This is a ``preproof'' accepted article for *Canadian Mathematical Bulletin*
This version may be subject to change during the production process.
DOI: 10.4153/S0008439525000128

CHARACTERIZATIONS AND MODELS FOR THE $C_{1,r}$ CLASS AND QUANTUM ANNULUS

SOURAV PAL AND NITIN TOMAR

ABSTRACT. For fixed $0<r<1$, let $A_r=\{z\in\mathbb{C}: r<|z|<1\}$ be the annulus with boundary $\partial\overline{A}_r=\mathbb{T}\cup r\mathbb{T}$, where $\mathbb{T}$ is the unit circle in the complex plane $\mathbb{C}$. An operator having $\overline{A}_r$ as a spectral set is called an A_r-*contraction*. Also, a normal operator with its spectrum lying in the boundary $\partial\overline{A}_r$ is called an A_r-*unitary*. The $C_{1,r}$ *class* was introduced by Bello and Yakubovich in the following way:

$$C_{1,r}=\{T: T \text{ is invertible and } \|T\|,\|rT^{-1}\|\leq 1\}.$$

McCullough and Pascoe defined the *quantum annulus* $\mathbb{QA}_r$ by

$$\mathbb{QA}_r=\{T\,:\,T \text{ is invertible and } \|rT\|,\|rT^{-1}\|\leq 1\}.$$

If $\mathscr{A}_r$ denotes the set of all A_r-contractions, then $\mathscr{A}_r\subsetneq C_{1,r}\subsetneq\mathbb{QA}_r$. We first find a model for an operator in $C_{1,r}$ and also characterize the operators in $C_{1,r}$ in several different ways. We prove that the classes $C_{1,r}$ and $\mathbb{QA}_r$ are equivalent. Then, via this equivalence, we obtain analogous model and characterizations for an operator in $\mathbb{QA}_r$.

1. INTRODUCTION

Throughout the paper, all operators are bounded linear operators acting on complex Hilbert spaces. We denote by $\mathbb{D},\mathbb{T},r\mathbb{D},r\mathbb{T}$ the unit disk, the unit circle, the disk with radius r and the circle with radius r respectively with center at the origin in the complex plane $\mathbb{C}$. For a Hilbert space $\mathscr{H}$, we mean by $\mathscr{B}(\mathscr{H})$ the algebra of operators acting on $\mathscr{H}$. A *contraction* is an operator whose norm is not greater than 1. For $0<r<1$, let us consider the following annuli:

$$\begin{aligned} A_r&=\{z\in\mathbb{C}\,:\,r<|z|<1\}\,,\\ \mathbb{A}_r&=\{z\in\mathbb{C}\,:\,r<|z|<r^{-1}\}. \end{aligned}\tag{1.1}$$

A Hilbert space operator T is said to be an A_r-*contraction* if $\overline{A}_r$ is a spectral set for T, which is to say that the spectrum $\sigma(T)\subseteq\overline{A}_r$ and for every rational function f with poles off $\overline{A}_r$ von Neumann's inequality holds, i.e. $\|f(T)\|\leq\sup|f(z)|$, where the supremum is taken over $\overline{A}_r$. Here $f(T)=p(T)q(T)^{-1}$, whence $f=p/q$ with $p,q\in\mathbb{C}[z]$ and q having no zeros in $\overline{A}_r$. An A_r-*unitary* is a normal operator having its spectrum in the boundary $\mathbb{T}\cup r\mathbb{T}$ of the annulus A_r. In the seminal paper [1], Agler proved that every A_r-contraction dilates to an A_r-unitary which announces the success of rational dilation on an annulus. Hence, for any A_r-contraction T acting on a Hilbert space $\mathscr{H}$, there is a Hilbert space $\mathscr{K}\supseteq\mathscr{H}$ and an A_r-unitary N such that $f(T)=P_{\mathscr{H}}f(N)|_{\mathscr{H}}$ for every rational function f with poles off $\overline{A}_r$. This path-breaking work due to Agler motivates numerous mathematicians to study further the functions and operators associated with an annulus which leads

2010 *Mathematics Subject Classification.* 47A20, 47A25, 47A63.

Key words and phrases. $C_{1,r}$ class, Quantum annulus, A_r-contraction, A_r-unitary.

The first named author is supported by the Seed Grant of IIT Bombay, the CDPA and the 'Core Research Grant' with Award No. CRG/2023/005223 of Science and Engineering Research Board (SERB), India. The second named author is supported by the Prime Minister's Research Fellowship (PMRF ID 1300140), Government of India.

to exciting works like [10, 11, 8, 9, 6, 7]. Also, see the references therein. Recently, Bello and Yakubovich [4] introduced two important classes of operators associated with an annulus, namely the C_α and $C_{1,r}$ classes which were defined in the following way:

$$C_\alpha = \{T : T \text{ is an invertible operator and } \alpha(T) = -T^{*2}T^2 + (1+r^2)T^*T - r^2 I \geq 0\},$$
$$C_{1,r} = \{T : T \text{ is an invertible operator and } \|T\|, \|rT^{-1}\| \leq 1\}.$$

We mention two important facts about these classes: first, the operators in $C_\alpha, C_{1,r}$ classes have their spectrums in $\overline{A}_r$ and second, if $\mathscr{A}_r$ denotes the set of all A_r-contractions, then $\mathscr{A}_r, C_\alpha, C_{1,r}$ form a strictly increasing chain as was proved in [4].

Theorem 1.1 ([4], Theorem 1.1)**.** $\mathscr{A}_r \subsetneq C_\alpha \subsetneq C_{1,r}$.

Also, an explicit model was constructed for an operator in the C_α class by Bello and Yakubovich, see Theorem 1.2 in [4]. Interestingly, McCullough and Pascoe [12] considered the annulus $\mathbb{A}_r$ as in (1.1) and introduced the *quantum annulus* $\mathbb{QA}_r$ which consists of invertible operators T such that both rT and rT^{-1} are contractions, i.e.

$$\mathbb{QA}_r = \{T : T \text{ is an invertible operator and } \|T\|, \|T^{-1}\| \leq r^{-1}\}.$$

Evidently, $C_{1,r} \subsetneq \mathbb{QA}_r$ which stretches the increasing chain of Theorem 1.1 one more step. In [12], McCullough and Pascoe found the following model theorem for an operator in $\mathbb{QA}_r$.

Theorem 1.2 ([12], Theorem 1.1)**.** *An invertible operator T acting on a Hilbert space $\mathscr{H}$ is in $\mathbb{QA}_r$ if and only if there exists an invertible operator J acting on a larger Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ such that $T^n = P_{\mathscr{H}} J^n|_{\mathscr{H}}$ for all $n \in \mathbb{Z}$ and J, up to unitary equivalence, takes the following form*

$$J = U \begin{bmatrix} rI_{\mathscr{K}_0} & 0 \\ 0 & r^{-1}I_{\mathscr{K}_1} \end{bmatrix}$$

where U is a unitary, $\mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_1$ and $P_{\mathscr{H}}$ denotes the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$.

These wider classes of operators generalizing the A_r-contractions have been extensively studied in recent past in [14, 16, 17] also.

