INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTIONS AND E-FUNCTIONS

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(Received 24th January, 1958)

1. Introductory. In § 2 a number of integrals in which the integrand contains a product of a hypergeometric function and an E-function will be evaluated. The following formulae will be employed in the proofs.

If $\rho+\sigma = \alpha+\beta+\gamma+1$, and if α, β or γ is zero or a negative integer,

$$F\begin{pmatrix}\alpha, \beta, \gamma\,; & 1\\ \rho, \sigma & \end{pmatrix} = \frac{\Gamma(\rho)\Gamma(\alpha-\sigma+1)\Gamma(\beta-\sigma+1)\Gamma(\gamma-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\rho-\alpha)\Gamma(\rho-\beta)\Gamma(\rho-\gamma)}. \qquad \text{.................(1)}$$

This is Saalschütz's theorem [1].

If $R(\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta) > -\frac{1}{2}$,

$$F\begin{pmatrix}\alpha, \beta, \gamma & ; & 1\\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}, 2\gamma & & \end{pmatrix} = \frac{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\gamma+\frac{1}{2})\Gamma(\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})\Gamma(\gamma-\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\gamma-\frac{1}{2}\beta+\frac{1}{2})}. \qquad \text{............(2)}$$

This theorem was given by Watson [2] for negative integral values of α, and later by Whipple [3] for general values of α.

$$F\begin{pmatrix}\alpha, \beta, \gamma & ; & 1\\ \alpha+\beta+\frac{1}{2}, \gamma+\frac{1}{2} & & \end{pmatrix} = \frac{\Gamma(\alpha+\beta+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\gamma+\frac{1}{2})\Gamma(\gamma-\alpha-\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(\gamma-\alpha+\frac{1}{2})\Gamma(\gamma-\beta+\frac{1}{2})}. \qquad \text{............(3)}$$

Formula (3) can be deduced from formula (2) by means of formula (10) in the appendix.

If $R(\gamma) > 0$,

$$F\begin{pmatrix}\alpha, 1-\alpha, \gamma & ; & 1\\ \rho, 2\gamma-\rho+1 & & \end{pmatrix} = \frac{2^{1-2\gamma}\pi\Gamma(\rho)\Gamma(2\gamma-\rho+1)}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\rho)\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\rho+\frac{1}{2}+\gamma)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}\rho)\Gamma(1-\frac{1}{2}\alpha-\frac{1}{2}\rho+\gamma)}. \qquad \text{...(4)}$$

This formula was given by Whipple [3].

If l is a positive integer,

$$\Gamma(z) = (2\pi)^{\frac{1}{2}-\frac{1}{2}l} l^{z-\frac{1}{2}} \Gamma\left(\frac{z}{l}\right)\Gamma\left(\frac{z+1}{l}\right)\dots\Gamma\left(\frac{z+l-1}{l}\right). \qquad \text{........................(5)}$$

2. Integrals. The first of the integrals to be proved is

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1}F(-n, \beta\,;\ \alpha+\beta-\rho-n+1\,;\ \lambda)E(p\,;\ \alpha_r : q\,;\ \rho_s : z/\lambda^l)\,d\lambda$$

$$= l^{\alpha-\rho}\Gamma(\rho-\alpha+n)[(\rho-\alpha-\beta\,;\ n)]^{-1}E(p+2l\,;\ \alpha_r : q+2l\,;\ \rho_s : z), \qquad \text{...........(6)}$$

where l and n are positive integers, $R(\rho) > R(\alpha) > 0$, $|\text{amp } z| < \pi$, $\alpha_{p+1+\nu} = (\alpha+\nu)/l$, $\alpha_{p+l+1+\nu} = (\rho-\beta+n+\nu)/l$, $\rho_{q+1+\nu} = (\rho-\beta+\nu)/l$, $\rho_{q+l+1+\nu} = (\rho+n+\nu)/l$ $(\nu = 0, 1, 2, , \dots l-1)$.

