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APPLICATIONS OF A MINIMAX INEQUALITY ON H-SPACES

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By applying a minimax inequality on H-spaces from our earlier work, new generalisations of well-known intersection theorems concerning sets with convex sections and minimax inequalities of von Neumann type are obtained. Our results generalise the corresponding results of Ben-El-Mechaiekh, Deguire and Granas, Fan, Liu, Shih-Tan, Sion and Tarafdar.

1. INTRODUCTION

In [**5**], we obtained a new generalisation of the Ky Fan minimax inequality [**11**] to non-compact H-spaces and gave some applications to fixed point theorems and system of inequalities which generalise the corresponding results of Browder [**3, 4**], Ding-Tan [**6**], Fan [**7**], Granas-Liu [**12**], Kneser [**13**], Shih-Tan [**16, 17**], Tarafdar [**19**] and Yen [**21**].

In this paper, we shall continue our earlier work to further apply our minimax inequality [**5**, Theorem 2] to obtain some new generalisations of well-known intersection theorems concerning sets with convex sections and minimax inequalities of von Neumann type. Our results generalise the corresponding results of Ben-El-Mechaiekh, Deguire and Granas [**2**], Fan [**8, 9, 10, 11**], Liu [**14**], Shih-Tan [**16**], Sion [**18**] and Tarafdar [**20**].

Let X be a non-empty set. We shall denote by $\mathcal{F}(X)$ the family of all non-empty finite subsets of X. A pair $(X, \{F_A\})$ is said to be an H-space [**1**] if X is a topological space (which need not be Hausdorff) and $\{F_A\}$ is a family of non-empty contractible subsets of X indexed by $A \in \mathcal{F}(X)$ such that $F_A \subset F_{A'}$ whenever $A \subset A'$. Let $X_1, \dots, X_n$ be n ($\geqslant 2$) topological spaces and $X = \prod_{i=1}^{n} X_i$. Let $i \in \{1, \dots, n\}$ be arbitrarily fixed. Let $\widehat{X}_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} X_j$ and let $P_i \colon X \to X_i$ and $\widehat{P}_i \colon X \to \widehat{X}_i$ be the projections. If $x \in X$, we write $P_i(x) = x_i$ and $\widehat{P}_i(x) = \widehat{x}_i$. Moreover, if $x_i \in X_i$ and

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$\widehat{x}_i \in \widehat{X}_i$, $[x_i, \widehat{x}_i]$ denotes the point $y \in X$ such that $P_i(y) = x_i$ and $\widehat{P}_i(y) = \widehat{x}_i$. If $A_i \subset X_i$ and $\widehat{A}_i \subset \widehat{X}_i$, $A_i \otimes \widehat{A}_i$ denotes the set $\{[x_i, \widehat{x}_i] : x_i \in A_i \text{ and } \widehat{x}_i \in \widehat{A}_i\}$.

We shall need the following minimax inequality which was obtained in our earlier paper [**5**, Theorem 2] and was a generalisation of Theorem 1 of Shih-Tan in [**17**] to H-spaces and hence also Theorem 1 of Fan in [**11**]:

Theorem A. *Let $(X, \{F_A\})$ be an H-space and $\phi, \psi: X \times X \to \mathbb{R} \cup \{-\infty, \infty\}$ be such that*

(a) *$\phi(x,y) \leqslant \psi(x,y)$ for each $(x,y) \in X \times X$ and $\psi(x,x) \leqslant 0$ for each $x \in X$;*

(b) *for each fixed $x \in X$, $\phi(x,y)$ is a lower semicontinuous function of y on X;*

(c) *for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, $\min_{x \in A} \psi(x,y) \leqslant 0$;*

(d) *there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$.*

Then there exists $\widehat{y} \in X$ such that $\phi(x, \widehat{y}) \leqslant 0$ for all $x \in X$.

2. Sets with H-convex sections

In this section, we shall apply our minimax inequality to obtain some new generalisations of well-known intersection theorems concerning sets with convex sections.