In this article, we further analyse the $C_{1,r}$ class and the quantum annulus. We find the following model-cum-characterization theorem for the $C_{1,r}$ class in terms of a pair of A_r-unitaries. Before we state the theorem, let us mention that any rational function f with poles off $\overline{A}_r$ can be represented as

$$f(z) = \frac{p(z)}{q_1(z)q_2(z)}, \tag{1.2}$$

where p, q_1 and q_2 are polynomials in $\mathbb{C}[z]$ such that the zeros of q_1 and q_2 lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $r\mathbb{D}$ respectively.

Theorem 1.3. *Let T be an invertible operator acting on a Hilbert space $\mathscr{H}$. Then $T \in C_{1,r}$ if and only if there is a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$, an A_r-unitary N on $\mathscr{K}$ and a self adjoint unitary F on $\mathscr{K}$ such that*

$$f(T) = P_{\mathscr{H}} \Big(p(N) q_1(N)^{-1} q_2(FNF)^{-1} \Big)\Big|_{\mathscr{H}}$$

for every rational function f with poles off $\overline{A}_r$ (as in (1.2)).

Note that both N and FNF^{-1} as in the model above are A_r-unitaries. Next, we obtain the following characterizations for an operator in the $C_{1,r}$ class. We mention that the equivalence of the conditions (1) and (2) of this theorem follows from Proposition 3.2 in [12]. However, we present a different proof for this part in this paper.

Theorem 1.4. *Let T be an invertible operator acting on a Hilbert space $\mathscr{H}$. Then the following are equivalent:*

(1) $T \in C_{1,r}$ *;*
(2) $(1+r^2)I_{\mathscr{H}} - T^*T - r^2T^{-1}(T^{-1})^* \geq 0$ *;*
(3) $(T^*T)^{1/2} \in C_\alpha$ *;*
(4) $(T^*T)^{1/2}$ *is an A_r-contraction ;*
(5) there exist a unitary U and an A_r-contraction P on $\mathscr{H}$ such that $T = UP$.

These results will be proved in Section 3. Though, we have $C_{1,r} \subsetneq \mathbb{QA}_r$ by definition, actually these two classes are comparable. To see this, let us consider the map $\varphi : \mathbb{A}_r \to A_{r^2}$ defined by $\varphi(z) = rz$, which is a biholomorphism with $\varphi^{-1}(z) = r^{-1}z$. Now one can easily prove the following lemma that establishes the equivalence of $C_{1,r}$ and $\mathbb{QA}_r$.

Lemma 1.5. *An operator $T \in C_{1,r}$ if and only if $r^{-1/2}T \in \mathbb{QA}_{\sqrt{r}}$. Also, $T \in \mathbb{QA}_r$ if and only if $rT \in C_{1,r^2}$.*

Hence, any result that holds for the $C_{1,r}$ class must have an analogue for the quantum annulus. So, we have the following model theorem and characterizations for $\mathbb{QA}_r$ that are analogous to Theorems 1.3 & 1.4 respectively.

Theorem 1.6. *Let T be an invertible operator acting on a Hilbert space $\mathscr{H}$. Then $T \in \mathbb{QA}_r$ if and only if there is a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$, a normal operator N on $\mathscr{K}$ with $\sigma(N) \subseteq \partial\overline{\mathbb{A}}_r$ and a self adjoint unitary F on $\mathscr{K}$ such that*

$$g(T) = P_{\mathscr{H}}\left(p(N)q_1(N)^{-1}q_2(FNF)^{-1}\right)\Big|_{\mathscr{H}}$$

for every rational function $g = p/q_1q_2$ with the zeros of q_1 and q_2 in $\mathbb{C} \setminus r^{-1}\overline{\mathbb{D}}$ and $r\mathbb{D}$ respectively.

Theorem 1.7. *Let T be an invertible operator on a Hilbert space $\mathscr{H}$. Then the following are equivalent:*

(1) $T \in \mathbb{QA}_r$*;*
(2) $(r^{-2}+r^2)I_{\mathscr{H}} - T^*T - T^{-1}(T^{-1})^* \geq 0$*;*
(3) $\overline{\mathbb{A}}_r$ *is a spectral set for* $(T^*T)^{1/2}$*;*
(4) there exist an operator P with $\overline{\mathbb{A}}_r$ as a spectral set and a unitary U on $\mathscr{H}$ such that $T = UP$.

Note that the model that we obtain for $\mathbb{QA}_r$ as in Theorem 1.6 consists of a pair of normal operators having their spectrums on the boundary of the annulus $\mathbb{QA}_r$. Since Lemma 1.5 allows us to move back and forth between $C_{1,r}$ and $\mathbb{QA}_r$, we do not want to miss the opportunity to achieve an alternative model for the $C_{1,r}$ class which goes parallel with the model for $\mathbb{QA}_r$ obtained by McCullough and Pascoe (i.e. Theorem 1.2).

Theorem 1.8. *An invertible operator T acting on a Hilbert space $\mathscr{H}$ is in $C_{1,r}$ if and only if there is an invertible operator J on a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ such that $T^n = P_{\mathscr{H}}J^n|_{\mathscr{H}}$ for every $n \in \mathbb{Z}$ and J admits the following form:*

$$J = U\begin{bmatrix} rI_{\mathscr{K}_0} & 0 \\ 0 & I_{\mathscr{K}_1} \end{bmatrix},$$

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

where U is a unitary and $\mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_1$.

We prove the results associated with $\mathbb{QA}_r$ in Section 4. In Section 2, we prove a few relevant and preparatory results.

2. Preparatory results

We begin with a famous result due to Ando which states that a pair of commuting contractions T_1, T_2 can always be lifted simultaneously to a pair of commuting unitaries U_1, U_2.

Theorem 2.1 (Ando, [3]). *Given a commuting pair of contractions (T_1, T_2) on a Hilbert space $\mathscr{H}$, there exists a commuting pair of unitaries (U_1, U_2) on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ such that*

$$p(T_1, T_2) = P_{\mathscr{H}} p(U_1, U_2)|_{\mathscr{H}}$$

for every polynomial p in two variables.

We now state and prove a few basic properties of an operator in $C_{1,r}$ class which will be used in the proof of the main theorems.

Lemma 2.2. *For every operator T in $C_{1,r}$ class, we have $r \leq \|T\| \leq 1$.*

A proof to this result follows from the fact that $\|T\|, \|rT^{-1}\| \leq 1$. The converse of Lemma 2.2 does not hold. Indeed, if we choose $r = 1/2$ and $T = \begin{bmatrix} 0 & 1 \\ \frac{1}{100} & 0 \end{bmatrix}$, then T is invertible and $r \leq \|T\| \leq 1$. However, T does not belong to $C_{1,r}$ as the spectrum $\sigma(T)$ is not contained in $\overline{A}_r$. Even more is true, the converse to Lemma 2.2 does not hold for A_r-contractions, i.e. an invertible operator T with $\sigma(T) \subseteq \overline{A}_r$ and $r \leq \|T\| \leq 1$ is not necessarily an A_r-contraction. The following example due to G. Misra [13] shows this clearly. Before going to the example let us state an interesting result due to Williams [18] showing an interplay between spectral set and complete non-normality. It is to mention that an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *completely non-normal* if there is no nonzero closed subspace $\mathscr{H}_1$ of $\mathscr{H}$ that reduces T and $T|_{\mathscr{H}_1}$ is normal.

