107.40 Four points, six distances

Given four points in the plane there are six distances between pairs of points. But given six positive numbers, can they always be the six distances between four points in the plane? Evidently not, since if all the numbers are 1 then three points will form an equilaterial triangle and there is no way to fit a fourth point with distance 1 from those three. However this is possible in three dimensions, as the vertices of a regular tetrahedron. As another example, if the six numbers are 1, 2, 4, 7, 12, 20 then even going into three dimensions won't help, since no three of these can be the sides of a triangle. Indeed it is evident that of the 20 triples chosen from the six numbers, at least four must be the sides of real triangles for the configuration to exist. We do allow three or even all four of the points to be collinear, in which case some triples of numbers may give degenerate, but nonetheless real, triangles. (I use the term 'real triangle' for one whose longest, or equal longest, side is $\leqslant$ the sum of the other two, and 'degenerate triangle' for the case of equality.)

1. *Four points in a plane*

Four distinct points in a euclidean plane have five 'degrees of freedom' in the sense that, placing one point at the origin and another on the x-axis, there are then five numbers which determine the positions of all points: the x-coordinate of the second point and the x, y coordinates of the other two. Six equations in five unknowns should leave a single condition for a solution to exist.

This condition is not hard to find.[*] Let A, B, C, D be given points in 3-space, and $P = (x, y, z)$ be a general point in 3-space. Then, for any real numbers a, b, c, d, e, the equation

$$a\,(PA)^2 + b\,(PB)^2 + c\,(PC)^2 + d\,(PD)^2 + e = 0 \qquad (1)$$

(where PA for example means the distance from P to A) is the equation of a sphere, since there are no 'cross terms' involving two of x, y, z and the coefficients of x^2, y^2 and z^2 are all equal to $a + b + c + d$. What we need is for the sphere to pass through all four points A, B, C, D and, crucially, for the sphere to be in actuality a plane which means that $a + b + c + d = 0$. So we now put $P = A$, $P = B$, $P = C$, $P = D$ in succession in (1) and add the equation $a + b + c + d = 0$. This gives five equations in five unknowns, which in matrix form is

$$\begin{vmatrix} 0 & (AB)^2 & (AC)^2 & (AD)^2 & 1 \\ (AB)^2 & 0 & (BC)^2 & (BD)^2 & 1 \\ (AC)^2 & (BC)^2 & 0 & (CD)^2 & 1 \\ (AD)^2 & (BD)^2 & (CD)^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \\ d \\ e \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}. \qquad (2)$$

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[*]   This argument was shown to me several decades ago by John Tyrrell.

The 5×5 matrix above will be called M in what follows. Then (2) has a solution with a, b, c, d, e not all zero if, and only if, the determinant of M is 0. For example, with all distances equal to 1 the determinant is 4, but if $AB = BC = AC = 1$ and $AD = BD = CD = \sqrt{3}/3$ (an equilateral triangle ABC and its centroid D) the determinant is 0.

Hence:

> A necessary condition for six distances to be realised by four points in a plane is $\det(M) = 0$ where M is the 5×5 matrix in (2).

The attractive feature about the necessary condition $\det(M) = 0$ is that it is completely symmetric in the six distances. An alternative approach is given in the next section, on the situation in 3-space. Unfortunately the necessary condition is not sufficient, since any three of the four points in the plane must form a real triangle, that is with the longest side less than or equal to the sum of the lengths of the other two sides. For example taking $AB = 1$, $BC = 2$, $AC = 4$ these cannot be the sides of a real triangle, so no fourth point can be added, keeping these distances between three of the four. Yet substituting these, and $AD = 4$, $BD = 5$ the condition $\det(M) = 0$ comes to $\left((CD)^2 - 84\right)^2 = 0$ which has two coincident positive solutions for CD, namely $2\sqrt{21}$. If on the other hand we choose $AB = 1$, $AC = 2$, $AD = 4$ (not the sides of a triangle) then there is a consistent solution, such as $BC = 3$, $CD = 5$ and $BD = \sqrt{\frac{29}{2}}$. In this case, B, A, C are in a straight line since $AB + AC = BC$. An example with two solutions for CD is $AB = 1$, $AC = 2$, $AD = 4$, $BC = 2$, $BD = \sqrt{13}$, $CD = \sqrt{15} \pm \sqrt{3}$ where the plus sign in the choice of CD just squeaks through the requirement that BCD is a real (and non-degenerate) triangle: $BD + BC - CD = 0.00052$ approximately.

