



# Nonselself-adjoint Semicrossed Products by Abelian Semigroups

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*Abstract.* Let  $\mathcal{S}$  be the semigroup  $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$ , where for each  $i \in I$ ,  $\mathcal{S}_i$  is a countable subsemigroup of the additive semigroup  $\mathbb{R}_+$  containing 0. We consider representations of  $\mathcal{S}$  as contractions  $\{T_s\}_{s \in \mathcal{S}}$  on a Hilbert space with the Nica-covariance property:  $T_s^* T_t = T_t T_s^*$  whenever  $t \wedge s = 0$ . We show that all such representations have a unique minimal isometric Nica-covariant dilation.

This result is used to help analyse the nonself-adjoint semicrossed product algebras formed from Nica-covariant representations of the action of  $\mathcal{S}$  on an operator algebra  $\mathcal{A}$  by completely contractive endomorphisms. We conclude by calculating the  $C^*$ -envelope of the isometric nonself-adjoint semicrossed product algebra (in the sense of Kakariadis and Katsoulis).

## 1 Introduction

The study of nonself-adjoint semicrossed products began with Arveson [1]. They were further studied by McAsey, Muhly, and Saito [19]. In both cases the algebras were described concretely. Peters [26] described the nonself-adjoint semicrossed products as universal algebras for covariant representations. In recent years, Davidson and Katsoulis have shown that nonself-adjoint semicrossed products are a particularly interesting and tractable class of operator algebras [4–8]. In particular, nonself-adjoint semicrossed product algebras have been shown to be a class where the  $C^*$ -envelope is often calculable.

The  $C^*$ -envelope of an operator algebra  $\mathcal{A}$  was introduced by Arveson [2, 3] as a non-commutative analogue of Shilov boundaries. The existence of the  $C^*$ -envelope was first discovered by Hamana [13]. Dritschel and McCullough [9] have since provided an alternative proof of the existence of the  $C^*$ -envelope. The viewpoint of Dritschel and McCullough has allowed for the explicit calculation of the  $C^*$ -envelope of many operator algebras. In particular, for nonself-adjoint semicrossed products the  $C^*$ -envelopes have been studied in [5, 10, 11, 14, 15].

In this paper we study the nonself-adjoint semicrossed product algebras by semigroups of the form  $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$ , where for each  $i \in I$  we have  $\mathcal{S}_i$  is a countable subsemigroup of the additive semigroup  $\mathbb{R}_+$  containing 0. Our algebras will be universal for Nica-covariant representations. The term Nica-covariant representation is used for isometric representations of quasi-lattice ordered semigroups that satisfy  $T_s T_s^* T_t T_t^* = T_{t \vee s} T_{t \vee s}^*$ ; see [23]. In the case of abelian lattice-ordered semigroups these are precisely the isometric representations satisfying  $\{T_s\}_{s \in \mathcal{S}}$  satisfying  $T_s^* T_t = T_t T_s^*$  when  $s \wedge t = 0$ . This property, particularly for the case  $\mathcal{S} = \mathbb{Z}_+^k$ , is also known

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as doubly commuting. Crossed product algebras associated with Nica-covariant representations have been widely studied in the  $C^*$ -algebra literature [12, 17, 23].

The paper is divided into three sections. In Section 2 Nica-covariant representations are studied independently from dynamical systems. The results of this section may be of interest even to those not concerned with nonself-adjoint semicrossed products. We show that contractive Nica-covariant representations can be dilated to isometric Nica-covariant representations. This result is well known for the case of the semigroup  $\mathbb{Z}_+^k$ ; see e.g., [32]. The proof of the existence of an isometric dilation presented here relies on the use of a generalisation of the Schur Product Theorem, and so it provides an alternative proof to what is usually presented for  $\mathbb{Z}_+^k$ .

In Section 3 nonself-adjoint semicrossed product algebras are introduced. In section 3.1 we extend our dilation result from Section 2 to representations of semicrossed products of  $C^*$ -algebras. This result allows us to conclude strong results comparing the different types of semicrossed product algebras. For example, Corollary 3.7, tells us that, in the case of a semicrossed product of a  $C^*$ -algebra, the universal algebra for *completely isometric* Nica-covariant representations is the same as the universal algebra for *completely contractive* Nica-covariant representations. If we were to work with completely isometric and completely contractive semicrossed algebras without imposing the condition of Nica-covariance on our semigroup representations, then an example due to Varopoulos [33] would show that the analogy of Corollary 3.7 would fail in this setting.

In the Section 3.2 we consider the  $C^*$ -envelope of the isometric semicrossed product algebras. In Theorem 3.15 we calculate the  $C^*$ -envelope of the isometric semicrossed product as

$$C_{\text{env}}^*(\mathcal{A} \rtimes_{\alpha}^{\text{iso}} \mathcal{S}) \cong C_{\text{env}}^*(\mathcal{A}) \rtimes_{\alpha} \mathcal{G},$$

where  $\mathcal{G}$  is the group generated by  $\mathcal{S}$ . This result generalises a recent result of Kakariadis and Katsoulis [15], where they worked with the semigroup  $\mathcal{S} = \mathbb{Z}_+$ .

When  $\mathcal{A}$  is a  $C^*$ -algebra the Nica-covariance requirement on our representations allows us to view the semicrossed product algebra  $\mathcal{A} \rtimes_{\alpha} \mathcal{S}$  as a tensor algebra for a product system of  $C^*$ -correspondences over  $\mathcal{S}$ . Thus, from this viewpoint we unite a recent result of Duncan and Peters [11] on the  $C^*$ -envelope of a tensor algebra associated with a dynamical system and the results of Kakariadis and Katsoulis on the  $C^*$ -envelope of the isometric semicrossed product for a dynamical system.

## 2 Nica-covariant Representations

Let  $\mathcal{S}$  be the semigroup  $\mathcal{S} = \sum_{i \in I}^{\oplus} \mathcal{S}_i$ , where for each  $i \in I$  we have  $\mathcal{S}_i$  is a subsemigroup in the additive semigroup  $\mathbb{R}_+$  containing 0. We further assume throughout that  $\mathcal{S}$  is the positive cone of the group  $\mathcal{G}$  it generates. Denote by  $\wedge$  and  $\vee$  the join and meet operations on the lattice group  $\mathcal{G}$ . In Section 3 we will be looking at the case when  $\mathcal{S}$  is countable. However, we will not need to assume that  $\mathcal{S}$  is countable in this section.

**Definition 2.1** A representation  $T: \mathcal{S} \rightarrow \mathcal{B}(\mathcal{H})$  of  $\mathcal{S}$  by contractions  $\{T_s\}_{s \in \mathcal{S}}$  on a Hilbert space  $\mathcal{H}$  is *Nica-covariant* when we have the following relation: if  $s \in \mathcal{S}_i$  and

$t \in \mathcal{S}_j$  where  $i \neq j$ , then  $T_s^* T_t = T_t T_s^*$ .

### 2.1 Isometric Dilations

An isometric representation  $V$  of a semigroup  $\mathcal{S}$  on a Hilbert space  $\mathcal{K}$  is a dilation of a contractive representation  $T$  on  $\mathcal{H} \subseteq \mathcal{K}$  if  $\mathcal{H}$  is invariant under each  $V_s^*$  and  $V_s^*|_{\mathcal{H}} = T_s^*$ . This is precisely when each  $V_s$  is of the form

$$V_s = \begin{bmatrix} T_s & 0 \\ * & * \end{bmatrix}$$

with respect to the decomposition  $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ . We say that  $V$  is a minimal isometric dilation of  $T$  if  $\mathcal{K} = \overline{\text{span}}\{V_s h : h \in \mathcal{H}\}$ .