Theorem 1. *Let $X_1, \ldots, X_n$ be n $(\geqslant 2)$ topological spaces and $X = \prod_{i=1}^{n} X_i$. If $(X, \{F_A\})$ is an H-space and $M_1, \ldots, M_n$, $N_1, \ldots, N_n$ are $2n$ subsets of X such that*

(1) *$M_i \subset N_i$ for each $i = 1, \ldots, n$;*

(2) *for each $i = 1, \ldots, n$ and for each $x_i \in X_i$, the section*

$$M_i(x_i) = \{\widehat{x}_i \in \widehat{X}_i : [x_i, \widehat{x}_i] \in M_i\}$$

is open in $\widehat{X}_i$ and for each $\widehat{x}_i \in \widehat{X}_i$, the section

$$M_i(\widehat{x}_i) = \{x_i \in X_i : [x_i, \widehat{x}_i] \in M_i\}$$

is non-empty;

(3) *for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exists $x \in A$ and $i \in \{1, \ldots, n\}$ such that $[x_i, \widehat{y}_i] \notin N_i$;*

(4) *there exist a non-empty closed and compact subset K of X and $x^0 \in X$ such that $X \setminus K \subset \bigcap_{i=1}^{n} X_i \otimes N_i(x_i^0)$.*

Then $\bigcap_{i=1}^{n} N_i \neq \emptyset$.

PROOF: Define $\phi, \psi \colon X \times X \to \mathbf{R}$ by

$$\phi(x,y) = \begin{cases} 1, & \text{if } x \in \prod_{i=1}^{n} M_i(\widehat{y}_i), \\ 0, & \text{if } x \notin \prod_{i=1}^{n} M_i(\widehat{y}_i), \end{cases}$$

$$\psi(x,y) = \begin{cases} 1, & \text{if } x \in \prod_{i=1}^{n} N_i(\widehat{y}_i), \\ 0, & \text{if } x \notin \prod_{i=1}^{n} N_i(\widehat{y}_i). \end{cases}$$

Then we have

(a) $\phi(x,y) \leqslant \psi(x,y)$ for each $(x,y) \in X \times X$ by (1);

(b) for each fixed $x \in X$ and for each $\lambda \in \mathbf{R}$, the set

$$\{y \in X : \phi(x,y) > \lambda\} = \begin{cases} X, & \text{if } \lambda < 0, \\ \{y \in X : x \in \prod_{i=1}^{n} M_i(\widehat{y}_i)\} = \bigcap_{i=1}^{n} X_i \otimes M_i(x_i), & \text{if } 0 \leqslant \lambda < 1, \\ \emptyset & \text{if } \lambda \geqslant 1, \end{cases}$$

is open in X by (2);

(c) by (3), for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$, there exist $x \in A$ and $i \in \{1, .., n\}$ such that $[x_i, \widehat{y}_i] \notin N_i$; thus $x_i \notin N_i(\widehat{y}_i)$ so that $x \notin \prod_{j=1}^{n} N_j(\widehat{y}_j)$; it follows that $\psi(x,y) = 0$ and hence $\min_{x \in A} \psi(x,y) = 0$;

(d) by (4), there exist a non-empty closed and compact subset K of X and $x^0 \in X$ such that $X \setminus K \subset \bigcap_{i=1}^{n} X_i \otimes N_i(x_i^0)$; it follows that for each $y \in X \setminus K$, $\widehat{y}_i \in N_i(x_i^0)$ for each $i = 1, \ldots, n$ so that $x_i^0 \in N_i(\widehat{y}_i)$ for each $i = 1, \ldots, n$ and hence $x^0 \in \prod_{i=1}^{n} N_i(\widehat{y}_i)$; it follows that $\psi(x^0, y) = 1 > 0$.