Definition: Three positive numbers such that the largest, or equal largest, is less than or equal to the sum of the other two will be said to have the *triangle property*.

Hence:

> To the above necessary condition we must add that, for the four triples of distances obtained by omitting in succession one of the points A, B, C, D, all have the triangle property.
>
> In terms of the list of six numbers to be used as distances this means that they can be divided into four sets of three numbers, each pair of these four sets having a single number in common, and each set of three having the triangle property.

We shall in fact usually think of distances as being assigned between definite pairs of points among A, B, C, D.
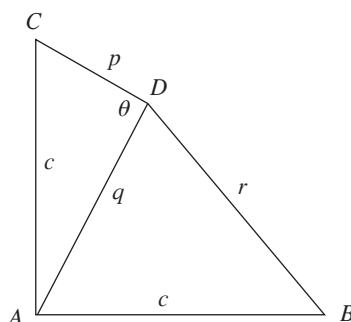
Example 1 (the *pqr* problem):

FIGURE 1

Let p, q, r be the distances of D from three corners of a square of side c, as shown. Given p, q, r (all > 0) when will there exist $c > 0$? The condition $\det(M) = 0$ becomes

$$2c^4 - 2c^2\left(p^2 + r^2\right) + \left(p^2 - q^2\right)^2 + \left(q^2 - r^2\right)^2 = 0,$$

which, as a quadratic in c^2, has real solutions if, and only if,

$$\left(p^2 + r^2\right)^2 > 2\left(\left(p^2 - q^2\right)^2 + \left(q^2 - r^2\right)^2\right).$$

When real the solutions for c are always positive since (i) they have the same sign, and (ii) the turning point of the graph of the above quadratic in c^2 occurs at $c^2 = \frac{1}{2}\left(p^2 + r^2\right) > 0$. (Alternatively the sum of the roots is > 0 and product of the roots is $\geqslant 0$.)

An interesting case is $p^2 + 2q^2 = r^2$ (for example $p = 1$, $q = 2$, $r = 3$), where the condition boils down to $c^2 = p^2 + q^2 \pm pq\sqrt{2}$ ($c^2 = 5 \pm 2\sqrt{2}$ in the numerical example). In fact some straightforward trigonometry (cosine and sine rules) shows that $\sin\theta + \cos\theta = 0$ and $\theta = 135°$ or $45°$ respectively for the $+$ and $-$ signs. In this case there is no problem with the existence of the triangles.

If $p = q = r$ then $c = p\sqrt{2}$ and $p^2 + q^2 = q^2 + r^2 = c^2$, so *CDB* is straight.

Taking $p = 1$, $q = 3$, $r = 2$ in $\det(M) = 0$ gives $2c^4 - 10c^2 + 89 = 0$, which has no real roots. A trigonometrical argument also shows that $\cos\theta + \sin\theta = \frac{15}{6} > 2$ which is impossible.

Example 2: Let $AB = x$, $BC = y$, $AC = x + y$ so that A, B, C are in a straight line. Then $\det(M) = 0$ gives, after some rearrangement, the standard formula (Stewart's Theorem, 1746)

$$(BD)^2 = \frac{x(CD)^2 + y(AD)^2}{x + y} - xy.$$

When $x = y$ so that BD is a median of triangle ACD then this reduces to Apollonius's theorem

$$(2BD)^2 = 2\left((AD)^2 + (CD)^2\right) - (AC)^2.$$

Example 3: Let $AB = BC = AC = 1$ so that A, B, C are the vertices of an equilateral triangle. Then the condition relating the distances of these points from D is

$$(AD)^4 + (BD)^4 + (CD)^4 - (AD)^2(BD)^2 - (AD)^2(CD)^2 - (BD)^2(CD)^2$$
$$-(AD)^2 - (BD)^2 - (CD)^2 + 1 = 0.$$

In this case it is geometrically evident that if AD and BD are chosen such that A, B, D form a proper triangle, then there will aways exist a positive solution to this equation for CD, indeed two solutions in general, realising this set of six numbers as distances between four points A, B, C, D. In fact, the condition for the equation, regarded as a quadratic equation in $(CD)^2$, to have real positive roots factorises as

$$(AD + 1 - BD)(BD + 1 - AD)(AD + BD - 1)(AD + BD + 1) \geqslant 0.$$

This certainly holds if ABD is a real triangle (recall $AB = 1$), and, if it is not, then the configuration of four points A, B, C, D cannot exist.

