)Kähler space.

If we first take a (pseudo-)Kähler space $\mathcal{Y}$, take its holomorphic space $\mathcal{Z}$ equipped with its (pseudo-)unitary structure, and then go to the (pseudo-)Kähler space $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ constructed as above, we return to the original structure.

If $\mathcal{Z}$ is complete, then the topological dual $\mathcal{Y}^{\#}$ can be identified with $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ by setting

$$\langle (z, \overline{z}) | (\overline{w}, w) \rangle := (z|w) + (\overline{z}|\overline{w}) = 2\mathrm{Re}(z|w).$$

With this identification we have

$$\omega(z, \overline{z}) = (-\mathrm{i}\overline{z}, \mathrm{i}z).$$

1.3.10 Conjugations on a (pseudo-)Kähler space

Proposition 1.90 *Let $(\mathcal{Y}, \nu, \omega, \mathrm{j})$ be a Kähler space. Let $\tau \in L(\mathcal{Y})$ be an involution. Then the following statements are equivalent:*

(1) *τ is anti-unitary on $\big(\mathcal{Y}^{\mathbb{C}}, (\cdot|\cdot)\big)$.*
(2) *$\tau \in O(\mathcal{Y}, \nu)$, $\tau\mathrm{j} = -\mathrm{j}\tau$.*
(3) *τ is anti-symplectic, $\tau\mathrm{j} = -\mathrm{j}\tau$.*

Definition 1.91 *If the conditions of Prop. 1.90 are satisfied we say that τ is a* conjugation of the Kähler space $\mathcal{Y}$.

Def. 1.91 is consistent with the definitions of a conjugation on a complex space, a symplectic space and a (pseudo-)unitary space.

Assume that $\mathcal{Y}$ is a complete Kähler space with a conjugation τ. Let $\mathcal{X} := \mathcal{Y}^{-\tau}$, which is a real Hilbert space for ν. We can identify $\mathcal{Y}$ with $\mathcal{X} \oplus \mathcal{X}$ by (1.22), as in Subsect. 1.3.2. Having in mind applications to CCR representations (see Subsect. 8.2.7), we prefer, however, to describe a more general identification. We fix a bounded, positive and invertible operator c on $\mathcal{X}$. Then the map

$$\mathcal{Y} \ni y \mapsto \left((2c)^{-\frac{1}{2}} \mathrm{j} \frac{\mathbb{1} + \tau}{2} y, (2c)^{\frac{1}{2}} \frac{\mathbb{1} - \tau}{2} y \right) \in \mathcal{X} \oplus \mathcal{X} \tag{1.38}$$

is bijective. With this identification we have

$$\begin{aligned}
&\tau = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \quad \mathrm{j} = \begin{bmatrix} 0 & -(2c)^{-1} \\ 2c & 0 \end{bmatrix}, \\
&(x_1^+, x_1^-) \cdot \nu (x_2^+, x_2^-) = x_1^+ \cdot \nu 2c x_2^+ + x_1^- \cdot \nu (2c)^{-1} x_1^-, \\
&(x_1^+, x_1^-) \cdot \omega (x_2^+, x_2^-) = x_1^+ \cdot \nu x_2^- - x_1^- \cdot \nu x_2^+, \quad (x_i^+, x_i^-) \in \mathcal{X} \oplus \mathcal{X}, \quad i = 1, 2.
\end{aligned}$$

1.3.11 Real representations of the group $U(1)$

Let $\mathcal{Y}$ be a real space. Consider the group $U(1) \simeq \mathbb{R}/2\pi\mathbb{Z}$ and its representation

$$U(1) \in \theta \mapsto u_\theta \in L(\mathcal{Y}). \tag{1.39}$$

Definition 1.92 *Let $n \in \{0, 1, \dots\}$. A representation (1.39) is called a* charge n representation *if there exists an anti-involution $\mathrm{j_{ch}}$ such that*

$$u_\theta = \cos(n\theta)\mathbb{1} + \sin(n\theta)\mathrm{j_{ch}}, \quad \theta \in U(1). \tag{1.40}$$

Proposition 1.93 (1) *If (1.39) is a charge 1 representation, then*

$$u_\theta y \neq y, \quad 0 \neq y \in \mathcal{Y}, \qquad 0 \neq \theta \in U(1), \tag{1.41}$$

and the operator $\mathrm{j_{ch}}$ in (1.40) coincides with $u_{\pi/2}$.