We wish to show that every Nica-covariant contractive representation of  $\mathcal{S}$  can be dilated to an isometric representation. Further, we will show that there is a unique minimal isometric dilation which is Nica-covariant. This result is well known in the discrete  $\mathcal{S} = \mathbb{Z}_+^k$  case. If each  $\mathcal{S}_i$  is *commensurable*, i.e., if for all  $s_1, \dots, s_n \in \mathcal{S}_i$  there exists  $s_0 \in \mathcal{S}_i$  and  $k_1, \dots, k_n \in \mathbb{N}$  such that  $s_i = k_i s_0$ , then these results have been described by Shalit [27]. We do not impose the condition of commensurability.

The key method to show the existence of the dilation is to use a generalisation of the Schur Product Theorem. To show that there is a minimal Nica-covariant isometric dilation we follow arguments similar to those of Solel [30].

**Definition 2.2** Let  $A = [A_{i,j}]_{1 \leq i,j \leq m}$  and  $B = [B_{i,j}]_{1 \leq i,j \leq m}$  be two matrices of operators where each  $A_{i,j}$  and  $B_{i,j}$  is a bounded operator on a Hilbert space  $\mathcal{H}$ . The *operator-valued Schur product* of  $A$  and  $B$  is defined by  $A \square B := [A_{i,j} B_{i,j}]_{1 \leq i,j \leq m}$ .

In the above definition, if  $\mathcal{H}$  is 1-dimensional, then the operation  $\square$  is simply the classical Schur product (or entry-wise product). In the following theorem we will generalise the Schur Product Theorem, which says that the Schur product of two positive matrices is positive. We follow a similar line of proof as in the classical case presented in [24, Chapter 3].

**Theorem 2.3** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two  $C^*$ -algebras in  $\mathcal{B}(\mathcal{H})$  such that  $\mathcal{A} \subseteq \mathcal{B}$ . Let  $A = [A_{i,j}]_{1 \leq i,j \leq m}$  and  $B = [B_{i,j}]_{1 \leq i,j \leq m}$  be operator matrices with all  $A_{i,j} \in \mathcal{A}$  and  $B_{i,j} \in \mathcal{B}$ . If  $A \geq 0$  and  $B \geq 0$ , then  $A \square B \geq 0$ .

**Proof** Let  $\tilde{A} = A \otimes I_m$  and  $\tilde{B} = [B_{i,j} \otimes I_m]_{1 \leq i,j \leq m}$ . Hence  $\tilde{A}$  and  $\tilde{B}$  are of the form

$$\tilde{A} = \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{bmatrix}$$

and

$$\tilde{B} = \begin{bmatrix} B_{1,1} & 0 & \dots & 0 & \dots & B_{1,m} & 0 & \dots & 0 \\ 0 & B_{1,1} & \dots & 0 & \dots & 0 & B_{1,m} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{1,1} & \dots & 0 & 0 & \dots & B_{1,m} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ B_{m,1} & 0 & \dots & 0 & \dots & B_{m,m} & 0 & \dots & 0 \\ 0 & B_{m,1} & \dots & 0 & \dots & 0 & B_{m,m} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{m,1} & \dots & 0 & 0 & \dots & B_{m,m} \end{bmatrix}$$

It follows that  $\tilde{A}$  and  $\tilde{B}$  are positive commuting operators. Hence  $\tilde{A}\tilde{B}$  is positive.

For each  $1 \leq k \leq m$ , let  $P_k$  be the projection onto the  $(m(k - 1) + k)$ -th copy of  $\mathcal{H}$  in  $\mathcal{H}^{(m^2)}$ , and let  $P = \sum_{k=1}^m P_k$ . Define  $R: \mathcal{H}^{(m)} \rightarrow \mathcal{H}^{(m^2)}$  by  $R\mathbf{h} = P(\mathbf{h}^{\otimes m})$ . Hence  $R$  is an isometry and for

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix} \in \mathcal{H}^{(m)}$$

we have

$$R\mathbf{h} = \begin{bmatrix} h_1 \\ 0 \\ \vdots \\ 0 \\ h_2 \\ 0 \\ \vdots \\ 0 \\ h_m \end{bmatrix},$$

with  $m$  zeroes between  $h_i$  and  $h_{i+1}$ ,  $1 \leq i < m$ . It follows that  $R^*(\tilde{A}\tilde{B})R = A \square B$ . Thus,  $A \square B$  is positive. ■

Let  $T$  be a Nica-covariant contractive representation of  $\mathcal{S}$  on  $\mathcal{H}$ . We extend  $T$  to a map on all of  $\mathcal{G}$  in the following way. Any element  $g \in \mathcal{G}$  can be written uniquely as  $g = g_+ - g_-$ , where  $g_-, g_+ \in \mathcal{S}$  and  $g_- \wedge g_+ = 0$ . Thus we extend  $T$  to  $\mathcal{G}$  by setting  $T_g = T_{g_-}^* T_{g_+} = T_{g_+} T_{g_-}^*$ . A well-known theorem of Sz.-Nagy says that  $T$  has an isometric dilation if and only if for  $s_1, \dots, s_n \in \mathcal{S}$  the operator matrix  $[T_{s_j - s_i}]_{1 \leq i, j \leq n}$  is positive (see e.g., [31, Theorem 7.1]). We will need to look more closely at the proof of this later.

In the case when  $\mathcal{S}$  is a subsemigroup of  $\mathbb{R}_+$  it has been proved by Mlak [20] that a contractive representation  $T$  has an isometric dilation. In the following theorem we will rely on the fact that the representation  $T$  restricted to  $\mathcal{S}_i$  has an isometric dilation for each  $i$ . Then an invocation of Theorem 2.3 will give us our result.

**Theorem 2.4** *Let  $T$  be a Nica-covariant contractive representation of the semigroup  $\mathcal{S} = \sum_{i \in I}^{\oplus} \mathcal{S}_i$ , where each  $\mathcal{S}_i$  is a subsemigroup of  $\mathbb{R}_+$  containing 0. Then  $T$  has an isometric dilation.*

**Proof** Take  $s_1, \dots, s_n$  in  $\mathcal{S}$ . By [31, Theorem 7.1] it suffices to show that the operator matrix  $[T_{s_j - s_i}]_{1 \leq i, j \leq n}$  is positive. Each  $s_j$  is of the form  $s_j = \sum_{i \in I} s_j^{(i)}$ , where  $s_j^{(i)}$  is in  $\mathcal{S}_i$ . We can choose a finite subset  $F \subseteq I$  such that  $s_j = \sum_{i \in F} s_j^{(i)}$  for  $j = 1, \dots, n$ . Since  $F$  is finite, we can and will relabel  $F$  by  $\{1, \dots, k\}$  for some  $k$ . Denote by  $T^{(j)}$  the restriction of  $T$  to  $\mathcal{S}_j$ .