Theorem 2.3 (Williams, [18]). *If an operator T on a finite dimensional space is completely non-normal and $\|T\| = 1$, then $\overline{\mathbb{D}}$ is a minimal spectral set for T, i.e. no proper closed subset of $\overline{\mathbb{D}}$ is a spectral set for T.*

Example 2.4. For $0 < r < 1$, the matrix $T = \begin{bmatrix} \sqrt{r} & 1-r \\ 0 & \sqrt{r} \end{bmatrix}$ is invertible and $\sigma(T) = \{\sqrt{r}\} \subseteq A_r$. Note that

$$T^*T = \begin{bmatrix} r & \sqrt{r}(1-r) \\ \sqrt{r}(1-r) & r + (1-r)^2 \end{bmatrix}$$

and thus it follows that $\sigma(T^*T) = \{1, r^2\}$. Therefore, $\|T\|^2 = \|T^*T\| = 1$. Hence, $r \leq \|T\| \leq 1$. It is not difficult to see that $T \in C_{1,r}$. Since $\|T\| = 1$ and T is completely non-normal, Theorem 2.3 implies that $\overline{\mathbb{D}}$ is a minimal spectral set for T. Hence $\overline{A}_r$ cannot be a spectral set for T. ■

Lemma 2.5. *Let T be an invertible operator acting on a Hilbert space $\mathscr{H}$. Then*

(a) T is an A_r-contraction if and only if rT^{-1} is an A_r-contraction ;
(b) T is in $C_{1,r}$ if and only if rT^{-1} is in $C_{1,r}$.

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

Proof. The part-(b) is obvious and so we prove only part-(a). Let T be an A_r-contraction. Then by Lemma 2.2, we have $\sigma(T) \subseteq \overline{A}_r$ and $r \leq \|T\| \leq 1$. So, it follows from the Spectral Mapping Theorem that
$$\sigma(rT^{-1}) = \{r/\lambda \ : \ \lambda \in \sigma(T)\} \subseteq \overline{A}_r.$$
Let f be any rational function with poles off $\overline{A}_r$. Then we define a rational function $g(z) = f(rz^{-1})$, where $z \mapsto rz^{-1}$ is an automorphism of the annulus A_r. Evidently, g has its poles off $\overline{A}_r$. Now,
$$\begin{aligned}\|f(rT^{-1})\| = \|g(T)\| &\leq \sup\{|g(z)| : r \leq |z| \leq 1\} \\ &= |g(w)| \quad \text{for some } w \in \mathbb{T} \cup r\mathbb{T} \quad \text{(by the Maximum-modulus principle)} \\ &= |f(rw^{-1})| \\ &\leq \sup\{|f(z)| : r \leq |z| \leq 1\}.\end{aligned}$$
Therefore, $\overline{A}_r$ is a spectral set for rT^{-1}. Again if $\overline{A}_r$ is a spectral set for $S = rT^{-1}$, then by previous part of the proof we have that $rS^{-1} = T$ is also an A_r-contraction and the proof is complete. ■

Let T be an operator on a Hilbert space $\mathscr{H}$ and let γ be a simple closed curve in $\mathbb{C}$ such that $\sigma(T)$ is contained in the interior of γ. If f is a holomorphic function on and in the interior of γ, then $f(T)$ can be defined in the following way:
$$f(T) := \frac{1}{2\pi i}\int_{\gamma} f(w)(w-T)^{-1}dw. \tag{2.1}$$
It is merely mentioned that the above integral and hence the definition of $f(T)$ is independent of the choice of γ. Before going to the next proposition we state a classic result whose proof is a routine exercise.

Lemma 2.6. *Let $T \in \mathscr{B}(\mathscr{H})$ and let Ω be an open set containing $\sigma(T)$. If a sequence $\{f_n\}$ of holomorphic functions on Ω converges uniformly to a function f on every compact subset of Ω, then f is holomorphic on Ω and $\{f_n(T)\}$ as in (2.1) converges to $f(T)$ in operator norm.*

Every rational function f with poles off $\overline{A}_r$ is analytic in an open neighbourhood containing $\overline{A}_r$ and thus has a unique Laurent series $f(z) = \sum_{j=-\infty}^{\infty} f_j z^j$. We show that for $T \in C_{1,r}$ acting on $\mathscr{H}$, the series $\sum_{j=-\infty}^{\infty} f_j T^j$ defines an operator on $\mathscr{H}$ and $f(T) = p(T)q(T)^{-1} = \sum_{j=-\infty}^{\infty} f_j T^j$, where $f = p/q$ with q having no zeros inside $\overline{A}_r$.

Proposition 2.7. *Given an operator $T \in C_{1,r}$ on a Hilbert space $\mathscr{H}$ and a rational function f with poles off $\overline{A}_r$, the series $\sum_{j=-\infty}^{\infty} f_j T^j$ defines a bounded linear operator on $\mathscr{H}$ and is same as $f(T)$.*

Proof. Since $T \in C_{1,r}$, it follows from the Spectral Mapping Theorem that $\sigma(T) \subseteq \overline{A}_r$. Thus $f(T)$ is well-defined. The sequence $\tilde{f}_n(z) = \sum_{j=-n}^{n} f_j z^j$ converges uniformly to $f(z)$ on $\overline{A}_r$ and we have by Lemma 2.6 that $\tilde{f}_n(T)$ converges to $f(T)$ in operator norm topology. Thus, $f(T) = \sum_{j=-\infty}^{\infty} f_j T^j$. ■

In order to prove that an A_r-contraction $T \in \mathscr{B}(\mathscr{H})$ admits an A_r-unitary dilation N on $\mathscr{K} \supseteq \mathscr{H}$, one needs to show that $f(T) = P_{\mathscr{H}} f(N)|_{\mathscr{H}}$ for every rational function f with poles off $\overline{A}_r$. The

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

next lemma shows that instead of all rational functions f it suffices to consider only the integral powers of z, i.e. the functions of the type z^j for $j \in \mathbb{Z}$.

Lemma 2.8. *For operators T and N in $C_{1,r}$ acting on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively with $\mathscr{K} \supseteq \mathscr{H}$, the following are equivalent:*

(1) $f(T) = P_{\mathscr{H}} f(N)|_{\mathscr{H}}$ for every rational function f with poles off $\overline{A}_r$;

(2) $T^j = P_{\mathscr{H}} N^j|_{\mathscr{H}}$ for every $j \in \mathbb{Z}$.

Proof. $(1) \implies (2)$ is obvious. We prove $(2) \implies (1)$. Let f be a rational function with poles off $\overline{A}_r$ and let $f(z) = \sum_{j=-\infty}^{\infty} f_j z^j$ be its Laurent series. We have by Proposition 2.7 that $f(T) = \sum_{j=-\infty}^{\infty} f_j T^j$ and $f(N) = \sum_{j=-\infty}^{\infty} f_j N^j$. For any $h \in \mathscr{H}$, it follows that

$$f(T)h = \sum_{j=-\infty}^{\infty} f_j T^j h = \sum_{j=-\infty}^{\infty} f_j (P_{\mathscr{H}} N^j h) = P_{\mathscr{H}} \sum_{j=-\infty}^{\infty} f_j N^j h = P_{\mathscr{H}} f(N)h.$$

■

Next, we show that the three classes $\mathscr{A}_r, C_\alpha, C_{1,r}$ as in Theorem 1.1 agree under subnormality condition. To do so, it suffices to show that every subnormal $C_{1,r}$ operator is an A_r-contraction. First we state an elementary result whose proof is a routine exercise.