(2) *If the representation (1.39) satisfies (1.41), then $u_{\pi/2}$ is an anti-involution.*

Proof (2) Clearly, $u_\pi^2 = \mathbb{1}$. Therefore, u_π is diagonalizable and $\frac{1}{2}(\mathbb{1} \pm u_\pi)$ are the projections onto its eigenvalues ± 1. By (1.41), $\mathrm{Ker}(\mathbb{1} - u_\pi) = \{0\}$. Therefore, $u_\pi = -\mathbb{1}$. Now $u_{\pi/2}^2 = u_\pi = -\mathbb{1}$. □

Proposition 1.94 *Assume that $\mathcal{Y}$ is either finite-dimensional or a real Hilbert space and the representation (1.39) is orthogonal. In both cases we suppose that the representation is strongly continuous. Then*

(1) $\mathcal{Y} = \overset{\infty}{\underset{n=0}{\oplus}} \mathcal{Y}_n$, *where $\mathcal{Y}_n$ are invariant and (1.39) restricted to $\mathcal{Y}_n$ is a charge n representation.*

(2) *The set of vectors $y \in \mathcal{Y}$ satisfying (1.41) equals $\mathcal{Y}_1$.*

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Proof We can complexify $\mathcal{Y}$ and write that $u_{\theta,\mathbb{C}} = \mathrm{e}^{\mathrm{i}\theta c}$ on $\mathbb{C}\mathcal{Y}$, for some operator c. Clearly, $\operatorname{spec} c \subset \mathbb{Z}$. Then $\mathcal{Y}_n := \operatorname{Ran} \mathbb{1}_{\{n,-n\}}(c) \cap \mathcal{Y}$. □

Charge 1 representations are related to (pseudo-)Kähler structures.

Proposition 1.95 *Consider a charge 1 representation*

$$u_\theta = \cos(\theta)\mathbb{1} + \sin(\theta)\mathrm{j}_{\mathrm{ch}}, \quad \theta \in U(1). \tag{1.42}$$

(1) *If $\mathcal{Y}$ is a real Hilbert space and $u_\theta \in O(\mathcal{Y})$, $\theta \in U(1)$, then j_{ch} is a Kähler anti-involution.*
(2) *If $\mathcal{Y}$ is a symplectic space and $u_\theta \in Sp(\mathcal{Y})$, $\theta \in U(1)$, then j_{ch} is a pseudo-Kähler anti-involution.*

1.4 Groups and Lie algebras

In this section we fix terminology and notation concerning groups and Lie algebras, mostly consisting of linear or affine transformations.

Throughout the section, $\mathcal{Y}$ and $\mathcal{W}$ denote finite-dimensional spaces.

1.4.1 General linear group and Lie algebra

Definition 1.96 *$GL(\mathcal{Y}, \mathcal{W})$ denotes the set of invertible elements in $L(\mathcal{Y}, \mathcal{W})$. The* general linear group of $\mathcal{Y}$ *is defined as $GL(\mathcal{Y}) := GL(\mathcal{Y}, \mathcal{Y})$.*

$$SL(\mathcal{Y}) := \{r \in GL(\mathcal{Y}) \ : \ \det r = 1\}$$

is its subgroup called the special linear group of $\mathcal{Y}$.

Definition 1.97 *The* general linear Lie algebra of $\mathcal{Y}$ *is denoted $gl(\mathcal{Y})$ and equals $L(\mathcal{Y})$ equipped with the bracket $[a, b] := ab - ba$.*

$$sl(\mathcal{Y}) := \{a \in gl(\mathcal{Y}) \ : \ \operatorname{Tr} a = 0\}$$

is its Lie sub-algebra called the special linear Lie algebra of $\mathcal{Y}$.