2. *Four points in* 3-*space*

Again we are given six numbers a, b, c, d, e, f, all positive, which we want to be the distances between four points, this time in 3-space. Four points in 3-space have six 'degrees of freedom' and there are six distances, so this time we can expect there to be no general equation connecting the six distances; however there will be inequalities among the distances which impose constraints. For example, the triangle property must be respected, but there might be others.

The 'sphere' argument used above for the planar case does not appear to yield anything interesting, so here is a direct approach. It is not so symmetrical as the 'sphere' argument, but has the advantage that given one real triangle, with sides a, b, c, which must exist for any hope of realising the whole of the six lengths, there is a single necessary and sufficient condition for the configuration to be possible.

Let us choose three numbers, say a, b, c of the six, all > 0, *which satisfy the triangle property* and use those to construct a triangle ABC, where we choose $A = (0, 0, 0)$, $B = (c, 0, 0)$, $c > 0$ and $C = (u, v, 0)$, where $v \neq 0$ (or else A, B, C are collinear and the figure of four points is planar). We also write $AC = b$ and $BC = a$. Thus

$$\text{(i)}: u^2 + v^2 = b^2; \qquad \text{(ii)}: (u - c)^2 + v^2 = a^2.$$

If no such triple exists then the six distances cannot be realised. Further let $D = (x, y, z)$. The remaining three distances are $AD = d$, $BD = e$ and

$CD = f$, say (see Figure 2a). Then

$$\text{(iii)} : x^2 + y^2 + z^2 = d^2, \text{(iv)} : (x - c)^2 + y^2 + z^2 = e^2,$$

$$\text{(v)} : (x - u)^2 + (y - v)^2 + z^2 = f^2.$$

Using (i) to simplify (ii) allows us to solve for u in terms of a, b, c, then (i) gives us v^2. Using (iv)-(iii) we have x (introducing d and e now), and (v)-(iii), using (i), gives $2ux + 2vy$, allowing a solution for y^2. In fact

$$x = \frac{c^2 + d^2 - e^2}{2c}, \quad 2ux + 2vy = b^2 + d^2 - f^2,$$

$$u = \frac{b^2 + c^2 - a^2}{2c}, \quad v^2 = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}.$$

Finally the consistency condition becomes $z^2 = d^2 - x^2 - y^2 \geqslant 0$, equality being the planar case as above. Making the substitutions for x^2 and y^2 results in rather a complicated condition, best found using software such as Maple! We find that $d^2 - x^2 - y^2$ has an interesting denominator, $(a - b + c)(a + b + c)(a - b - c)(a + b - c)$, which is negative for a non-degenerate triangle ABC. Taking this sign into account then some experimentation shows that it is possible to do some grouping of terms: the condition $z^2 \geqslant 0$ is

$$a^2d^2\left(-a^2 + b^2 + c^2\right) + b^2e^2\left(a^2 - b^2 + c^2\right) + c^2f^2\left(a^2 + b^2 - c^2\right)$$

$$-a^2\left(d^2 - e^2\right)\left(d^2 - f^2\right) - b^2\left(e^2 - d^2\right)\left(e^2 - f^2\right) - c^2\left(f^2 - d^2\right)\left(f^2 - e^2\right) - a^2b^2c^2 \geqslant 0. \quad (3)$$

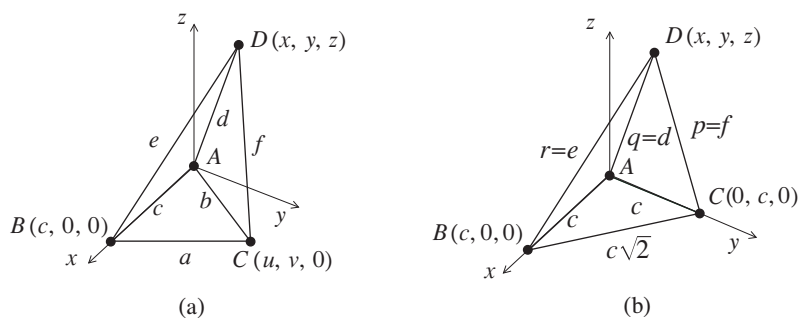(a)   (b)

FIGURE 2

As an example, if $a = b = c = 1$ (equilateral triangle in the plane $z = 0$) and $d = e = f$ then the condition becomes $d \geqslant \frac{1}{3}\sqrt{3}$. Equality here is realised by an equilaterial triangle with its centroid (the planar case) but in three dimensions the other three equal lengths can be any number greater than $\frac{1}{3}\sqrt{3}$.