Suppose $\psi(x,x) \leqslant 0$ for each $x \in X$; then all hypotheses of Theorem A are satisfied so that there exists $w \in X$ such that $\phi(x,w) \leqslant 0$ for all $x \in X$. Thus $x \notin \prod_{i=1}^{n} M_i(\widehat{w}_i)$ for all $x \in X$ and hence $\prod_{i=1}^{n} M_i(\widehat{w}_i) = \emptyset$, which contradicts (2). Therefore there exists $z \in X$ such that $\psi(z,z) > 0$ which implies that $z \in \prod_{i=1}^{n} N_i(\widehat{z}_i)$ so that $z_i \in N_i(\widehat{z}_i)$ for each $i = 1, \ldots, n$ and hence $z = [z_i, \widehat{z}_i] \in N_i$ for each $i = 1, \ldots, n$. Thus $\bigcap_{i=1}^{n} N_i \neq \emptyset$. ☐

Corollary 1. *Let $X_1, \dots, X_n$ be $n\ (\geqslant 2)$ topological spaces and $X = \prod_{i=1}^{n} X_i$. If $(X, \{F_A\})$ is an H-space and $M_1, \dots, M_n$, $N_1, \dots, N_n$ are $2n$ subsets of X such that*

(1) *for each $i = 1, \dots, n$, $M_i \subset N_i$;*

(2) *for each $i = 1, \dots, n$ and for each $x_i \in X_i$, the section*

$$M_i(x_i) = \{\widehat{x}_i \in \widehat{X}_i : [x_i, \widehat{x}_i] \in M_i\}$$

is open in $\widehat{X}_i$ and for each $\widehat{x}_i \in \widehat{X}_i$, the section

$$M_i(\widehat{x}_i) = \{x_i \in X_i : [x_i, \widehat{x}_i] \in M_i\}$$

is non-empty;

(3)′ *for each $i = 1, \dots, n$ and for each $\widehat{x}_i \in \widehat{X}_i$, the section*

$$N_i(\widehat{x}_i) = \{x_i \in X_i : [x_i, \widehat{x}_i] \in N_i\}$$

has the following property: for each $A \in \mathcal{F}(X)$, if $P_i(A) \subset N_i(\widehat{x}_i)$, then $P_i(F_A) \subset N_i(\widehat{x}_i)$;

(4) *there exist a non-empty closed and compact subset K of X and $x^0 \in X$ such that $X \setminus K \subset \bigcap_{i=1}^{n} X_i \otimes N_i(x_i^0)$.*

Then $\bigcap_{i=1}^{n} N_i \neq \emptyset$.

Proof: Suppose the condition (3) of Theorem 1 does not hold. Then there exist $A \in \mathcal{F}(X)$, $y \in F_A$ such that $[x_i, \widehat{y}_i] \in N_i$ for all $x \in A$ and for all $i = 1, \dots, n$; it follows that $P_i(x) = x_i \in N_i(\widehat{y}_i)$ for all $x \in A$ and for all $i = 1, \dots, n$ so that $P_i(A) \subset N_i(\widehat{y}_i)$ for all $i = 1, \dots, n$. By (3)′, $P_i(F_A) \subset N_i(\widehat{y}_i)$ for all $i = 1, \dots, n$. Thus $y_i = P_i(y) \in P_i(F_A) \subset N_i(\widehat{y}_i)$ for all $i = 1, \dots, n$ so that $y = [y_i, \widehat{y}_i] \in N_i$ for all $i = 1, \dots, n$ and hence $\bigcap_{i=1}^{n} N_i \neq \emptyset$. On the other hand, if the condition (3) of Theorem 1 also holds, then the conclusion that $\bigcap_{i=1}^{n} N_i \neq \emptyset$ follows from Theorem 1. ∎

If X is a non-empty convex subset of a topological vector space, by taking $F_A = co(A)$, the convex hull of A for each $A \in \mathcal{F}(X)$, we see that Corollary 1 (and hence also Theorem 1) are generalisations of Theorem 1 of Fan in [**8**] (see also Theorem 1 in [**9, 10**] and Theorem 8 in [**11**]), Theorem 7 of Shih-Tan in [**16**] and Theorem 4.1 of Tarafdar in [**20**].