By the Nica-covariance property

$$T_{s_j - s_i} = T_{s_j^{(1)} - s_i^{(1)}}^{(1)} \cdots T_{s_j^{(k)} - s_i^{(k)}}^{(k)}.$$

Thus, we can factor the operator matrix  $[T_{s_j - s_i}]$  as

$$\begin{aligned} [T_{s_j - s_i}]_{i,j} &= \left[ T_{s_j^{(1)} - s_i^{(1)}}^{(1)} \cdots T_{s_j^{(k)} - s_i^{(k)}}^{(k)} \right]_{i,j} \\ &= \left[ T_{s_j^{(1)} - s_i^{(1)}}^{(1)} \right]_{i,j} \square \cdots \square \left[ T_{s_j^{(k)} - s_i^{(k)}}^{(k)} \right]_{i,j}. \end{aligned}$$

Since  $[T_{s_j^{(l)} - s_i^{(l)}}^{(l)}]_{i,j}$  is a positive matrix for  $1 \leq l \leq k$  [20] and since the representation is Nica-covariant, it follows by Theorem 2.3 that  $[T_{s_j - s_i}]_{i,j}$  is positive. ■

In the above we made use of [31, Theorem 7.1] to guarantee the existence of a dilation. We will now pay closer attention to how the dilation there is constructed. Then, following similar arguments of [30], we will show that there is a unique minimal Nica-covariant isometric dilation.

**Theorem 2.5** *Let  $T$  be a Nica-covariant contractive representation of the semigroup  $\mathcal{S} = \sum_{i \in I}^{\oplus} \mathcal{S}_i$ , where each  $\mathcal{S}_i$  is subsemigroup of  $\mathbb{R}_+$  containing 0. Then  $T$  has a minimal isometric dilation that is Nica-covariant. Furthermore, this dilation is unique.*

**Proof** We first sketch the details of the construction of an isometric dilation. Let  $\mathcal{H}$  be the space on which the representation  $T$  acts. Let  $\mathcal{K}_0$  denote the space of all finitely nonzero functions  $f: \mathcal{S} \rightarrow \mathcal{H}$ . For  $f, g \in \mathcal{K}_0$  we define

$$\langle f, g \rangle = \sum_{s,t \in \mathcal{S}} \langle T_{t-s} f(t), g(s) \rangle.$$

By Theorem 2.4 this defines a positive semidefinite sesquilinear form on  $\mathcal{K}_0$ . Let

$$\begin{aligned} \mathcal{N} &= \{f \in \mathcal{K}_0 : \langle f, f \rangle = 0\} \\ &= \{f \in \mathcal{K}_0 : \langle f, g \rangle = 0 \text{ for all } g \in \mathcal{K}_0\}, \end{aligned}$$

and set  $\mathcal{K} = \overline{\mathcal{K}_0 / \mathcal{N}}$ , where the closure is taken with respect to the norm induced by  $\langle \cdot, \cdot \rangle$ . We isometrically embed  $\mathcal{H}$  in  $\mathcal{K}$  by the map  $h \mapsto \widehat{h}$ , where  $\widehat{h}(s) = \delta_0(s)h$ , where  $\delta_0(s) = 1$  if  $s = 0$  and 0 otherwise.

Now define maps  $V_s$  on  $\mathcal{K}_0$  by  $(V_s f)(t) = f(t - s)$  if  $t - s \in \mathcal{S}$  and  $(V_s f)(t) = 0$  otherwise. Note that for  $f \in \mathcal{K}_0$  and  $u \in \mathcal{S}$  we have

$$\begin{aligned} \langle V_u f, V_u f \rangle &= \sum_{s,t} \langle T_{t-s} f(t - u), f(s - u) \rangle = \sum_{s,t \geq u} \langle T_{t-s} f(t - u), f(s - u) \rangle \\ &= \sum_{s,t \geq 0} \langle T_{(t+u)-(s+u)} f(t), f(s) \rangle = \sum_{s,t} \langle T_{t-s} f(t), f(s) \rangle = \langle f, f \rangle. \end{aligned}$$

Hence each  $V_u$  is isometric on  $\mathcal{K}_0$  and leaves  $\mathcal{N}$  invariant. It follows that we can extend  $V_u$  to an isometry on  $\mathcal{K}$ , and we have that  $\{V_s\}_{s \in \mathcal{S}}$  is an isometric representation of  $\mathcal{S}$ .

Furthermore, note that for  $g \in \mathcal{G}$  and  $h, k \in \mathcal{H}$  we have

$$\begin{aligned} \langle V_g \widehat{h}, \widehat{k} \rangle &= \langle V_{g_-}^* V_{g_+} \widehat{h}, \widehat{k} \rangle = \langle V_{g_+} \widehat{h}, V_{g_-} \widehat{k} \rangle \\ &= \sum_{s,t \in \mathcal{S}} \langle T_{t-s} \widehat{h}(t - g_+), \widehat{k}(s - g_-) \rangle \\ &= \langle T_{g_+ - g_-} h, k \rangle = \langle T_g h, k \rangle. \end{aligned}$$

Thus we have  $P_{\mathcal{H}} V_g|_{\mathcal{H}} = T_g$  for all  $g \in \mathcal{G}$ . In particular  $\{V_s\}_{s \in \mathcal{S}}$  is an isometric dilation of  $\{T_s\}_{s \in \mathcal{S}}$ . It is easily seen to be a minimal isometric dilation. Dilations with the property that  $P_{\mathcal{H}} V_g|_{\mathcal{H}} = T_g$  are called *regular* dilations. We want to show that this dilation is Nica-covariant.

Next we will show that if we have  $s \in \mathcal{S}_i$  and  $\mu \in \mathcal{S}$  such that  $s \wedge \mu = 0$ . then  $V_s^* V_\mu|_{\mathcal{H}} = V_\mu V_s^*|_{\mathcal{H}}$ . Take  $s, \mu$  as described,  $\nu \in \mathcal{S}$  and  $h, k \in \mathcal{H}$ . By the minimality of the dilation it suffices to show that

$$\langle V_s^* V_\mu \widehat{h}, V_\nu \widehat{k} \rangle = \langle V_\mu V_s^* \widehat{h}, V_\nu \widehat{k} \rangle.$$

We calculate

$$\begin{aligned} \langle V_s^* V_\mu \widehat{h}, V_\nu \widehat{k} \rangle &= \langle V_\nu^* V_s^* V_\mu \widehat{h}, \widehat{k} \rangle = \langle V_{(\mu - \nu - s)_-}^* V_{(\mu - \nu - s)_+} \widehat{h}, \widehat{k} \rangle \\ &= \langle T_{(\mu - \nu - s)_-}^* T_{(\mu - \nu - s)_+} h, k \rangle. \end{aligned}$$

Note that, by our choice of  $s$  and  $\mu$ , we have that  $(\mu - \nu - s)_+ = (\mu - \nu)_+$  and  $(\mu - \nu - s)_- = s + (\mu - \nu)_-$ . Also  $s \wedge (\mu - \nu)_+ = 0$ . Thus,

$$\begin{aligned} \langle V_s^* V_\mu \widehat{h}, V_\nu \widehat{k} \rangle &= \langle T_{(\mu - \nu - s)_-}^* T_{(\mu - \nu - s)_+} h, k \rangle = \langle T_{s + (\mu - \nu)_-}^* T_{(\mu - \nu)_+} h, k \rangle \\ &= \langle T_{(\mu - \nu)_-}^* T_{(\mu - \nu)_+} T_s^* h, k \rangle = \langle P_{\mathcal{H}} V_{(\mu - \nu)_-}^* V_{(\mu - \nu)_+} P_{\mathcal{H}} V_s^* \widehat{h}, \widehat{k} \rangle \\ &= \langle V_{(\mu - \nu)_-}^* V_{(\mu - \nu)_+} V_s^* \widehat{h}, \widehat{k} \rangle = \langle V_\mu V_s^* \widehat{h}, V_\nu \widehat{k} \rangle. \end{aligned}$$

This tells us that the representation  $V$  has the Nica-covariant property when restricted to  $\mathcal{H}$ . We will now extend this to all of  $\mathcal{K}$ .