To prove this, consider first the case $p = q = 0$, noting that

$$E(::z/\lambda^l) \equiv \exp(-\lambda^l/z).$$

Now expand in powers of $1/z$ and integrate term by term, so obtaining

$$\sum_{t=0}^{\infty} \frac{(-1/z)^t}{t!} B(\alpha+tl, \rho-\alpha)F(-n, \beta, \alpha+tl\,;\ \alpha+\beta-\rho-n+1, \rho+tl\,;\ 1).$$

https://doi.org/10.1017/S2040618500033712 Published online by Cambridge University Press

Here apply formula (1), and get

$$\sum_{t=0}^{\infty} \frac{(-1/z)^t}{t!} \frac{\Gamma(\alpha+tl)\Gamma(\rho-\alpha+n)\Gamma(\rho-\beta+n+tl)}{\Gamma(\rho+n+tl)\,(\rho-\alpha-\beta;\ n)\Gamma(\rho-\beta+tl)}$$

$$= \frac{\Gamma(\alpha)\Gamma(\rho-\alpha+n)\Gamma(\rho-\beta+n)}{\Gamma(\rho+n)\Gamma(\rho-\beta)\,(\rho-\alpha-\beta;\ n)} F\begin{pmatrix}\alpha_{p+1}, \ldots, \alpha_{p+l}, \alpha_{p+l+1}, \ldots, \alpha_{p+2l}\,; & -\dfrac{1}{z}\\ \rho_{q+1}, \ldots, \rho_{q+l}, \rho_{q+l+1}, \ldots, \rho_{q+2l} & \end{pmatrix}.$$

On applying (5) this can be written

$$l^{\alpha-\rho}\frac{\Gamma(\rho-\alpha+n)}{(\rho-\alpha-\beta;\ n)} E\begin{pmatrix}\alpha_{p+1}, \ldots, \alpha_{p+2l} : z\\ \rho_{q+1}, \ldots, \rho_{q+2l}\end{pmatrix};$$

and from this, on generalising, (6) is obtained.

The following integrals (7), (8) and (9) can be derived in the same way from formulae (2), (3) and (4) respectively.

If l is a positive integer and if $R(\gamma)>0$, $R(\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta) > -\frac{1}{2}$, $|\mathrm{amp}\, z| < \pi$,

$$\int_0^1 \lambda^{\gamma-1}(1-\lambda)^{\gamma-1}F(\alpha, \beta;\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2};\ \lambda)\,E\{p;\ \alpha_r : q;\ \rho_s : z\lambda^{-l}(1-\lambda)^{-l}\}\,d\lambda$$

$$= \frac{\pi l^{-\frac{1}{2}}2^{1-2\gamma}\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})}E(p+2l;\ \alpha_r : q+2l;\ \rho_s : 2^{2l}z), \quad \ldots(7)$$

where $\alpha_{p+1+\nu} = (\gamma+\nu)/l$, $\alpha_{p+l+1+\nu} = (\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}+\nu)/l$, $\rho_{q+1+\nu} = (\gamma-\frac{1}{2}\alpha+\frac{1}{2}+\nu)/l$, $\rho_{q+l+1+\nu} = (\gamma-\frac{1}{2}\beta+\frac{1}{2}+\nu)/l$ $(\nu = 0, 1, 2, \ldots, l-1)$.

If l is a positive integer and if $R(\gamma)>0$, $|\mathrm{amp}\, z| < \pi$,

$$\int_0^1 \lambda^{\gamma-1}(1-\lambda)^{-\frac{1}{2}}F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ \lambda)\,E(p;\ \alpha_r : q;\ \rho_s : z/\lambda^l)\,d\lambda$$

$$= \frac{\pi l^{-\frac{1}{2}}\Gamma(\alpha+\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})}E(p+2l;\ \alpha_r : q+2l;\ \rho_s : z), \quad \ldots\ldots\ldots\ldots\ldots(8)$$

where $\alpha_{p+1+\nu} = (\gamma+\nu)/l$, $\alpha_{p+l+1+\nu} = (\gamma-\alpha-\beta+\frac{1}{2}+\nu)/l$, $\rho_{q+1+\nu} = (\gamma-\alpha+\frac{1}{2}+\nu)/l$, $\rho_{q+l+1+\nu} = (\gamma-\beta+\frac{1}{2}+\nu)$ $(\nu = 0, 1, 2, \ldots, l-1)$.