The following result is an analytic formulation of Theorem 1.

THEOREM 2. *Let $X_1, \dots, X_n$ be $n\ (\geqslant 2)$ topological spaces and $X = \prod_{i=1}^{n} X_i$. If $(X, \{F_A\})$ is an H-space, $f_1, \dots, f_n$, $g_1, \dots, g_n$ are $2n$ real-valued functions on X and $t_1, \dots, t_n \in \mathrm{R}$ such that*

(a) *for each $i = 1, \dots, n$, $f_i \leqslant g_i$;*
(b) *for each $i = 1, \dots, n$ and for each fixed $x_i \in X_i$, the map $\widehat{x}_i \to f_i[x_i, \widehat{x}_i]$ is lower semicontinuous on $\widehat{X}_i$;*
(c) *for each $A \in \mathcal{F}(X)$ and for each $y \in F_A$ there exist $x \in A$ and $i \in \{1, \dots, n\}$ such that $g_i[x_i, \widehat{y}_i] \leqslant t_i$;*
(d) *there exist a non-empty closed and compact subset K of X and $x^0 \in X$ such that $g_i[x_i^0, \widehat{y}_i] > t_i$ for all $y \in X \setminus K$ and for all $i = 1, \dots, n$;*
(e) *for each $i = 1, \dots, n$ and for each $\widehat{x}_i \in \widehat{X}_i$, there exists $x_i \in X_i$ such that $f_i[x_i, \widehat{x}_i] > t_i$.*

Then there exists $z \in X$ such that $g_i(z) > t_i$ for all $i = 1, \dots, n$.

PROOF OF "THEOREM 1 $\Rightarrow$ THEOREM 2": For each $i = 1, \dots, n$, let M_i and N_i be subsets of X defined by

$$M_i = \{u \in X : f_i(u) > t_i\},$$
$$N_i = \{u \in X : g_i(u) > t_i\}.$$

Apply Theorem 1; the result follows. ∎

PROOF OF "THEOREM 2 $\Rightarrow$ THEOREM 1": For each $i = 1, \dots, n$, let f_i and g_i be the characteristic functions of M_i and N_i respectively. Apply Theorem 2 with $t_1 = \dots = t_n = 0$, the result follows. ∎

An argument similar to that of proving Corollary 1 can be used to prove the following and is omitted.

COROLLARY 2. *Let $X_1, \dots, X_n$ be $n\ (\geqslant 2)$ topological spaces and $X = \prod_{i=1}^{n} X_i$. If $(X, \{F_A\})$ is an H-space, $f_1, \dots, f_n, g_1, \dots, g_n$ are $2n$ real-valued functions on X and $t_1, \dots, t_n \in \mathrm{R}$ such that*

(a) *for each $i = 1, \dots, n$, $f_i \leqslant g_i$;*
(b) *for each $i = 1, \dots, n$ and for each fixed $x_i \in X_i$, the map $\widehat{x}_i \to f_i[x_i, \widehat{x}_i]$ is lower semicontinuous on $\widehat{X}_i$;*
(c)′ *for each $i = 1, \dots, n$ for each $\widehat{x}_i \in \widehat{X}_i$ and for each $A \in \mathcal{F}(X)$, if $P_i(A) \subset \{x_i \in X_i : g_i[x_i, \widehat{x}_i] > t_i\}$, then $P_i(F_A) \subset \{x_i \in X_i : g_i[x_i, \widehat{x}_i] > t_i\}$;*
(d) *there exist a non-empty closed and compact subset K of X and $x^0 \in X$ such that $g_i[x_i^0, \widehat{y}_i] > t_i$ for all $y \in X \setminus K$ and for all $i = 1, \dots, n$;*
(e) *for each $i = 1, \dots, n$ and for each $\widehat{x}_i \in \widehat{X}_i$, there exists $x_i \in X_i$ such that $f_i[x_i, \widehat{x}_i] > t_i$.*

Then there exists $z \in X$ *such that* $g_i(z) > t_i$ *for all* $i = 1, \ldots, n$.