By the minimality of the representation  $V$  it suffices to show that for  $s \in S_i, t \in S_j$ , where  $i \neq j, \mu, \nu \in \mathcal{S}$  and  $h, k \in \mathcal{H}$  that

$$\langle V_s^* V_t V_\mu \widehat{h}, V_\nu \widehat{k} \rangle = \langle V_t V_s^* V_\mu \widehat{h}, V_\nu \widehat{k} \rangle.$$

The right-hand side of the above is

$$\begin{aligned} \langle V_t V_s^* V_\mu \widehat{h}, V_\nu \widehat{k} \rangle &= \langle V_\nu^* V_t V_s^* V_\mu \widehat{h}, \widehat{k} \rangle = \langle V_\nu^* V_t V_{(\mu-s)_-}^* V_{(\mu-s)_+} \widehat{h}, \widehat{k} \rangle \\ &= \langle V_\nu^* V_t V_{(\mu-s)_+} V_{(\mu-s)_-}^* \widehat{h}, \widehat{k} \rangle. \end{aligned}$$

Note that  $t + (\mu - s)_+ = (t + \mu - s)_+$  and  $(\mu - s)_- = (t + \mu - s)_-$ , hence we have

$$\begin{aligned} \langle V_t V_s^* V_\mu \widehat{h}, V_\nu \widehat{k} \rangle &= \langle V_\nu^* V_{(t+\mu-s)_-}^* V_{(t+\mu-s)_+} \widehat{h}, \widehat{k} \rangle = \langle V_{\nu+(t+\mu-s)_-}^* V_{(t+\mu-s)_+} \widehat{h}, \widehat{k} \rangle \\ &= \langle V_{(t+\mu-\nu-s)_-}^* V_{(t+\mu-\nu-s)_+} \widehat{h}, \widehat{k} \rangle, \end{aligned}$$

with the last equality coming from the fact that

$$((t + \mu - s)_+ - (t + \mu - s)_- - \nu)_- = (t + \mu - s - \nu)_-$$

and

$$((t + \mu - s)_+ - (t + \mu - s)_- - \nu)_+ = (t + \mu - s - \nu)_+.$$

Hence,

$$\begin{aligned} \langle V_t V_s^* V_\mu \widehat{h}, V_\nu \widehat{k} \rangle &= \langle V_{(t+\mu-\nu-s)_-}^* V_{(t+\mu-\nu-s)_+} \widehat{h}, \widehat{k} \rangle \\ &= \langle V_\nu^* V_s^* V_t V_\mu \widehat{h}, \widehat{k} \rangle = \langle V_s^* V_t V_\mu \widehat{h}, V_\nu \widehat{k} \rangle. \end{aligned}$$

It follows that  $V$  is Nica-covariant.

To show that the dilation is unique we follow a standard argument. Suppose  $V$  and  $W$  are two minimal isometric Nica-covariant dilations of  $T$  on  $\mathcal{K}_1$  and  $\mathcal{K}_2$  respectively. Take  $h_1, h_2 \in \mathcal{H}$  and  $\nu, \mu \in \mathcal{S}$ . Then

$$\langle V_\mu h_1, V_\nu h_2 \rangle = \langle V_\nu^* V_\mu h_1, h_2 \rangle = \langle V_{(\mu-\nu)_-}^* V_{(\mu-\nu)_+} h_1, h_2 \rangle = \langle T_{\mu-\nu} h_1, h_2 \rangle.$$

Similarly  $\langle W_\mu h_1, W_\nu h_2 \rangle = \langle T_{\mu-\nu} h_1, h_2 \rangle$ . Thus the map  $U : V_\nu h \mapsto W_\nu h$  extends to a unitary from  $\mathcal{K}_1$  to  $\mathcal{K}_2$  that fixes  $\mathcal{H}$ , and the two dilations  $V$  and  $W$  are unitarily equivalent. ■

### 3 Semicrossed Product Algebras

Throughout let  $\mathcal{S}$  be the semigroup  $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$ , where each  $\mathcal{S}_i$  is a countable sub-semigroup of  $\mathbb{R}_+$  containing 0. We further suppose that  $\mathcal{S}$  is the positive cone of the group  $\mathcal{G}$  generated by  $\mathcal{S}$ .

**Definition 3.1** Let  $\mathcal{A}$  be a unital operator algebra. If  $\alpha = \{\alpha_s : s \in \mathcal{S}\}$  is a family of completely isometric unital endomorphisms of  $\mathcal{A}$  forming an action of  $\mathcal{S}$  on  $\mathcal{A}$ , then we call the triple  $(\mathcal{A}, \mathcal{S}, \alpha)$  a *semigroup dynamical system*.

**Definition 3.2** Let  $(\mathcal{A}, \mathcal{S}, \alpha)$  be a semigroup dynamical system. An *isometric (contractive) Nica-covariant representation* of  $(\mathcal{A}, \mathcal{S}, \alpha)$  on a Hilbert space  $\mathcal{H}$  consists of a pair  $(\sigma, V)$ , where  $\sigma$  is a completely contractive representation  $\sigma: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  and  $V = \{V_s\}_{s \in \mathcal{S}}$  is an isometric (contractive) Nica-covariant representation of  $\mathcal{S}$  on  $\mathcal{H}$  such that

$$\sigma(A)V_s = V_s\sigma(\alpha_s(A))$$

for all  $A \in \mathcal{A}$  and  $s \in \mathcal{S}$ .

We will be interested in two nonself-adjoint semicrossed product algebras associated with a semigroup dynamical system  $(\mathcal{A}, \mathcal{S}, \alpha)$ . We define  $\mathcal{A}_N \times_\alpha \mathcal{S}$  to be the universal algebra for all contractive Nica-covariant representations of  $(\mathcal{A}, \mathcal{S}, \alpha)$  in the sense that contractive Nica-covariant representations are uniquely determined by representations of  $\mathcal{A}_N \times_\alpha \mathcal{S}$ . We define  $\mathcal{A}_N \times_\alpha^{\text{iso}} \mathcal{S}$  similarly as the universal algebra for all isometric Nica-covariant representations.

The algebras  $\mathcal{A} \times_\alpha^{\text{iso}} \mathbb{Z}_+$  were introduced by Kakariadis and Katsoulis [15] and have proven to be a more tractable class of algebras than  $\mathcal{A} \times_\alpha \mathbb{Z}$ . While in general one expects  $\mathcal{A}_N \times_\alpha \mathcal{S}$  and  $\mathcal{A}_N \times_\alpha^{\text{iso}} \mathcal{S}$  to be different, there are times when the two algebras coincide. For example, when  $\mathcal{A} = \mathfrak{A}_n$  is the noncommutative disc algebra and  $\mathcal{S} = \mathbb{Z}_+$  it follows from [5] that

$$\mathfrak{A}_n \times_\alpha^{\text{iso}} \mathbb{Z}_+ \cong \mathfrak{A}_n \times_\alpha \mathbb{Z}_+.$$

Further examples of when the semicrossed product and the isometric semicrossed product are the same for the case  $\mathcal{S} = \mathbb{Z}_+$  can be found in [8, Section 12]. When  $\mathcal{A}$  is a unital  $C^*$ -algebra we will see (Corollary 3.7) that

$$\mathcal{A}_N \times_\alpha^{\text{iso}} \mathcal{S} \cong \mathcal{A}_N \times_\alpha \mathcal{S}.$$