Lemma 2.9. *Let N be a normal operator with $\sigma(N) \subseteq \overline{A}_r$. Then N is an A_r-contraction.*

The next two results are also important in the context of this article.

Proposition 2.10. *A subnormal operator $T \in C_{1,r}$ if and only if T is an A_r-contraction.*

Proof. Let $T \in C_{1,r}$ be subnormal. We have that $\sigma(T) \subseteq \overline{A}_r$. Let N be the minimal normal extension of T. It follows from Theorem 2.11 in Chapter 2 of [5] that $\sigma(N) \subseteq \sigma(T)$ and so, $\sigma(N) \subseteq \overline{A}_r$. Note that N is invertible and since $T = N|_{\mathscr{H}}$, we have that $T^{-1} = N^{-1}|_{\mathscr{H}}$. Therefore, $T^m = N^m|_{\mathscr{H}}$ for every $m \in \mathbb{Z}$. It follows from Lemma 2.8 that $f(T) = f(N)|_{\mathscr{H}}$ for every rational function f with poles outside $\overline{A}_r$. By Lemma 2.9, N is an A_r-contraction and so, $\|f(T)\| \leq \|f(N)\| \leq \sup\{|f(z)| : z \in \overline{A}_r\}$ for every rational function f with poles off $\overline{A}_r$. Thus, T is an A_r-contraction. The converse follows from Theorem 1.1. ■

Proposition 2.11. *A subnormal operator $T \in \mathbb{QA}_r$ if and only if $\overline{\mathbb{A}}_r$ is a spectral set for T.*

Proof. Let $T \in \mathbb{QA}_r$ be subnormal. We have by Lemma 1.5 that $rT \in C_{1,r^2}$ is also subnormal. It follows from Proposition 2.10 that rT is an A_{r^2}-contraction. Since $\varphi : \mathbb{A}_r \to A_{r^2}, \varphi(z) = rz$ is a biholomorphism, we have that $\varphi^{-1}(rT) = T$ has $\varphi^{-1}(\overline{A}_{r^2}) = \overline{\mathbb{A}}_r$ as a spectral set. The converse is trivial. ■

However, an operator in $\mathbb{QA}_r$ may not always have $\overline{\mathbb{A}}_r$ as a spectral set. Actually, the class of operators having $\overline{\mathbb{A}}_r$ as a spectral set is contained in $\mathbb{QA}_r$ and it follows trivially from the von Neumann's inequality. The following example shows that the containment is strict.

Example 2.12. For $0 < r < 1$, consider the matrix $T = \begin{bmatrix} 1 & -r+r^{-1} \\ 0 & 1 \end{bmatrix}$. It is not difficult to see that T is invertible and $\|T\| = \|T^{-1}\| = r^{-1}$. Therefore, $T \in \mathbb{QA}_r$. Since $\|rT\| = 1$ and rT is completely non-normal, Theorem 2.3 implies that $\overline{\mathbb{D}}$ is a minimal spectral set for rT. Consequently, $r^{-1}\overline{\mathbb{D}}$ is a minimal spectral set for T. Hence, $\overline{\mathbb{A}}_r$ cannot be a spectral set for T. ■

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

Lemma 2.13. *Let $T \in C_{1,r}$. If f as in (1.2) is any rational function with poles off $\overline{A}_r$, then*

$$f(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^{-n}\right) \text{ and } f(T) = \left(\sum_{n=0}^{\infty} a_n T^n\right)\left(\sum_{n=0}^{\infty} b_n T^{-n}\right),$$

for some scalar coefficients a_n, b_n.

Proof. For any rational function f with poles off $\overline{A}_r$ we have from (1.2) that

$$f(z) = \frac{p(z)}{q_1(z)q_2(z)},$$

where p, q_1 and q_2 are polynomials with zeros of q_1 in $\mathbb{C}\setminus\overline{\mathbb{D}}$ and zeros of q_2 in $r\mathbb{D}$. Suppose $\alpha_1,\dots,\alpha_k \in \mathbb{C}\setminus\overline{\mathbb{D}}$ are the zeros of q_1 and $\beta_1,\dots,\beta_l \in r\mathbb{D}$ are the zeros of q_2. Then

$$q_1(z) = \alpha(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_k) \quad \text{and} \quad q_2(z) = \beta(z-\beta_1)(z-\beta_2)\dots(z-\beta_l),$$

for some $\alpha, \beta \in \mathbb{C}$. Now for each α_j, β_i and for any $z \in \overline{A}_r$, we have

$$\frac{1}{z-\alpha_j} = \frac{-1}{\alpha_j(1-z/\alpha_j)} = -\sum_{n=0}^{\infty}\frac{z^n}{\alpha_j^{n+1}} \quad \text{and} \quad \frac{1}{z-\beta_i} = \frac{1}{z(1-\beta_i/z)} = \sum_{n=0}^{\infty}\beta_i^n z^{-(n+1)},$$

where both the series converge uniformly on $\overline{A}_r$. Thus, for any $z \in \overline{A}_r$ we have that

$$\frac{1}{q_1(z)} = \sum_{n=0}^{\infty} q_{n1} z^n \quad \text{and} \quad \frac{1}{q_2(z)} = \sum_{n=0}^{\infty}\frac{q_{n2}}{z^n}$$

for some scalar coefficients q_{n1}, q_{n2}. Consequently,

$$f(z) = p(z)\left(\sum_{n=0}^{\infty} q_{n1} z^n\right)\left(\sum_{n=0}^{\infty} q_{n2} z^{-n}\right).$$

Evidently it follows from Lemma 2.6 that

$$f(T) = p(T)\left(\sum_{n=0}^{\infty} q_{n1} T^n\right)\left(\sum_{n=0}^{\infty} q_{n2} T^{-n}\right)$$

and the proof is complete. ∎

We conclude this Section by recalling from the literature a useful result on joint spectrum.

Theorem 2.14 ([15], Theorem 4.9)**.** *If $\underline{T} = (T_1,\dots,T_n)$ is a tuple of commuting operators on a Hilbert space X and if $\sigma_T(\underline{T}) = K_1 \cup K_2$, where K_1 and K_2 are disjoint compact sets in $\mathbb{C}^n$, then there are closed linear subspaces X_1 and X_2 of X such that*

(1) $X = X_1 \oplus X_2$;
(2) X_1, X_2 are invariant under any operator which commutes with each T_k;
(3) $\sigma_T(\underline{T}|_{X_1}) = K_1$ and $\sigma_T(\underline{T}|_{X_2}) = K_2$, where $\underline{T}|_{X_i} = (T_1|_{X_i},\dots,T_n|_{X_i})$ for $i = 1, 2$.