Corollary 2 (and hence also Theorem 2) generalises Theorem 2 of Fan in [**8**] (see also Theorem 3 in [**9**], Theorem 2 in [**10**] and Theorem 7 in [**11**]), Theorem 6 of Shih-Tan in [**16**] and Theorem 4.3 of Tarafdar in [**20**].

Since the case $n = 2$ of Theorem 1 and Corollary 1 is most useful, we shall state that case explicitly as follows:

THEOREM 3. *Let* $(X \times Y, \{F_A\})$ *be an* H*-space and* M_1, M_2, N_1, N_2 *be subsets of* $X \times Y$. *Suppose that*

(1) *for each* $i = 1, 2$, $M_i \subset N_i$;
(2) *for each fixed* $x \in X$, $M_1(x) = \{y \in Y : (x, y) \in M_1\}$ *is open in* Y *and* $M_2(x) = \{y \in Y : (x, y) \in M_2\} \neq \emptyset$;
(3) *for each fixed* $y \in Y$, $M_2(y) = \{x \in X : (x, y) \in M_2\}$ *is open in* X *and* $M_1(y) = \{x \in X : (x, y) \in M_1\} \neq \emptyset$;
(4) *for each* $A \in \mathcal{F}(X \times Y)$ *and for each* $(x, y) \in F_A$, *there exists* $(w, z) \in A$ *such that* $(w, y) \notin N_1$ *or* $(x, z) \notin N_2$;
(5) *there exist a non-empty closed and compact subset* K *of* $X \times Y$ *and* $(x_0, y_0) \in X \times Y$ *such that* $X \times Y \setminus K \subset \{x \in X : (x, y_0) \in N_2\} \times \{y \in Y : (x_0, y) \in N_1\}$.

Then $N_1 \cap N_2 \neq \emptyset$.

COROLLARY 3. *Let* $(X \times Y, \{F_A\})$ *be an* H*-space and* M_1, M_2, N_1, N_2 *be subsets of* $X \times Y$. *Suppose that*

(1) *for each* $i = 1, 2$, $M_i \subset N_i$;
(2) *for each fixed* $x \in X$, $M_1(x) = \{y \in Y : (x, y) \in M_1\}$ *is open in* Y *and* $M_2(x) = \{y \in Y : (x, y) \in M_2\} \neq \emptyset$;
(3) *for each fixed* $y \in Y$, $M_2(y) = \{x \in X : (x, y) \in M_2\}$ *is open in* X *and* $M_1(y) = \{x \in X ; (x, y) \in M_1\} \neq \emptyset$;
(4) *for each* $y \in Y$, *the section* $N_1(y) = \{x \in X : (x, y) \in N_1\}$ *has the property: for each* $A \in \mathcal{F}(X \times Y)$, *if* $P_1(A) \subset N_1(y)$, *then* $P_1(F_A) \subset N_1(y)$;
(5) *for each* $x \in X$, *the section* $N_2(x) = \{y \in Y : (x, y) \in N_2\}$ *has the property: for each* $A \in \mathcal{F}(X \times Y)$, *if* $P_2(A) \subset N_2(x)$, *then* $P_2(F_A) \subset N_2(x)$;
(6) *there exist a non-empty closed and compact subset* K *of* $X \times Y$ *and* $(x_0, y_0) \in X \times Y$ *such that* $X \times Y \setminus K \subset \{x \in X : (x, y_0) \in N_2\} \times \{y \in Y : (x_0, y) \in N_1\}$.

Then $N_1 \cap N_2 \neq \emptyset$.

3. Minimax inequalities of von Neumann type

Minimax inequalities treated in this section evolve from the von Neumann minimax principle [15]. We shall show that such inequalities are consequences of Theorem 3 or Corollary 3.