Let  $\mathcal{P}(\mathcal{A}, \mathcal{S})$  be the algebra of all formal polynomials  $p$  of the form

$$p = \sum_{i=1}^n \mathcal{V}_{s_i} A_{s_i},$$

where  $s_1, \dots, s_n$  are in  $\mathcal{S}$ , with multiplication defined by  $A\mathcal{V}_s = \mathcal{V}_s\alpha(A)$ . If  $(\sigma, T)$  is a contractive Nica-covariant representation of  $(\mathcal{A}, \mathcal{S}, \alpha)$ , then we can define a representation  $\sigma \times T$  of  $\mathcal{P}(\mathcal{A}, \mathcal{S})$  by

$$(\sigma \times T)\left(\sum_{i=1}^n \mathcal{V}_{s_i} A_{s_i}\right) = \sum_{i=1}^n T_{s_i} \sigma(A_{s_i}).$$

We define two norms on  $\mathcal{P}(\mathcal{A}, \mathcal{S})$  as follows. For  $p \in \mathcal{P}(\mathcal{A}, \mathcal{S})$  let

$$\|p\| = \sup_{\substack{(\sigma, T) \text{ contractive} \\ \text{Nica-covariant}}} \{ \|(\sigma \times T)(p)\| \}$$



and

$$\|p\|_{\text{iso}} = \sup_{\substack{(\sigma, V) \text{ isometric} \\ \text{Nica-covariant}}} \{ \|(\sigma \times V)(p)\| \}.$$

We can realise our semicrossed product algebras as

$$\mathcal{A}_{N \times_{\alpha} \mathcal{S}} = \overline{\mathcal{P}(\mathcal{A}, \mathcal{S})}^{\|\cdot\|} \quad \text{and} \quad \mathcal{A}_{N \times_{\alpha}^{\text{iso}} \mathcal{S}} = \overline{\mathcal{P}(\mathcal{A}, \mathcal{S})}^{\|\cdot\|_{\text{iso}}}.$$

If  $(\mathcal{B}, \mathcal{G}, \beta)$  is a dynamical system where  $\beta$  is an action of the group  $\mathcal{G}$  on the  $C^*$ -algebra  $\mathcal{B}$  by automorphisms, there is an adjoint operation on  $\mathcal{P}(\mathcal{B}, \mathcal{G})$  given by  $(\mathcal{V}_g \mathcal{B})^* := \mathcal{V}_{-g} \beta_g^{-1}(\mathcal{B}^*)$ . If  $(\pi, U)$  is covariant representation of  $(\mathcal{B}, \mathcal{G}, \beta)$ , then  $\{U_s\}_{s \in \mathcal{S}}$  is necessarily a family of commuting unitaries, and hence  $\{U_s\}_{s \in \mathcal{S}}$  is automatically Nica-covariant.

**Example 3.3** Let  $(\mathcal{A}, \mathcal{S}, \alpha)$  be a semigroup dynamical system. Let  $\sigma$  be a completely contractive representation of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$ . Define a completely contractive representation  $\tilde{\sigma}$  of  $\mathcal{A}$  on  $\mathcal{H} \otimes \ell^2(\mathcal{S})$  by

$$\tilde{\sigma}(A)(h_s)_{s \in \mathcal{S}} = (\sigma(\alpha_s(A))h_s)_{s \in \mathcal{S}}$$

for all  $A \in \mathcal{A}$  and  $(h_s)_{s \in \mathcal{S}} \in \mathcal{H} \otimes \ell^2(\mathcal{S})$ .

For each  $s \in \mathcal{S}$  define an operator  $W_s$  on  $\mathcal{H} \otimes \ell^2(\mathcal{S})$  by  $W_s(h)_t = (h)_{s+tt}$ , where  $h \in \mathcal{H}$  and  $(h)_s \in \mathcal{H} \otimes \ell^2(\mathcal{S})$  is the vector with  $h$  in the  $s$ -th position and 0 everywhere else. Then  $(\tilde{\sigma}, W)$  is an isometric Nica-covariant representation of  $(\mathcal{A}, \mathcal{S}, \alpha)$ .

Note that in the case where each  $\alpha_s$  is an automorphism on  $\mathcal{A}$  then we can extend this idea to give a Nica-covariant representation  $(\hat{\sigma}, U)$  on  $\mathcal{H} \otimes \ell^2(\mathcal{G})$ , where each  $U_s$  is unitary.

**Definition 3.4** The isometric Nica-covariant representation  $(\tilde{\sigma}, W)$  constructed above is called an *induced representation* of  $(\mathcal{A}, \mathcal{S}, \alpha)$ .

### 3.1 Dilations of Nica-covariant representations

We now consider some dilation results for Nica-covariant representations of a semigroup dynamical system  $(\mathcal{A}, \mathcal{S}, \alpha)$  in the case when  $\mathcal{A}$  is a  $C^*$ -algebra.

In the case that  $\mathcal{S} = \mathbb{Z}_+^k$  the following theorem is a special case of a theorem of Solel’s [30, Theorem 3.1] that deals with representations of product systems of  $C^*$ -correspondences. The result has also been shown by Ling and Muhly [18] for the case  $\mathcal{S} = \mathbb{Z}_+^k$  and  $\alpha$  is an action on  $\mathcal{A}$  by automorphisms.

**Theorem 3.5** Let  $\mathcal{S} = \sum_{i \in I}^{\oplus} \mathcal{S}_i$ , where each  $\mathcal{S}_i$  is a subsemigroup of  $\mathbb{R}_+$  containing 0 and let  $(\mathcal{A}, \mathcal{S}, \alpha)$  be a semigroup dynamical system, where  $\mathcal{A}$  is a unital  $C^*$ -algebra. Let  $(\sigma, T)$  be a contractive Nica-covariant representation of  $(\mathcal{A}, \mathcal{S}, \alpha)$  on  $\mathcal{H}$ . Then there is an isometric Nica-covariant representation  $(\pi, V)$  of  $(\mathcal{A}, \mathcal{S}, \alpha)$  on  $\mathcal{K} \supseteq \mathcal{H}$  such that

- (i)  $\pi(A)|_{\mathcal{H}} = \sigma(A)$  for all  $A \in \mathcal{A}$

(ii)  $P_{\mathcal{H}}V_s|_{\mathcal{H}} = T_s$  for all  $s \in \mathcal{S}$ .

Furthermore,  $\mathcal{K}$  is minimal in the sense that  $\mathcal{K} = \bigvee_{s \in \mathcal{S}} V_s \mathcal{H}$ .