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

3. CHARACTERIZATIONS AND OPERATOR MODEL FOR THE $C_{1,r}$ CLASS

While investigating the success or failure of rational dilation on the closure of a domain Ω, a primary step towards the endeavour is to study the normal operators having their spectrum in the boundary $\partial\overline{\Omega}$. Such operators constitute an analogue of unitaries, which are normal operators associated with the boundary of the unit disk. For an annulus A_r, they are A_r-unitaries. In this Section, we first characterize an A_r-unitary as a direct sum $U_1 \oplus rU_2$ for a pair of unitaries U_1, U_2. Using this characterization, we frame a model (see Theorem 1.3) for an operator in $C_{1,r}$ class and the model consists of a pair of A_r-unitaries. For the sake of brevity, we fix the following notations for a contraction $T \in \mathscr{B}(\mathscr{H})$:

$$T(n) = T^n \ \ (n \geq 1), \quad T(0) = I_{\mathscr{H}}, \quad T(n) = T^{*|n|} \ \ (n \leq -1).$$

Evidently, $\|T\| \leq 1$ if and only if $I - T^*T \geq 0$. Let $D_T = (I - T^*T)^{1/2}$ be the unique positive square root of a contraction T.

Theorem 3.1. *An operator $T \in \mathscr{B}(\mathscr{H})$ is an A_r-unitary if and only if $\mathscr{H}$ decomposes into an orthogonal sum $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ such that $\mathscr{H}_1, \mathscr{H}_2$ reduce T and $T_1 = T|_{\mathscr{H}_1}, T_2 = rT^{-1}|_{\mathscr{H}_2}$ are unitaries. This decomposition is uniquely determined. Indeed, we have that*

$$\mathscr{H}_1 = \{h \in \mathscr{H} \ : \ \|T^n h\| = \|h\| = \|T^{*n}h\|, \ n = 1, 2, \dots\}$$

and

$$\mathscr{H}_2 = \{h \in \mathscr{H} \ : \ \|T^n h\| = r^n\|h\| = \|T^{*n}h\|, \quad n = -1, -2, \dots\}.$$

The space $\mathscr{H}_1$ or $\mathscr{H}_2$ may coincide with the trivial space $\{0\}$. With respect to the decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, T has the following block-matrix form:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & rT_2^{-1} \end{bmatrix}.$$

Proof. Since T is an A_r-unitary, we have by Theorem 1.1 that T is in $C_{1,r}$ and thus T and rT^{-1} are contractions. Hence, $T(n)$ and $(rT^{-1})(n)$ are contractions for every integer n. For each fixed $n \in \mathbb{Z}$, it is evident that

$$\operatorname{Ker} D_{T(n)} = \{h \in \mathscr{H} \ : \ \|T(n)h\| = \|h\|\} \quad \text{and} \quad \operatorname{Ker} D_{(rT^{-1})(n)} = \{h \in \mathscr{H} \ : \ \|rT^{-1}(n)h\| = \|h\|\}.$$

Therefore, we have

$$\mathscr{H}_1 = \bigcap_{n \in \mathbb{Z} \setminus \{0\}} \text{Ker } D_{T(n)} \quad \text{and} \quad \mathscr{H}_2 = \bigcap_{n \in \mathbb{Z} \setminus \{0\}} \text{Ker } D_{(rT^{-1})(n)}.$$

It is obvious that $\mathscr{H}_1$ and $\mathscr{H}_2$ are closed linear subspaces of $\mathscr{H}$. For any $h \in \mathscr{H}_1$, we have

$$\|T^n Th\| = \|T^{n+1}h\| = \|h\| = \|Th\| \ \ (n = 0, 1, 2, \dots),$$

$$\|T^{*n}Th\| = \|T^{*n-1}T^*Th\| = \|T^{*n-1}h\| = \|h\| = \|Th\| \ \ (n = 1, 2, \dots),$$

which follow from the fact that for a contraction T, $\|Th\| = \|h\|$ if and only if $T^*Th = h$. Hence, $Th \in \mathscr{H}_1$. Similarly, one can show that $T^*h \in \mathscr{H}_1$. Thus $\mathscr{H}_1$ reduces T. A similar argument implies that $\mathscr{H}_2$ reduces T. If we set $T_1 = T|_{\mathscr{H}_1}$ and $T_2 = rT^{-1}|_{\mathscr{H}_2}$, then it follows from the definition of $\mathscr{H}_1$ and $\mathscr{H}_2$ that T_1 and T_2 are unitaries on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Consequently, we have that

$$\langle h_1, h_2 \rangle = \langle T^*Th_1, (rT^{-1})(rT^{-1})^*h_2 \rangle = \langle T^{-1}T^{-1*}T^*Th_1, r^2h_2 \rangle = r^2\langle h_1, h_2 \rangle$$

for $h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2$ and so, $\langle h_1, h_2 \rangle = 0$ as $0 < r < 1$. Hence, $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal. Consider the subspace $\mathscr{H}_3 = \mathscr{H} \ominus \left(\mathscr{H}_1 \oplus \mathscr{H}_2\right)$ which reduces T and thus, $T_3 = T|_{\mathscr{H}_3}$ is normal.

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

Since $\sigma(T_3) \subseteq \sigma(T) \subseteq \mathbb{T} \cup r\mathbb{T}$, we have that $\sigma(T_3) = K_1 \cup K_2$, where $K_1 = \sigma(T_3) \cap \mathbb{T}$ and $K_2 = \sigma(T_3) \cap r\mathbb{T}$. Since T_3 is normal, we have by Theorem 2.14 that there are closed subspaces $\mathscr{H}_3'$ and $\mathscr{H}_3''$ of $\mathscr{H}_3$ reducing T_3 such that

$$\mathscr{H}_3 = \mathscr{H}_3' \oplus \mathscr{H}_3'', \quad \sigma(T|_{\mathscr{H}_3'}) = K_1 \subseteq \mathbb{T} \quad \text{and} \quad \sigma(T|_{\mathscr{H}_3''}) = K_2 \subseteq r\mathbb{T}.$$

It shows that T and rT^{-1} are unitaries on $\mathscr{H}_3'$ and $\mathscr{H}_3''$ respectively. Thus, $\mathscr{H}_3' \subseteq \mathscr{H}_1$ and $\mathscr{H}_3'' \subseteq \mathscr{H}_2$ implying that $\mathscr{H}_3 = \mathscr{H}_3' \oplus \mathscr{H}_3'' \subseteq \mathscr{H}_1 \oplus \mathscr{H}_2$. Consequently, $\mathscr{H}_3 = \{0\}$ and so, $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Also, it is clear from the construction that $\mathscr{H}_1$ and $\mathscr{H}_2$ are the maximal closed reducing subspaces of $\mathscr{H}$ on which T and rT^{-1} act as unitaries respectively. Hence, $\mathscr{H}_1, \mathscr{H}_2$ are uniquely determined and the proof is complete. ■

Now we are in a position to give a proof to Theorem 1.3, one of the main results of this article.