1.4.2 Homogeneous linear differential equations

Assume that $\mathbb{R} \ni t \mapsto a_t \in gl(\mathcal{Y})$ is continuous, and $t \geq s$.

Definition 1.98 *We define the* time-ordered exponential *by the following convergent series:*

$$\mathrm{Texp} \int_s^t a_u \mathrm{d}u := \sum_{n=0}^{\infty} \underset{t \geq u_n \geq \cdots \geq u_1 \geq s}{\int \cdots \int} a_{u_n} \cdots a_{u_1} \mathrm{d}u_n \ldots \mathrm{d}u_1.$$

For $y \in \mathcal{Y}$, $s \in \mathbb{R}$, there exists a unique solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} y_t = a_t y_t, \qquad y_s = y. \tag{1.43}$$

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It can be expressed in terms of the time-ordered exponential as

$$y_t = \mathrm{Texp} \int_s^t a_u \mathrm{d}u \, y.$$

Clearly, if $a_t = a \in gl(\mathcal{Y})$ does not depend on t, we can use the usual exponential instead of the time-ordered exponential:

$$\mathrm{Texp} \int_s^t a \mathrm{d}u = \mathrm{e}^{(t-s)a}.$$

1.4.3 Affine transformations

Definition 1.99 *$AL(\mathcal{Y}, \mathcal{W})$ will denote $\mathcal{W} \times L(\mathcal{Y}, \mathcal{W})$ acting on $\mathcal{Y}$ as follows: if $(w, a) \in AL(\mathcal{Y}, \mathcal{W})$ and $y \in \mathcal{Y}$, then $(w, a)y := w + ay$. Elements of $AL(\mathcal{Y}, \mathcal{W})$ are called* affine *maps from $\mathcal{Y}$ to $\mathcal{W}$. We set $AL(\mathcal{Y}) := AL(\mathcal{Y}, \mathcal{Y})$.*

Definition 1.100 *If $G \subset L(\mathcal{Y}, \mathcal{W})$, we set $AG := \mathcal{W} \times G$ as a subset of $AL(\mathcal{Y}, \mathcal{W})$.*

In particular, if $G \subset L(\mathcal{Y})$ is a group, then so is AG. The multiplication in $AG(\mathcal{Y})$ is

$$(y_2, r_2)(y_1, r_1) = (y_2 + r_2 y_1, r_2 r_1).$$

Thus $AG(\mathcal{Y})$ is an example of a semi-direct product of $\mathcal{Y}$ and G, determined by the natural action of G on $\mathcal{Y}$, and is often denoted by $\mathcal{Y} \rtimes G$.

Definition 1.101 *The* general affine Lie algebra *of $\mathcal{Y}$ is $agl(\mathcal{Y}) := \mathcal{Y} \times L(\mathcal{Y})$ equipped with the bracket*

$$[(y_2, a_2), (y_1, a_1)] = (a_2 y_1 - a_1 y_2, a_2 a_1 - a_1 a_2).$$

Definition 1.102 *If $g \subset gl(\mathcal{Y})$, then we set $ag := \mathcal{Y} \times g$ as a subset of $agl(\mathcal{Y})$.*

Clearly, if g is a Lie algebra, then so is ag. It is an example of the semi-direct product of $\mathcal{Y}$ and g, determined by the natural action of g on $\mathcal{Y}$, and is often denoted by $\mathcal{Y} \rtimes g$.