**Proof** Let  $\mathcal{K}_0$ ,  $\mathcal{K}$ , and  $\mathcal{N}$  be as in the proof of Theorem 2.5. For each  $A \in \mathcal{A}$  we define  $\pi_0(A)$  on  $\mathcal{K}_0$  by

$$(\pi_0(A)f)(s) = \sigma(\alpha_s(A))f(s),$$

for each  $f \in \mathcal{K}_0$  and  $s \in \mathcal{S}$ . Note that, for  $A \in \mathcal{A}$  and  $t, s \in \mathcal{S}$  we have

$$\begin{aligned} T_{t-s}\sigma(\alpha_t(A)) &= T_{(t-s)_+}T_{(t-s)_-}^*\sigma(\alpha_t(A)) = T_{(t-s)_+}\sigma(\alpha_{t+(t-s)_-}(A))T_{(t-s)_-}^* \\ &= \sigma(\alpha_{t+(t-s)_--(t-s)_+}(A))T_{(t-s)_-}^*T_{(t-s)_+} = \sigma(\alpha_s(A))T_{t-s}. \end{aligned}$$

It follows that, if  $f \in \mathcal{N}$  and  $g \in \mathcal{K}_0$  then for each  $A \in \mathcal{A}$ ,

$$\begin{aligned} \langle \pi_0(A)f, g \rangle &= \sum_{s,t} \langle T_{t-s}\sigma(\alpha_t(A))f(t), g(s) \rangle \\ &= \sum_{s,t} \langle T_{t-s}f(t), \sigma(\alpha_s(A^*))g(s) \rangle = 0, \end{aligned}$$

we thus can extend  $\pi_0$  to a representation  $\pi$

$$\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K}).$$

It is easy to check that  $(\pi, V)$  form a Nica-covariant representation with the desired properties. ■

**Remark 3.6** In the case when  $\mathcal{S} = \sum_{i \in I}^{\oplus} \mathcal{S}_i$ , where each  $\mathcal{S}_i$  is a subsemigroup of  $\mathbb{R}_+$  containing 0 and each  $\mathcal{S}_i$  has the extra condition of being commensurable then the statement of Theorem 3.5 is a special case of [28, Theorem 4.2]. However, in the proof there, the only place where the commensurable condition is used is in ensuring that contractive Nica-covariant representation of  $\mathcal{S}$  has minimal Nica-covariant isometric dilation. As Theorems 2.4 and 2.5 provide the existence of minimal Nica-covariant isometric dilations in the case when each  $\mathcal{S}_i$  is not necessarily commensurable the proof given in [28] provides an alternate proof of Theorem 3.5.

**Corollary 3.7** Let  $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$ , where each  $\mathcal{S}_i$  is a countable subsemigroup of  $\mathbb{R}_+$  containing 0 and let  $(\mathcal{A}, \mathcal{S}, \alpha)$  be a semigroup dynamical system where  $\mathcal{A}$  is a unital  $C^*$ -algebra. Then the norms  $\|\cdot\|$  and  $\|\cdot\|_{\text{iso}}$  on  $\mathcal{P}(\mathcal{A}, \mathcal{S})$  are the same. Hence

$$\mathcal{A}_{N \times_{\alpha}^{\text{iso}} \mathcal{S}} = \mathcal{A}_{N \times_{\alpha} \mathcal{S}}.$$

**Proof** Take any  $p \in \mathcal{P}(\mathcal{A}, \mathcal{S})$ . Since an isometric Nica-covariant representation is itself contractive it follows that  $\|p\|_{\text{iso}} \leq \|p\|$ . Now take a contractive Nica-covariant representation  $(\sigma, T)$  on a Hilbert space  $\mathcal{H}$ . Let  $(\pi, V)$  be the minimal isometric Nica-covariant dilation of  $(\sigma, T)$ . Then

$$\|(\sigma \times T)(p)\| = \|P_{\mathcal{H}}(\pi \times V)(p)P_{\mathcal{H}}\| \leq \|(\pi \times V)(p)\|.$$

Hence  $\|p\| \leq \|p\|_{\text{iso}}$ . ■

**Remark 3.8** Let  $(\mathcal{A}, \mathcal{S}, \alpha)$  be a semigroup dynamical system. If  $\mathcal{A}$  is a  $C^*$ -algebra, then  $(\mathcal{A}, \mathcal{S}, \alpha)$  can be used to describe a product system of  $C^*$ -correspondences over  $\mathcal{S}$ . Fowler constructs a concrete  $C^*$ -algebra that is universal for Nica-covariant completely contractive representations of this product system [12]. It was observed by Solel [30] that the nonself-adjoint Banach algebra formed by the left regular representation of the product system is universal for Nica-covariant completely contractive representations (while Solel was working in  $\mathbb{Z}_+^k$  the same reasoning works for countable  $\mathcal{S}$ ). Thus  $\mathcal{A}_{N \times_\alpha \mathcal{S}}$  can also be realised as the concrete tensor algebra in the sense of Solel; see [30, Corollary 3.17].

Furthermore, if  $\sigma$  is a faithful representation of  $\mathcal{A}$ , it follows that the induced representation  $(\tilde{\sigma}, W)$  is a completely isometric representation of  $\mathcal{A}_{N \times_\alpha \mathcal{S}}$ .

The following theorem can be proved by a standard argument in dynamical systems using direct limits of  $C^*$ -algebras. As stated below, the result is a special case of [16, Theorem 2.1] and [22, Section 2].

**Theorem 3.9** Let  $(\mathcal{A}, \mathcal{S}, \alpha)$  be a semigroup dynamical system, where  $\mathcal{A}$  is a  $C^*$ -algebra and each  $\alpha_s$  is injective. Then there exists a  $C^*$ -dynamical system  $(\mathcal{B}, \mathcal{G}, \beta)$  where each  $\beta_s$  is an automorphism, unique up to isomorphism, together with an embedding  $i: \mathcal{A} \rightarrow \mathcal{B}$  such that

- (i)  $\beta_s \circ i = i \circ \alpha_s$ , i.e.,  $\beta$  dilates  $\alpha$
- (ii)  $\bigcup_{s \in \mathcal{S}} \beta_s^{-1}(i(\mathcal{A}))$  is dense in  $\mathcal{B}$ , i.e.,  $\mathcal{B}$  is minimal.

**Definition 3.10** Let  $(\mathcal{A}, \mathcal{S}, \alpha)$  and  $(\mathcal{B}, \mathcal{G}, \beta)$  be as in Theorem 3.9, then we call  $(\mathcal{B}, \mathcal{G}, \beta)$  the *minimal automorphic dilation* of  $(\mathcal{A}, \mathcal{S}, \alpha)$ .

The minimal automorphic dilation of a dynamical system is frequently utilised in the literature. Group crossed product  $C^*$ -algebras have a long history and are well understood objects. Thus it is beneficial if one can relate a semicrossed algebra to a crossed product algebra, often the crossed product algebra of the minimal automorphic dilation. We will see in Theorem 3.15 that the minimal automorphic dilation plays an important role when calculating the  $C^*$ -envelope of crossed product algebras. First we will show now that  $\mathcal{A}_{N \times_\alpha \mathcal{S}}$  sits nicely inside  $\mathcal{B} \times_\beta \mathcal{G}$ . In the case where  $\mathcal{S} = \mathbb{Z}_+$  the following has been shown by Kakariadis and Katsoulis [15] and Peters [26].

**Theorem 3.11** Let  $(\mathcal{A}, \mathcal{S}, \alpha)$  be a semigroup dynamical system, where  $\mathcal{A}$  is a  $C^*$ -algebra and each  $\alpha_s$  is injective. Let  $(\mathcal{B}, \mathcal{G}, \beta)$  be the minimal automorphic dilation of  $(\mathcal{A}, \mathcal{S}, \alpha)$ . The  $\mathcal{A}_{N \times_\alpha \mathcal{S}}$  is completely isometrically isomorphic to a subalgebra of  $\mathcal{B} \times_\beta \mathcal{G}$ .

Furthermore,  $\mathcal{A}_{N \times_\alpha^{\text{iso}} \mathcal{S}}$  generates  $\mathcal{B} \times_\beta \mathcal{G}$  as a  $C^*$ -algebra.