Proof of Theorem 1.3. Let $T \in C_{1,r}$. Then (T, rT^{-1}) is a commuting pair of contractions acting on $\mathscr{H}$. It follows from Ando's dilation, Theorem 2.1, that there are commuting unitaries U_1, rU_2^{-1} on a Hilbert space $\mathscr{K}_0 \supseteq \mathscr{H}$ such that

$$p(T, rT^{-1})h = P_{\mathscr{H}}\, p(U_1, rU_2^{-1})h \tag{3.1}$$

for every $h \in \mathscr{H}$ and for every polynomial $p \in \mathbb{C}[z_1, z_2]$. Consequently, we have that

$$T^j h = P_{\mathscr{H}} U_1^j h \quad \text{and} \quad T^{-j} h = P_{\mathscr{H}} U_2^{-j} h, \quad \text{for } h \in \mathscr{H} \text{ and } j = 0, 1, 2, \dots. \tag{3.2}$$

Let f be a rational function with poles off $\overline{A}_r$. Then, $f(z) = p(z)q_1(z)^{-1}q_2(z)^{-1}$ as in (1.2), where q_1, q_2 have their zeros in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $r\mathbb{D}$ respectively. For any $h \in \mathscr{H}$, we have from Lemma 2.13 that

$$\begin{aligned}
f(T)h &= p(T)\left(\sum_{n=0}^{\infty} q_{n1} T^n\right)\left(\sum_{n=0}^{\infty} q_{n2} T^{-n}\right)h \\
&= \lim_{m\to\infty} \left[p(T)\left(\sum_{n=0}^{m} q_{n1} T^n\right)\left(\sum_{n=0}^{m} q_{n2} T^{-n}\right)\right]h \\
&= \lim_{m\to\infty} \left[P_{\mathscr{H}}\, p(U_1)\left(\sum_{n=0}^{m} q_{n1} U_1^n\right)\left(\sum_{n=0}^{m} q_{n2} U_2^{-n}\right)\right]h \quad [\text{by } (3.1)\ \&\ (3.2)] \\
&= P_{\mathscr{H}}\, p(U_1)\left(\sum_{n=0}^{\infty} q_{n1} U_1^n\right)\left(\sum_{n=0}^{\infty} q_{n2} U_2^{-n}\right)h \\
&= P_{\mathscr{H}}\, p(U_1) q_1(U_1)^{-1} q_2(U_2)^{-1} h.
\end{aligned}$$

Set

$$N = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \text{ and } F = \begin{bmatrix} 0 & I_{\mathscr{K}_0} \\ I_{\mathscr{K}_0} & 0 \end{bmatrix} \quad \text{on } \mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_0.$$

Note that F is a self-adjoint unitary. Since U_1 and $r^{-1}U_2$ are unitaries on $\mathscr{K}_0$, it follows from Theorem 3.1 that N is an A_r-unitary. We have that $\mathscr{H} \subseteq \mathscr{K}_0$. Let $V : \mathscr{H} \to \mathscr{K}$ be defined as $Vh = (h, 0)$. Evidently, V is an isometric embedding and $V^*(x_1, x_2) = P_{\mathscr{H}} x_1$ for every $(x_1, x_2) \in \mathscr{K}$.

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

So, for any $h \in \mathscr{H}$ we have that

$$\begin{aligned}
V^*\Big(p(N)q_1(N)^{-1}Fq_2(N)^{-1}F\Big)Vh &= V^*\left(p(N)q_1(N)^{-1}Fq_2(N)^{-1}\begin{bmatrix}0 & I_{\mathscr{K}_0}\\ I_{\mathscr{K}_0} & 0\end{bmatrix}\right)\begin{bmatrix}h\\0\end{bmatrix}\\
&= V^*\left(p(N)q_1(N)^{-1}F\begin{bmatrix}q_2(U_1)^{-1} & 0\\ 0 & q_2(U_2)^{-1}\end{bmatrix}\right)\begin{bmatrix}0\\h\end{bmatrix}\\
&= V^*\left(p(N)q_1(N)^{-1}\begin{bmatrix}0 & I_{\mathscr{K}_0}\\ I_{\mathscr{K}_0} & 0\end{bmatrix}\right)\begin{bmatrix}0\\ q_2(U_2)^{-1}h\end{bmatrix}\\
&= V^*\begin{bmatrix}p(U_1)q_1(U_1)^{-1} & 0\\ 0 & p(U_2)q_1(U_2)^{-1}\end{bmatrix}\begin{bmatrix}q_2(U_2)^{-1}h\\ 0\end{bmatrix}\\
&= V^*\begin{bmatrix}p(U_1)q_1(U_1)^{-1}q_2(U_2)^{-1}h\\ 0\end{bmatrix}\\
&= P_{\mathscr{H}}p(U_1)q_1(U_1)^{-1}q_2(U_2)^{-1}h\\
&= f(T)h.
\end{aligned}$$

Also, we have

$$FNF = \begin{bmatrix}0 & I_{\mathscr{K}_0}\\ I_{\mathscr{K}_0} & 0\end{bmatrix}\begin{bmatrix}U_1 & 0\\ 0 & U_2\end{bmatrix}\begin{bmatrix}0 & I_{\mathscr{K}_0}\\ I_{\mathscr{K}_0} & 0\end{bmatrix} = \begin{bmatrix}U_2 & 0\\ 0 & U_1\end{bmatrix}$$

and hence $q_2(FNF) = Fq_2(N)F$. Note that FNF is unitarily equivalent to N and hence is an A_r-unitary. Since F is a self-adjoint unitary, we have that $q_2(FNF)^{-1} = Fq_2(N)^{-1}F$. Putting everything together, we have that

$$f(T)h = V^*\Big(p(N)q_1(N)^{-1}q_2(FNF)^{-1}\Big)Vh$$

for any rational function f with poles off $\overline{A}_r$ and for every $h \in \mathscr{H}$. To see the converse, assume that there is a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$, an A_r-unitary N on $\mathscr{K}$ and a self adjoint unitary F on $\mathscr{K}$ such that

$$f(T) = P_{\mathscr{H}}\Big(p(N)q_1(N)^{-1}q_2(FNF)^{-1}\Big)\Big|_{\mathscr{H}}$$

for every rational function f with poles off $\overline{A}_r$ (as in (1.2)). Then $T = P_{\mathscr{H}}N|_{\mathscr{H}}$ and $T^{-1} = P_{\mathscr{H}}FN^{-1}F|_{\mathscr{H}}$. Since N is an A_r-unitary, it follows from Theorem 1.1 that $N \in C_{1,r}$. Thus $\|T\| \leq \|N\| \leq 1$ and $\|T^{-1}\| \leq \|FN^{-1}F\| \leq \|N^{-1}\| \leq r^{-1}$. Consequently, $T \in C_{1,r}$. The proof is now complete. ■

Remark 3.2. In Theorem 1.3, if we denote the A_r-unitary FNF by $\widetilde{N}$, then it follows as a special case that for every A_r-contraction $T \in \mathscr{B}(\mathscr{H})$, there is a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ and an A_r-unitary $\widetilde{N} \in \mathscr{B}(\mathscr{K})$ such that

$$f(T) = P_{\mathscr{H}}f(\widetilde{N})|_{\mathscr{H}}$$

for every rational function of the form $f = 1/q$ such that the zeros of q lie inside $r\mathbb{D}$. Also, if $g = p/q_1$ with q_1 having its zeros inside $\mathbb{C}\setminus\overline{\mathbb{D}}$, then also we have

$$g(T) = P_{\mathscr{H}}g(N)|_{\mathscr{H}},$$

for some A_r-unitary $N \in \mathscr{B}(\mathscr{K}_1)$ such that $\mathscr{K}_1 \supseteq \mathscr{H}$.

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

We conclude this Section by providing Theorem 1.4, another main theorem of this paper that characterizes an operator in $C_{1,r}$ in different ways. Once again we mention that the equivalence of parts (1) & (2) of this theorem follows from Proposition 3.2 in [12]. However, we present here a different proof for this part also.