1.4.4 Inhomogeneous linear differential equations

Consider a continuous function $\mathbb{R} \ni t \mapsto (w_t, a_t) \in agl(\mathcal{Y})$. Then, for $y \in \mathcal{Y}$, $s \in \mathbb{R}$, there exists a unique solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} y_t = w_t + a_t y_t, \qquad y_s = y. \tag{1.44}$$

It can be written as

$$y_t = \int_s^t \left(\mathrm{Texp} \int_v^t a_u \mathrm{d}u \right) w_v \mathrm{d}v + \left(\mathrm{Texp} \int_s^t a_u \mathrm{d}u \right) y. \tag{1.45}$$

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If $(w_t, a_t) = (a, w) \in agl(\mathcal{Y})$ does not depend on t, then (1.45) reduces to

$$y_t = a^{-1}(\mathrm{e}^{(t-s)a} - \mathbb{1})w + \mathrm{e}^{(t-s)a}y.$$

This motivates setting

$$\mathrm{e}^{(w,a)} := (a^{-1}(\mathrm{e}^a - \mathbb{1})w, \mathrm{e}^a) \in AGL(\mathcal{Y}).$$

Note in particular that

$$\mathrm{e}^{(0,a)} = (0, \mathrm{e}^a), \quad \mathrm{e}^{(w,0)} = (w, \mathbb{1}).$$

1.4.5 Exact sequences

Let $\pi : F \to G$, $\rho : G \to H$ be homomorphisms between groups.

Definition 1.103 *By saying that*

$$F \xrightarrow{\pi} G \xrightarrow{\rho} H \tag{1.46}$$

is an exact sequence, we mean that $\mathrm{Ran}\,\pi = \mathrm{Ker}\,\rho$*.*

Often, if they are obvious from the context, π, ρ are omitted from (1.46).

The one-element group is often denoted by 1. Therefore,

$$1 \to F \to G \to H \to 1 \tag{1.47}$$

means that F is a normal subgroup of G and we have a natural isomorphism $H \simeq G/F$.

1.4.6 Cayley transform

Let $\mathcal{Y}$ be a vector space. Let $r \in L(\mathcal{Y})$ and $r + \mathbb{1}$ be invertible.

Definition 1.104 *We define the* Cayley transform of r *as*

$$\gamma := (\mathbb{1} - r)(\mathbb{1} + r)^{-1}.$$

Note that $\gamma + \mathbb{1}$ is again invertible and

$$r = (\mathbb{1} - \gamma)(\mathbb{1} + \gamma)^{-1}.$$

Hence the Cayley transform is an involution of

$$\big\{a \in L(\mathcal{Y}) \ : \ r + \mathbb{1} \text{ is invertible}\big\}. \tag{1.48}$$

Let r_1, r_2, r belong to (1.48) with $r = r_1 r_2$. Let $\gamma_1, \gamma_2, \gamma$ be their Cayley transforms. Then we have the identity

$$\mathbb{1} + \gamma = (\mathbb{1} + \gamma_2)(\mathbb{1} + \gamma_1\gamma_2)^{-1}(\mathbb{1} + \gamma_1). \tag{1.49}$$

Suppose that $\mathcal{Y}$ is a finite-dimensional symplectic space. Then the Cayley transform is a bijection of

$$\big\{r \in Sp(\mathcal{Y}) \ : \ r + \mathbb{1} \text{ is invertible}\big\}$$

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onto

$$\{\gamma \in sp(\mathcal{Y}) \ : \ \gamma + \mathbb{1} \text{ is invertible}\}.$$

If $\mathcal{Y}$ is a Euclidean space, then the same is true with $Sp(\mathcal{Y})$, $sp(\mathcal{Y})$ replaced with $O(\mathcal{Y})$, $o(\mathcal{Y})$.

If $\mathcal{Y}$ is a unitary space, then the same is true with $Sp(\mathcal{Y})$, $sp(\mathcal{Y})$ replaced with $U(\mathcal{Y})$, $u(\mathcal{Y})$.

1.5 Notes

Most of the material in this section is a collection of concepts and facts from any basic linear algebra course, after a minor "cleaning up". The need for a particularly precise terminology in this area is especially important in differential geometry. Therefore, in the literature such concepts as Kähler, symplectic and complex structures typically appear in the context of differentiable manifolds; see e.g. Guillemin–Sternberg (1977). They are rarely considered in the (much simpler) context of linear algebra.