**Proof** Let  $\sigma$  be a faithful representation of  $\mathcal{A}$  on  $\mathcal{H}$ . Then the induced representation  $\tilde{\sigma} \times W$  is a completely isometric representation of  $\mathcal{A}_{N \times_\alpha \mathcal{S}}$ , by Remark 3.8. We will embed this completely isometric copy of  $\mathcal{A}_{N \times_\alpha \mathcal{S}}$  into a completely isometric representation of  $\mathcal{B} \times_\beta \mathcal{G}$  by suitably dilating the representation  $(\tilde{\sigma}, W)$ .

Let  $i$  be the embedding of  $\mathcal{A}$  into  $\mathcal{B}$  as in Theorem 3.9. The representation  $\sigma$  also defines a faithful representation of  $i(\mathcal{A})$ , which we will also denote by  $\sigma$ . We can

thus find a representation  $\pi$  of  $\mathcal{B}$  on  $\mathcal{K} \supseteq \mathcal{H}$  such that  $\pi(A)|_{\mathcal{H}} = \sigma(i(A))$  for all  $A \in \mathcal{A}$ , see e.g., [25, Proposition 4.1.8]. We thus have an induced representation  $\widehat{\pi} \times U$  of  $\mathcal{B} \times_{\beta} \mathcal{G}$ . Restricting  $\pi$  to  $\mathcal{A}$  we see that  $(\widehat{\pi} \circ i) \times U$  is a completely isometric representation of  $\mathcal{A}_{N \times_{\alpha} \mathcal{S}}$ , since  $\widetilde{\sigma} \times W$  is. Further note that  $\widehat{\pi}$  is faithful on  $\bigcup_{s \in \mathcal{S}} \beta_s^{-1}(\mathcal{A})$ . By the construction of  $\mathcal{B}$ ,  $\widehat{\pi}$  is also faithful representation of  $\mathcal{B}$ . Now, by [25, Theorem 7.7.5],  $\widetilde{\sigma} \times W$  is a faithful representation of  $\mathcal{B} \times_{\beta} \mathcal{G}$ . Hence  $\mathcal{A}_{N \times_{\alpha} \mathcal{S}}$  sits completely isometrically inside  $\mathcal{B} \times_{\beta} \mathcal{G}$ .

That  $\mathcal{A}_{N \times_{\alpha} \mathcal{S}}$  generates  $\mathcal{B} \times_{\beta} \mathcal{G}$  as a  $C^*$ -algebra follows immediately after considering the algebra  $\text{Alg}\{\mathcal{P}(\mathcal{A}, \mathcal{S}), (\mathcal{P}(\mathcal{A}, \mathcal{S}))^*\}$  inside  $\mathcal{P}(\mathcal{B}, \mathcal{G})$ . ■

### 3.2 $C^*$ -Envelopes

Our goal in this subsection is to calculate the  $C^*$ -envelope of  $\mathcal{A}_{N \times_{\alpha}^{\text{iso}} \mathcal{S}}$  in the case when  $\alpha$  is a family of completely isometric automorphisms on a unital operator algebra  $\mathcal{A}$ .

If  $\mathcal{C}$  is a  $C^*$ -algebra that completely isometrically contains  $\mathcal{A}$  such that  $\mathcal{C} = C^*(\mathcal{A})$  then we call  $\mathcal{C}$  a  $C^*$ -cover of  $\mathcal{A}$ . If  $\mathcal{A}$  is a  $C^*$ -algebra, Theorem 3.11 says that  $\mathcal{B} \times_{\beta} \mathcal{G}$  is a  $C^*$ -cover of  $\mathcal{A}_{N \times_{\alpha}^{\text{iso}} \mathcal{S}}$  when  $(\mathcal{B}, \mathcal{G}, \beta)$  is the minimal automorphic dilation of  $(\mathcal{A}, \mathcal{S}, \alpha)$ .

**Definition 3.12** Let  $\mathcal{A}$  be an operator algebra and let  $\mathcal{C}$  be a  $C^*$ -cover of  $\mathcal{A}$ . Let  $\alpha$  define an action of  $\mathcal{S}$  on  $\mathcal{C}$  by faithful  $*$ -endomorphisms that leave  $\mathcal{A}$  invariant. We define the *relative semicrossed product*  $\mathcal{A}_{N \times_{\mathcal{C}, \alpha} \mathcal{S}}$  to be the subalgebra of  $\mathcal{C}_{N \times_{\alpha} \mathcal{S}}$  generated by the natural copy of  $\mathcal{A}$  inside  $\mathcal{C}_{N \times_{\alpha} \mathcal{S}}$  and the universal isometries  $\{\mathcal{V}_s\}_{s \in \mathcal{S}}$ .

The idea of a relative semicrossed product was introduced by Kakariadis and Katsoulis [15] when studying semicrossed products by the semigroup  $\mathbb{Z}_+$ . The key idea is to realise the universal algebra  $\mathcal{A}_{N \times_{\alpha}^{\text{iso}} \mathcal{S}}$  as a relative semicrossed algebra. This allows a concrete place in which to try and discover the  $C^*$ -envelope.

It is important to note that if  $\alpha$  is an action of  $\mathcal{S}$  on an operator algebra  $\mathcal{A}$  by completely isometric automorphisms which extend to completely isometric automorphisms of a  $C^*$ -cover  $\mathcal{C}$  of  $\mathcal{A}$ , then each  $\alpha_s$  necessarily leaves the Shilov boundary ideal  $\mathcal{J}$  of  $\mathcal{A}$  in  $\mathcal{C}$  invariant, see e.g., [8, Proposition 10.6]. We will write  $\{\alpha_s\}_{s \in \mathcal{S}}$  for the automorphisms on  $\mathcal{A}/\mathcal{J}$  induced by the automorphisms  $\{\alpha_s\}_{s \in \mathcal{S}}$  on  $\mathcal{A}$ .

The proof of the following proposition follows the same reasoning as the proof of [15, Proposition 2.3]. It is an application of Ditschel and McCullough’s [9] result that any representation can be dilated to a maximal representation and Muhly and Solel’s [21] result that any maximal representation extends to a  $*$ -representation of any  $C^*$ -cover.

**Proposition 3.13** Let  $\mathcal{A}$  be an operator algebra and let  $\mathcal{C}$  be a  $C^*$ -cover of  $\mathcal{A}$ . Let  $\alpha$  be an action of  $\mathcal{S}$  on  $\mathcal{C}$  by automorphisms that restrict to automorphisms of  $\mathcal{A}$ . Let  $\mathcal{J}$  be the Shilov boundary ideal of  $\mathcal{A}$  in  $\mathcal{C}$ . Then the relative semicrossed products  $\mathcal{A}_{N \times_{\mathcal{C}, \alpha} \mathcal{S}}$  and  $\mathcal{A}/\mathcal{J}_{N \times_{\mathcal{C}/\mathcal{J}, \alpha} \mathcal{S}}$  are completely isometrically isomorphic.

Let  $(\mathcal{C}, \mathcal{S}, \alpha)$  be a semigroup dynamical system, where  $\mathcal{C}$  is a  $C^*$ -algebra and each

$\alpha_s$  is an automorphism on  $\mathcal{C}$ . Then it is immediate that the minimal automorphic dilation of  $(\mathcal{C}, \mathcal{S}, \alpha)$  is simply  $(\mathcal{C}, \mathcal{G}, \alpha)$ . If we view  $\mathcal{G}$  as being a discrete group, then  $\mathcal{G}$  has a compact dual  $\widehat{\mathcal{G}}$ . Recall that for every character  $\gamma$  in  $\widehat{\mathcal{G}}$  we can define an automorphism  $\tau_\gamma$  on  $\mathcal{P}(\mathcal{C}, \mathcal{G})$  by

$$\tau_\gamma\left(\sum_{i=1}^n \mathcal{V}_{s_i} A_{s_i}\right) = \sum_{i=1}^n \gamma(s_i) \mathcal{V}_{s_i} A_{s_i}.$$

The automorphism  $\tau_\gamma$  extends to an automorphism of  $\mathcal{C} \times_\alpha \mathcal{G}$  with  $\mathcal{C}$  as its fixed-point set [25, Proposition 7.8.3.]. We call  $\tau_\gamma$  a *gauge automorphism*. The gauge automorphisms restrict to automorphisms of  $\mathcal{C}_N \times_\alpha \mathcal{S}$ .