Theorem 4. *Let $(X \times Y, \{F_A\})$ be an H-space, $f, s, t, g\colon X \times Y \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that*

(a) *$f \leqslant s \leqslant t \leqslant g$ on $X \times Y$;*
(b) *for each fixed $x \in X$, $y \to f(x,y)$ is lower semicontinuous on Y;*
(c) *for each fixed $y \in Y$, $x \to g(x,y)$ is upper semicontinuous on X;*
(d) *for each $A \in \mathcal{F}(X \times Y)$ and for each $(x,y) \in F_A$, there exists $(w,z) \in A$ such that $s(w,y) \leqslant \gamma$ or $t(x,z) \geqslant \gamma$;*
(e) *there exist a non-empty closed and compact subset K of $X \times Y$ and $(x_0, y_0) \in X \times Y$ such that $s(x_0,y) > \gamma$ and $t(x,y_0) < \gamma$ for all $(x,y) \in X \times Y \setminus K$.*

Then either there exists $\widehat{y} \in Y$ such that $f(x,\widehat{y}) \leqslant \gamma$ for all $x \in X$ or there exists $\widehat{x} \in X$ such that $g(\widehat{x},y) \geqslant \gamma$ for all $y \in Y$.

Proof: Suppose that the conclusion were not true. Let

$$M_1 = \{(x,y) \in X \times Y : f(x,y) > \gamma\},\ M_2 = \{(x,y) \in X \times Y : g(x,y) < \gamma\},$$
$$N_1 = \{(x,y) \in X \times Y : s(x,y) > \gamma\},\ N_2 = \{(x,y) \in X \times Y : t(x,y) < \gamma\}.$$

Then for each $y \in Y$, $M_1(y) = \{x \in X : f(x,y) > \gamma\} \neq \emptyset$ and for each $x \in X$, $M_2(x) = \{y \in Y : g(x,y) < \gamma\} \neq \emptyset$. Moreover,

(i) for each $i = 1,2$, $M_i \subset N_i$ by (a);
(ii) for each fixed $x \in X$, $M_1(x) = \{y \in Y : (x,y) \in M_1\}$ is open in Y by (b);
(iii) for each fixed $y \in Y$, $M_2(y) = \{x \in X : (x,y) \in M_2\}$ is open in X by (c);
(iv) by (d), for each $A \in \mathcal{F}(X \times Y)$ and for each $(x,y) \in F_A$, there exists $(w,z) \in A$ such that $(w,y) \notin N_1$ or $(x,z) \notin N_2$;
(v) by (e), there exist a non-empty closed and compact subset K of $X \times Y$ and $(x_0,y_0) \in X \times Y$ such that $(x_0,y) \in N_1$ and $(x,y_0) \in N_2$ for all $(x,y) \in X \times Y \setminus K$ so that $X \times Y \setminus K \subset \{x \in X : (x,y_0) \in N_2\} \times \{y \in Y : (x_0,y) \in N_1\}$. Thus all hypotheses of Theorem 3 are satisfied and hence $N_1 \cap N_2 \neq \emptyset$. Take any $(\widehat{x},\widehat{y}) \in N_1 \cap N_2$, then $s(\widehat{x},\widehat{y}) > \gamma$ and $t(\widehat{x},\widehat{y}) < \gamma$, which contradicts (a). Therefore the conclusion must hold.

∎

COROLLARY 4. *Let $(X \times Y, \{F_A\})$ be an H-space, $f, s, t, g: X \times Y \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that*

(i) *$f \leqslant s \leqslant t \leqslant g$ on $X \times Y$;*
(ii) *for each fixed $x \in X$, $y \to f(x,y)$ is lower semicontinuous on Y;*
(iii) *for each fixed $y \in Y$, $x \to g(x,y)$ is upper semicontinuous on X;*
(iv) *for each fixed $y \in Y$ and for each $A \in \mathcal{F}(X \times Y)$, if $P_1(A) \subset \{x \in X : s(x,y) > \gamma\}$, then $P_1(F_A) \subset \{x \in X : s(x,y) > \gamma\}$;*
(v) *for each fixed $x \in X$ and for each $A \in \mathcal{F}(X \times Y)$, if $P_2(A) \subset \{y \in Y : t(x,y) < \gamma\}$, then $P_2(F_A) \subset \{y \in Y : t(x,y) < \gamma\}$;*
(vi) *there exist a non-empty closed and compact subset K of $X \times Y$ and $(x_0, y_0) \in X \times Y$ such that $s(x_0, y) > \gamma$ and $t(x, y_0) < \gamma$ for all $(x,y) \in X \times Y \setminus K$.*