Proof of Theorem 1.4. (1) $\implies$ (2). Let $T \in C_{1,r}$. It is easy to see that

$$D_T^2 D^2_{(rT^{-1})^*} = D^2_{(rT^{-1})^*} D_T^2 = (1+r^2)I_{\mathscr{H}} - T^*T - r^2T^{-1}(T^{-1})^*. \tag{3.3}$$

Consequently, $p(D_T^2)p(D^2_{(rT^{-1})^*}) = p(D^2_{(rT^{-1})^*})p(D_T^2)$ for every polynomial $p \in \mathbb{C}[z]$. Choose a sequence of polynomials $p_n(x)$ that converges uniformly to $x^{1/2}$ on the interval $0 \le x \le 1$. It follows from the spectral theorem that the sequence of operators $p_n(B)$ converges to $B^{1/2}$ for any positive operator B such that $0 \le B \le I_{\mathscr{H}}$. Applying (3.3) to these polynomials and taking the limit as $n \to \infty$, we have

$$D_T D_{(rT^{-1})^*} = D_{(rT^{-1})^*} D_T.$$

Thus, for any $h \in \mathscr{H}$, we have that

$$\left\langle \left((1+r^2)I_{\mathscr{H}} - T^*T - r^2T^{-1}(T^{-1})^*\right) h, h\right\rangle = \left\langle D_T^2 D^2_{(rT^{-1})^*} h, h\right\rangle = \|D_T D_{(rT^{-1})^*} h\|^2 \ge 0.$$

(2) $\implies$ (3). Let $\Delta_T = (1+r^2)I_{\mathscr{H}} - T^*T - r^2T^{-1}(T^{-1})^* \ge 0$ and let $P = (T^*T)^{1/2}$. Note that P is invertible as T is invertible. Moreover, we have

$$0 \le P^*\Delta_T P = P((1+r^2)I_{\mathscr{H}} - P^2 - r^2P^{-2})P = (I_{\mathscr{H}} - P^2)(P^2 - r^2 I_{\mathscr{H}}) = \alpha(P^*, P).$$

(3) $\implies$ (4). Let $P = (T^*T)^{1/2} \in C_\alpha$. Then $\alpha(P^*, P) = (I_{\mathscr{H}} - P^2)(P^2 - r^2 I_{\mathscr{H}}) \ge 0$. Let $\lambda \in \sigma(P)$. It follows from the spectral theorem that $(1-\lambda^2)(\lambda^2 - r^2) \ge 0$ and this holds if and only if $r \le \lambda \le 1$. Therefore, $\sigma(P) \subseteq \overline{A}_r$. Consequently, we have by Lemma 2.9 that P is an A_r-contraction.

(4) $\implies$ (5). Let $P = (T^*T)^{1/2}$ be an A_r-contraction. For $U = TP^{-1}$, we have that

$$U^*U = P^{-1}T^*TP^{-1} = P^{-1}P^2P^{-1} = I_{\mathscr{H}} \quad \text{and} \quad UU^* = TP^{-2}T^* = T(T^*T)^{-1}T^* = I_{\mathscr{H}}.$$

Hence, U is a unitary on $\mathscr{H}$ and $T = UP$.

(5) $\implies$ (1). Let $T = UP$ for a unitary U and an A_r-contraction P on $\mathscr{H}$. Then $rT^{-1} = (rP^{-1})U^*$. Consequently, we have that $\|T\| \le \|P\| \le 1$ and $\|rT^{-1}\| \le \|rP^{-1}\| \le 1$. The proof is now complete. ∎

4. The quantum annulus

In this Section, we provide a model for operators in the quantum annulus $\mathbb{QA}_r$. Recall that

$$\mathbb{QA}_r = \{T : T \text{ is invertible and } \|rT\|, \|rT^{-1}\| \le 1\},$$

which is the quantization of the closed annulus $\overline{\mathbb{A}}_r = \{z \in \mathbb{C} : z \neq 0 \text{ and } |rz|, |rz^{-1}| \le 1\}$ in the sense that the scalars in the annulus are replaced by operators with similar norm-bounds. With this terminology, $C_{1,r}$ is nothing but the quantization of $\overline{A}_r$. It is evident that $C_{1,r}$ is a proper subset of $\mathbb{QA}_r$. However, Lemma 1.5 shows that these two classes are actually equivalent. Thus, in light of Theorem 1.3, we have an analogous model for $\mathbb{QA}_r$ in Theorem 1.6 whose proof is given below.

https://doi.org/10.4153/S0008439525000128 Published online by Cambridge University Press

Proof of Theorem 1.6. The converse is straightforward. We assume that $T \in \mathbb{QA}_r$. We have by Lemma 1.5 that $rT \in C_{1,r^2}$. It follows from Theorem 1.3 that there is a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$, an A_{r^2}-unitary $\tilde{N}$ on $\mathscr{K}$ and a self adjoint unitary F on $\mathscr{K}$ such that

$$f(rT) = P_{\mathscr{H}}\left(f_0(\tilde{N}) f_1(\tilde{N})^{-1} f_2(F\tilde{N}F)^{-1} \right)\Big|_{\mathscr{H}} \tag{4.1}$$

for every rational function $f = f_0/f_1 f_2$ with zeros of f_1 and f_2 lying in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $r^2\mathbb{D}$ respectively. Let $g = p/q_1 q_2$ with zeros of q_1 and q_2 in $\mathbb{C} \setminus r^{-1}\overline{\mathbb{D}}$ and $r\mathbb{D}$ respectively. We define

$$f(z) = g(r^{-1}z) = \frac{p(r^{-1}z)}{q_1(r^{-1}z) q_2(r^{-1}z)},$$

which is holomorphic on $\overline{A}_{r^2}$ with zeros of $q_1(r^{-1}z)$ and $q_2(r^{-1}z)$ lying in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $r^2\mathbb{D}$ respectively. We have by (4.1) that

$$g(T) = f(rT) = P_{\mathscr{H}}\left(p(r^{-1}\tilde{N}) q_1(r^{-1}\tilde{N})^{-1} q_2(r^{-1}F\tilde{N}F)^{-1} \right)\Big|_{\mathscr{H}}.$$

Let $N = r^{-1}\tilde{N}$. Then N is a normal operator and we have that

$$\sigma(N) = \{r^{-1}\lambda : \lambda \in \sigma(\tilde{N})\} \subseteq \{r^{-1}\lambda : |\lambda| = 1 \text{ or } |\lambda| = r^2\} = \partial\overline{\mathbb{A}}_r.$$

The proof is now complete. ∎

Since we have an equivalence of the two classes $C_{1,r}$ and $\mathbb{QA}_r$ by Lemma 1.5, we have a model in Theorem 1.8 for $C_{1,r}$ analogous to the model for $\mathbb{QA}_r$ due to McCullough and Pascoe. We present a brief proof to this below.