**Lemma 3.14** *Let  $\mathcal{A}$  be a unital operator algebra. Let  $\mathcal{C}$  be a  $C^*$ -cover of  $\mathcal{A}$  and let  $\mathcal{J}$  be the Shilov boundary ideal of  $\mathcal{A}$  in  $\mathcal{C}$ . Let  $\alpha$  be an action of  $\mathcal{S}$  on  $\mathcal{C}$  by automorphisms which restrict to completely isometric automorphisms of  $\mathcal{A}$ . Then*

$$C_{\text{env}}^*(\mathcal{A}_N \times_{\mathcal{C}, \alpha} \mathcal{S}) \cong C_{\text{env}}^*(\mathcal{A}) \times_{\dot{\alpha}} \mathcal{G}.$$

**Proof** By the preceding proposition it suffices to show that

$$C_{\text{env}}^*(\mathcal{A}/\mathcal{J}_N \times_{\mathcal{C}/\mathcal{J}, \dot{\alpha}} \mathcal{S}) \cong \mathcal{C}/\mathcal{J} \times_{\dot{\alpha}} \mathcal{G}.$$

The algebra  $\mathcal{A}/\mathcal{J}_N \times_{\mathcal{C}/\mathcal{J}, \dot{\alpha}} \mathcal{S}$  embeds completely isometrically into  $\mathcal{C}/\mathcal{J} \times_{\dot{\alpha}} \mathcal{G}$  and generates it as a  $C^*$ -algebra. Let  $\mathcal{J}$  be the Shilov boundary ideal of  $\mathcal{A}/\mathcal{J}_N \times_{\mathcal{C}/\mathcal{J}, \dot{\alpha}} \mathcal{S}$  in  $\mathcal{C}/\mathcal{J} \times_{\dot{\alpha}} \mathcal{G}$ . Suppose that  $\mathcal{J} \neq \{0\}$ .

The ideal  $\mathcal{J}$  is invariant under automorphisms of  $\mathcal{C}/\mathcal{J} \times_{\dot{\alpha}} \mathcal{G}$  and hence by the gauge automorphisms of  $\mathcal{C}/\mathcal{J} \times_{\dot{\alpha}} \mathcal{G}$ . Therefore  $\mathcal{J}$  has non-trivial intersection with the fixed points of the gauge automorphisms, *i.e.*,  $\mathcal{J} \cap \mathcal{C}/\mathcal{J} \neq \{0\}$ . But  $\mathcal{J} \cap \mathcal{C}/\mathcal{J}$  is a boundary ideal for  $\mathcal{A}$  in  $\mathcal{C}/\mathcal{J}$ . Hence  $\mathcal{J} = \{0\}$ . This proves the result. ■

We can now prove the main result of this section. This theorem generalises the result of Kakariadis and Katsoulis [15] from the semigroup  $\mathbb{Z}_+$  to our more general semigroups  $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$ . From another viewpoint, in the case when  $\mathcal{A}$  is a  $C^*$ -algebra and  $\mathcal{A}_N \times_{\alpha}^{\text{iso}} \mathcal{S} \cong \mathcal{A}_N \times_{\alpha} \mathcal{S}$  we have that the  $C^*$ -envelope of an associated tensor algebra is a crossed product algebra, by Remark 3.8 and Corollary 3.7. This was shown for abelian  $C^*$ -algebras by Duncan and Peters [11].

By [8, Proposition 10.1] the group  $\text{Aut}(\mathcal{A})$  of completely isometric automorphisms on the unital operator algebra  $\mathcal{A}$  is isomorphic to the group of completely isometric automorphisms on  $C_{\text{env}}^*(\mathcal{A})$  which leave  $\mathcal{A}$  invariant. Thus, if  $\{\alpha_s\}_{s \in \mathcal{S}}$  a family of completely isometric automorphisms defining an action of  $\mathcal{S}$  on  $\mathcal{A}$ , then they can be extended to a family completely isometric automorphisms defining an action of  $\mathcal{S}$  on  $C_{\text{env}}^*(\mathcal{A})$ .

**Theorem 3.15** *Let  $\mathcal{A}$  be a unital operator algebra. Let  $\alpha$  be an action of  $\mathcal{S}$  on  $\mathcal{A}$  by completely isometric automorphisms. Denote also by  $\dot{\alpha}$  the extension of this action to  $C_{\text{env}}^*(\mathcal{A})$ . Then*

$$C_{\text{env}}^*(\mathcal{A}_N \times_{\alpha}^{\text{iso}} \mathcal{S}) \cong C_{\text{env}}^*(\mathcal{A}) \times_{\dot{\alpha}} \mathcal{G}.$$

**Proof** We will show that  $\mathcal{A}_N \times_{\alpha}^{\text{iso}} \mathcal{S}$  is isomorphic to a relative semicrossed product. The result will then follow by Lemma 3.14.

Let  $\{\mathcal{V}_s\}_{s \in \mathcal{S}}$  be the universal isometries in  $\mathcal{A}_N \times_{\alpha}^{\text{iso}} \mathcal{S}$  acting on a Hilbert space  $\mathcal{H}$ . For each  $s \in \mathcal{S}$  let  $\mathcal{H}_s = \mathcal{H}$  and define maps  $\mathcal{V}^{s,t}$  when  $s \leq t$ ,

$$\mathcal{V}^{s,t} : \mathcal{H}_s \longrightarrow \mathcal{H}_t$$

by  $\mathcal{V}^{s,t} = \mathcal{V}_{t-s}$ . Let  $\mathcal{K}$  be the Hilbert space inductive limit of the directed system  $(\mathcal{H}_s)_{s \in \mathcal{S}}$ .

For each  $A \in \mathcal{A}$  the commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathcal{V}_s} & \mathcal{H} \\ A \downarrow & & \alpha_s^{-1}(A) \downarrow \\ \mathcal{H} & \xrightarrow{\mathcal{V}_s} & \mathcal{H} \end{array}$$

defines an operator  $\pi(A)$  on  $\mathcal{K}$ . Thus we have a completely isometric representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$ .

Now for each  $s, t \in \mathcal{S}$  define operator  $U_t^s : \mathcal{H}_s \rightarrow \mathcal{H}_t$  by  $U_t^s = \mathcal{V}_t$ . Passing to the direct limit we get a family of commuting unitaries  $\{U_s\}_{s \in \mathcal{S}}$  on  $\mathcal{K}$  satisfying

$$\pi(A)U_s = U_s\pi(\alpha_s(A)).$$

The unitaries  $\{U_s\}_{s \in \mathcal{S}}$  thus define \*-automorphisms of  $\mathcal{C} := C^*(\pi(A))$  extending  $\alpha$ . Thus

$$\mathcal{A}_N \times_{\alpha}^{\text{iso}} \mathcal{S} \cong \mathcal{A}_N \times_{\mathcal{C}, \alpha} \mathcal{S}.$$

The result now follows by Lemma 3.14. ■

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