Then either there exists $\widehat{y} \in Y$ such that $f(x, \widehat{y}) \leqslant \gamma$ for all $x \in X$ or there exists $\widehat{x} \in X$ such that $g(\widehat{x}, y) \geqslant \gamma$ for all $y \in Y$.

PROOF: Suppose the condition (d) of Theorem 4 does not hold. Then there exist $A \in \mathcal{F}(X \times Y)$ and $(x_0, y_0) \in F_A$ such that $s(w, y_0) > \gamma$ and $t(x_0, z) < \gamma$ for all $(w,z) \in A$. It follows that $P_1(A) \subset \{x \in X : s(x, y_0) > \gamma\}$ and $P_2(A) \subset \{y \in Y : t(x_0, y) < \gamma\}$ so that by (iv) and (v), $P_1(F_A) \subset \{x \in X; s(x, y_0) > \gamma\}$ and $P_2(F_A) \subset \{y \in Y : t(x_0, y) < \gamma\}$. As $(x_0, y_0) \in F_A$, we must have $s(x_0, y_0) > \gamma$ and $t(x_0, y_0) < \gamma$ which contradicts (i). Hence the condition (d) of Theorem 4 must hold. The conclusion follows from Theorem 4. ∎

When X and Y are compact, the condition (vi) of Corollary 4 (respectively, condition (e) of Theorem 4) is satisfied by setting $K = X \times Y$. Thus Corollary 4 (and hence also Theorem 4) is a generalisation of Theorem 5.4 of Ben-El-Mechaiekh, Deguire and Granas in [**2**] to H-spaces in non-compact setting.

THEOREM 5. *Let $(X \times Y, \{F_A\})$ be an H-space, $f, s, t, g: X \times Y \to \mathbb{R}$ such that*

(1) *$f \leqslant s \leqslant t \leqslant g$ on $X \times Y$;*
(2) *for each fixed $x \in X$, $y \to f(x,y)$ is lower semicontinuous on Y;*
(3) *for each fixed $y \in Y$, $x \to g(x,y)$ is upper semicontinuous on X;*
(4) *for each $A \in \mathcal{F}(X \times Y)$, for each $(x,y) \in F_A$ and for each $\lambda \in \mathbb{R}$, there exists $(w,z) \in A$ such that $s(w,y) \leqslant \lambda$ or $t(x,z) \geqslant \lambda$;*
(5) *there exist non-empty closed and compact subsets M of X and N of Y such that*

(I) $$\inf_{y \in Y} \sup_{x \in X} f(x,y) \leqslant \sup_{x \in X} \inf_{y \in Y \setminus N} s(x,y);$$

(II) $$\inf_{y \in Y} \sup_{x \in X \setminus M} t(x,y) \leqslant \sup_{x \in X} \inf_{y \in Y} g(x,y).$$

Then

$$\alpha \equiv \inf_{y\in Y} \sup_{x\in X} f(x,y) \leqslant \sup_{x\in X} \inf_{y\in Y} g(x,y) \equiv \beta.$$

PROOF: Without loss of generality, we may assume that $\alpha \neq -\infty$ and $\beta \neq +\infty$. Assume to the contrary that $\alpha > \beta$. Choose a real number γ such that $\alpha > \gamma > \beta$. By (I) and (II),

$$\gamma < \alpha = \inf_{y\in Y} \sup_{x\in X} f(x,y) \leqslant \sup_{x\in X} \inf_{y\in Y\setminus N} s(x,y),$$
$$\gamma > \beta = \sup_{x\in X} \inf_{y\in Y} g(x,y) \geqslant \inf_{y\in Y} \sup_{x\in X\setminus M} t(x,y),$$

so that there exists $(x_0, y_0) \in X \times Y$ such that

$$s(x_0,y) > \gamma \text{ and } t(x,y_0) < \gamma \text{ for all } (x,y) \in X \times Y \setminus M \times N.$$