Example 4 (*the pqr problem in* 3D):

See Figure 2b. Here, $b = c$ and $a = c\sqrt{2}$, while $f = p, d = q, e = r$ in the general discussion above. Writing C for c^2 the condition (3) becomes

$$2C^2 - 2\left(p^2 + r^2\right)C + \left(p^2 - q^2\right)^2 + \left(q^2 - r^2\right)^2 \leqslant 0.$$

This quadratic equation in C has real roots provided

$$\left|p - r\right| \leqslant q\sqrt{2} \leqslant p + r. \tag{4}$$

The sum and product of the real roots being both positive (from the coefficient of C and the constant term respectively), this implies that both roots are positive, hence yield real values of c. Because the quadratic is 'U-shaped' the interval of real values of $C = c^2$ will be between the real roots, assuming these exist, that is if (4) holds.

For example, if $p = 1, q = 2, r = 3$ then the interval of possible values of c^2 is $5 - 2\sqrt{2} \leqslant c^2 \leqslant 5 + 2\sqrt{2}$, the two endpoints representing the two values for the planar situation as in Example 1.

Note that $p = 1, q = 3, r = 2$ fails to satisfy (4) so is not realisable in three dimensions.

If $p = q = r$ then one of the solutions for $C = c^2$ is 0, which we are not allowing; the other solution is $C = 2p^2$. For example if $p = q = r = 1$ the solutions for C are 0, 2, so that c can now lie in the half-open interval $0 < c \leqslant \sqrt{2}$.

3. *Higher numbers of points*?

Five points have 10 mutual distances, and in 3-space they have nine degrees of freedom, three more than for four points. So there should be a single relationship between the 10 distances which, together with some inequality requirements, determines whether a configuration of five points exists. Indeed an argument exactly parallel to that in Section 1 gives a 6×6 determinantal criterion which is necessary but not sufficient. This time the sphere is in 4-space and needs to contain the five points and to have 'infinite radius' so that the sphere is really a 'flat' 3-space. The same applies to higher dimensions, $n + 2$ points in n-space: there will be a necessary determinantal criterion for the $\frac{1}{2}(n + 1)(n + 2)$ distances to be realisable.

On the other hand, $n + 1$ distinct points in n-dimensional space (such as 4 points in 3-space) have between them $n(n + 1)$ coordinates. For the purpose of considering just the distances between these points—of which there are $\frac{1}{2}n(n + 1)$ altogether—we need to allow for distance-preserving transformations (isometries) of n-space. The 'special orthogonal group' SO (n) of all isometries in n-space preserving orientation and fixing a given point has dimension $\frac{1}{2}n(n - 1)$, a standard result which can be found in many internet sources. Including the translations, which have dimension n, gives $\frac{1}{2}n(n + 1)$ dimensions for all isometries. So we need to subtract this from the number $n(n + 1)$ of coordinates to obtain the number of 'degrees of freedom' of the $n + 1$ points, leaving $\frac{1}{2}n(n + 1)$ degrees of freedom. For

example, with $n = 2$ this gives 3 degrees of freedom for 3 points in the plane, and with $n = 3$ it gives 6 degrees of freedom for 4 points in 3-space, as noted above. So the number of degrees of freedom for $n + 1$ points in n-space is always the same as the number of mutual distances between them. The result for $n = 3$ outlined in Section 2 above is therefore typical: there will not be an identity to be satisfied by the $\frac{1}{2}n(n + 1)$ distances, but there will be one or more inequalities to ensure that distances correspond with configurations of points.

Question: what can be said about the case of four *concyclic* points and their six distances?

107.41   Langley's Problem: 100 years on

Edward Mann Langley was the founder and first editor of *The Mathematical Gazette*. Among his many contributions is a short note [1]. In just two lines of text, squeezed in at the bottom of a page, he sets a tantalising problem. Figure 1 shows it in diagrammatic form. Given that $AB = AC$, $a = 20°, b = 50°, c = 60°$ readers were asked to prove that $\theta = 30°$.
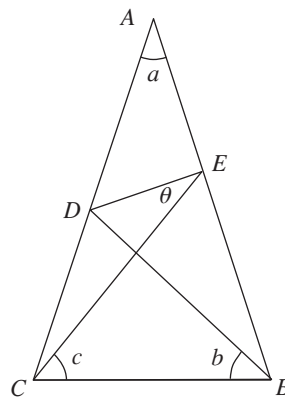

FIGURE 1

Solutions were published in a subsequent issue [2]. Some made use of trigonometry, others only the most elementary geometry.