If l is a positive integer and if $R(\beta)>0$, $R(\beta-\rho) > -1$, $|\mathrm{amp}\, z| < \pi$,

$$\int_0^1 \lambda^{\beta-1}(1-\lambda)^{\beta-\rho}F(\alpha, 1-\alpha;\ \rho;\ \lambda)\,E\{p;\ \alpha_r : q;\ \rho_s : z\lambda^{-l}(1-\lambda)^{-l}\}\,d\lambda$$

$$= \frac{\pi l^{-\frac{1}{2}}2^{1-2\beta}\Gamma(\rho)}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\rho)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}\rho)}E(p+2l;\ \alpha_r : q+2l;\ \rho_s : 2^{2l}z), \quad \ldots\ldots(9)$$

where $\alpha_{p+1+\nu} = (\beta+\nu)/l$, $\alpha_{p+l+1+\nu} = (\beta-\rho+1+\nu)/l$, $\rho_{q+1+\nu} = (\beta+\frac{1}{2}\alpha-\frac{1}{2}\rho+\frac{1}{2}+\nu)/l$, $\rho_{q+l+1+\nu} = (\beta-\frac{1}{2}\alpha-\frac{1}{2}\rho+1+\nu)/l$ $(\nu = 0, 1, 2, \ldots, l-1)$.

Note. The condition $|\mathrm{amp}\, z| < \pi$ in (6), (7), (8) and (9) can be replaced by the following wider conditions.

If $p < q+1$, $z \neq 0$; if $p = q+1$, $|z| > 1$; if $p > q+1$, $|\mathrm{amp}\, z| < \frac{1}{2}(p-q+1)\pi$.

Appendix. The formula

$$F\begin{pmatrix}\alpha, \beta, \gamma & ;\ 1\\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}, 2\gamma & \end{pmatrix} = F\begin{pmatrix}\frac{1}{2}\alpha, \frac{1}{2}\beta, \gamma & ;\ 1\\ \frac{1}{2}\beta+\frac{1}{2}\beta+\frac{1}{2}, \gamma+\frac{1}{2} & \end{pmatrix}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where $R(\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta) > -\frac{1}{2}$, can be derived from the formula

$$F\begin{pmatrix}\alpha, \beta & ; & z\\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2} & & \end{pmatrix} = F\begin{Bmatrix}\tfrac{1}{2}\alpha, \tfrac{1}{2}\beta & ; & 4z(1-z)\\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha+\tfrac{1}{2} & & \end{Bmatrix}, \qquad \text{.........................(11)}$$

where $|z| < 1$.

For

$$\int_0^1 t^{\gamma-1}(1-t)^{\gamma-1}F(\alpha, \beta ; \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2} ; zt)\,dt$$

$$= \int_0^1 t^{\gamma-1}(1-t)^{\gamma-1}F\{\tfrac{1}{2}\alpha, \tfrac{1}{2}\beta ; \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2} ; 4zt(1-zt)\}\,dt,$$

and from this it follows that

$$F(\alpha, \beta, \gamma ; \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}, 2\gamma ; z)$$

$$= \sum_{r=0}^{\infty} \frac{(\tfrac{1}{2}\alpha ; r)(\tfrac{1}{2}\beta ; r)}{r!(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2} ; r)}(4z)^r \frac{(\gamma ; r)}{(2\gamma ; r)} F\begin{pmatrix}-r, \gamma+r ; & z\\ 2\gamma+r & \end{pmatrix}.$$

On letting $z \to 1$ and applying Gauss's theorem, formula (10) is obtained.

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https://doi.org/10.1017/S2040618500033712 Published online by Cambridge University Press