Let $K = M \times N$; then K is a non-empty closed and compact subset of $X \times Y$. By Theorem 4, either there exists $\widehat{y} \in Y$ such that $f(x,\widehat{y}) \leqslant \gamma$ for all $x \in X$ or there exists $\widehat{x} \in X$ such that $g(\widehat{x},y) \geqslant \gamma$ for all $y \in Y$; it follows that either $\sup_{x\in X} f(x,\widehat{y}) \leqslant \gamma$ or $\inf_{y\in Y} g(\widehat{x},y) \geqslant \gamma$ which contradicts the assumption that $\alpha > \gamma > \beta$. Therefore the conclusion must hold. ∎

COROLLARY 5. *Let $(X \times Y, \{F_A\})$ be an H-space, $f,s,t,g\colon X \times Y \to \mathbb{R}$ such that*

(i) $f \leqslant s \leqslant t \leqslant g$ *on* $X \times Y$;
(ii) *for each fixed* $x \in X$, $y \to f(x,y)$ *is lower semicontinuous on* Y;
(iii) *for each fixed* $y \in Y$, $x \to g(x,y)$ *is upper semicontinuous on* X;
(iv) *for each fixed* $y \in Y$, *for each* $A \in \mathcal{F}(X \times Y)$ *and for each* $\lambda \in \mathbb{R}$, *if* $P_1(A) \subset \{x \in X : s(x,y) > \lambda\}$, *then* $P_1(F_A) \subset \{x \in X : s(x,y) > \lambda\}$;
(v) *for each fixed* $x \in X$, *for each* $A \in \mathcal{F}(X \times Y)$ *and for each* $\lambda \in \mathbb{R}$, *if* $P_2(A) \subset \{y \in Y : t(x,y) < \lambda\}$, *then* $P_2(F_A) \subset \{y \in Y : t(x,y) < \lambda\}$;
(vi) *there exist non-empty closed and compact subsets* M *of* X *and* N *of* Y *such that*

(I) $$\inf_{y\in Y} \sup_{x\in X} f(x,y) \leqslant \sup_{x\in X} \inf_{y\in Y\setminus N} s(x,y);$$
(II) $$\inf_{y\in Y} \sup_{x\in X\setminus M} t(x,y) \leqslant \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

Then

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \leqslant \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

PROOF: Suppose the condition (4) of Theorem 5 does not hold. Then there exist $A \in \mathcal{F}(X \times Y)$, $(x_0, y_0) \in F_A$ and $\lambda \in \mathbb{R}$ such that $s(w, y_0) > \lambda$ and $t(x_0, z) < \lambda$

for all $(w, z) \in A$; it follows that $P_1(A) \subset \{x \in X : s(x, y_0) > \lambda\}$ and $P_2(A) \subset \{y \in Y : t(x_0, y) < \lambda\}$ so that by (iv) and (v), $P_1(F_A) \subset \{x \in X : s(x, y_0) > \lambda\}$ and $P_2(F_A) \subset \{y \in Y : t(x_0, y) < \lambda\}$. As $(x_0, y_0) \in F_A$, we must have $s(x_0, y_0) > \lambda$ and $t(x_0, y_0) < \lambda$ which contradicts (i). Hence the condition (4) of Theorem 5 must hold. The conclusion follows from Theorem 5. □

When $f \equiv s \equiv t \equiv g$, the conclusion of Corollary 5 (respectively Theorem 5) implies the following minimax equality, which generalises the minimax principle of the von Neumann type due to Sion [**18**]:

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

When $f \equiv s$ and $t \equiv g$, Corollary 5 (and hence also Theorem 5) contains a minimax inequality of Liu [**14**].

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