Proof of Theorem 1.8. The converse is trivial. Let us assume that $T \in C_{1,r}$. By Lemma 1.5, $r^{-1/2}T \in \mathbb{QA}_{\sqrt{r}}$. It follows from Theorem 1.2 that there is a unitary U on some larger Hilbert space $\mathscr{K} \supseteq \mathscr{H}$, an invertible operator $\tilde{J}$ on $\mathscr{K}$ such that $(r^{-1/2}T)^n = P_{\mathscr{H}}\tilde{J}^n|_{\mathscr{H}}$ for all $n \in \mathbb{Z}$ and that

$$\tilde{J} = U \begin{bmatrix} r^{1/2} I_{\mathscr{K}_0} & 0 \\ 0 & r^{-1/2} I_{\mathscr{K}_1} \end{bmatrix},$$

where U is a unitary and $\mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_1$. Take $J = r^{1/2}\tilde{J}$ and the desired conclusion follows. ∎

Theorem 1.8 gives a model for an operator in the $C_{1,r}$ class. Therefore, the model operator J as in Theorem 1.8 cannot be an A_r-unitary. The reason is obvious; an A_r-unitary can provide a model for the A_r-contractions and the $C_{1,r}$ class is strictly bigger than that of the A_r-contractions. So, a natural question arises: when the model operator J becomes an A_r-unitary so that the initial operator $T \in C_{1,r}$ becomes an A_r-contraction ? Below we provide a necessary and sufficient condition for the same.

Proposition 4.1. *Let J acting on a Hilbert space $\mathscr{K}$ be as in Theorem 1.8. Then the following are equivalent:*

(1) J is an A_r-unitary ;
(2) J is normal ;
(3) $UJ_r = J_rU$ where $J_r = \begin{bmatrix} rI_{\mathscr{K}_0} & 0 \\ 0 & I_{\mathscr{K}_1} \end{bmatrix}$.

Proof. (1) $\implies$ (2) follows trivially. We shall prove (2) $\implies$ (3) $\implies$ (1).

(2) $\implies$ (3). Let $J = UJ_r$ be normal. Then $J^*J = JJ^*$ implies that $UJ_r^2 = J_r^2U$. Since J_r is self-adjoint, one can prove that $UJ_r = J_rU$ by an application of spectral theorem as discussed in the proof of Theorem 1.4.

(3) $\implies$ (1). Let $UJ_r = J_rU$ and let $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ with respect to $\mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_1$. Then

$$0 = UJ_r - J_rU = \begin{bmatrix} rU_{11} & U_{12} \\ rU_{21} & U_{22} \end{bmatrix} - \begin{bmatrix} rU_{11} & rU_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} 0 & (1-r)U_{12} \\ (r-1)U_{21} & 0 \end{bmatrix},$$

which is possible if and only if $U_{12} = U_{21} = 0$ as $r < 1$. Hence, U_{11} and U_{22} are unitaries on $\mathscr{K}_0$ and $\mathscr{K}_1$ respectively. Furthermore, we have that

$$J = \begin{bmatrix} rU_{11} & 0 \\ 0 & U_{22} \end{bmatrix}.$$

It follows from Theorem 3.1 that J is an A_r-unitary which completes the proof. ■

Proof of Theorem 1.7. Follows directly from Theorem 1.4 and Lemma 1.5. ■

Concluding remark. The model operators by Bello-Yakubovich [4] for C_α class or by McCullough-Pascoe [12] for $\mathbb{QA}_r$ cannot be normal or subnormal operators as a subnormal operator in $C_{1,r}$ is an A_r-contraction by Proposition 2.10 and a subnormal operator in $\mathbb{QA}_r$ has $\overline{\mathbb{A}}_r$ as a spectral set by Proposition 2.11. The chain $\mathscr{A}_r \subsetneq C_\alpha \subsetneq C_{1,r}$ clearly shows that an operator in $\mathscr{A}_r$ cannot be a model for $C_{1,r}$. Also Example 2.12 confirms that the operators having $\overline{\mathbb{A}}_r$ as a spectral set is properly contained in $\mathbb{QA}_r$ and thus cannot be a model for the $\mathbb{QA}_r$. However, our models for both $C_{1,r}$ and $\mathbb{QA}_r$ consist of a pair of normal operators having their spectrums on the boundary of the annuli A_r and $\mathbb{A}_r$ respectively. ■

5. Data Availability Statement

(1) Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

(2) In case any datasets are generated during and/or analysed during the current study, they must be available from the corresponding author on reasonable request.

6. Declarations

Ethical Approval. This declaration is not applicable.

Competing interests. There are no competing interests.

Authors' contributions. This declaration is not applicable.

REFERENCES

[1] J. Agler, *Rational dilation on an annulus,* Ann. of Math., 121 (1985), 537 – 563.

[2] J. Agler, J. Harland and B. J. Raphael, *Classical function theory, operator dilation theory, and machine computation on multiply-connected domains*, Mem. Amer. Math. Soc., 191 (2008), 289 – 312.

[3] T. Ando, *On a pair of commutative contractions*, Acta Sci. Math., 24 (1963), 88 – 90.

[4] G. Bello and D. Yakubovich, *An operator model in the annulus*, J. Operator Theory, 90 (2023), 25 – 40.

[5] J. B. Conway, *The theory of subnormal operators*, Amer. Math. Soc., Providence, Rhode Island, 1991.

[6] M. A. Dritschel, M. T. Jury and S. McCullough, *Dilations and constrained algebras*, Oper. Matrices, 10 (2016), 829 – 861.

[7] M. Dritschel and B. Undrakh, *Rational dilation problems associated with constrained algebras*, J. Math. Anal. Appl., 467 (2018), 95 – 131.

[8] B. A. Mair and S. McCullough, *Invariance of extreme harmonic functions on an annulus; applications to theta functions*, Houston J. Math., 20 (1994), 453 – 473.

[9] S. McCullough and L. Shen, *On the Szegő kernel of an annulus*, Proc. Amer. Math. Soc., 121 (1994), 1111 – 1121.

[10] S. McCullough, *Matrix functions of positive real part on an annulus*, Houston J. Math., 21 (1995), 489 – 506.

[11] S. McCullough and S. Sultanic, *Agler-commutant lifting on an annulus*, Integral Equations Operator Theory, 72 (2012), 449 – 482.

[12] S. McCullough and J. E. Pascoe, *Geometric dilations and operator annuli*, J. Funct. Anal., 285 (2023), Paper No. 110035, 20 pp.

[13] G. Misra, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory, 11 (1984), 305 – 317.

[14] M. Mittal, *Function theory on the quantum annulus and other domains*, Thesis (Ph.D.)-University of Houston, ISBN: 978-1124-46385-8, 2010, 141 pp.

[15] J. L. Taylor, *The analytic-functional calculus for several commuting operators*, Acta Math., 125 (1970), 1 – 38.

[16] G. Tsikalas, *A von Neumann type inequality for an annulus*, J. Math. Anal. Appl., 506 (2022), Paper No. 125714, 12 pp.

[17] G. Tsikalas, *A note on a spectral constant associated with an annulus*, Oper. Matrices, 16 (2022), 95 – 99.

[18] J. P. Williams, *Minimal spectral sets of compact operators*, Acta Sci. Math., 28 (1967), 93 – 106.

(Sourav Pal) MATHEMATICS DEPARTMENT, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI - 400076, INDIA.

Email address: `sourav@math.iitb.ac.in`

(Nitin Tomar) MATHEMATICS DEPARTMENT, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI-400076, INDIA.
Email address: tnitin@math.iitb.ac.in