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THE FINITE SUM OF THE PRODUCTS OF TWO TOEPLITZ OPERATORS

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Abstract

We consider in this paper the question of when the finite sum of products of two Toeplitz operators is a finite-rank perturbation of a single Toeplitz operator on the Hardy space over the unit disk. A necessary condition is found. As a consequence we obtain a necessary and sufficient condition for the product of three Toeplitz operators to be a finite-rank perturbation of a single Toeplitz operator.

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1. Introduction

Let D be the open unit disk in the complex plane and ∂D the unit circle. Let $d\sigma(z)$ be the normalized Lebesgue measure on the unit circle ∂D. Let L^q denote the Lebesgue q-square integrable functions on the unit circle and let L^∞ be the space of essentially bounded functions on the unit circle. The Hardy space H^2 is the Hilbert space consisting of the analytic functions on the unit disk D that are also in L^2. H^∞ denotes the set of bounded analytic functions on the unit disk. Let P be the orthogonal projection from L^2 onto H^2. For $f \in L^\infty$, the Toeplitz operator T_f and the Hankel operator H_f with symbol f are defined by $T_f h = P(fh)$ and $H_f h = (1-P)(fh)$ for h in H^2. A bounded linear operator on the Hilbert space H is said to have finite rank if the closure of the range of the operator has finite dimension. As is well known, Hankel and Toeplitz operators are closely related by the following important fact:

$$T_{fg} - T_f T_g = H^*_{\bar{f}} H_g.$$

Studying the Toeplitz algebra has shed light on the theory of Toeplitz operators [3, 4, 8]. We know that the algebra of finite sums of finite products of Toeplitz

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operators is dense in the Toeplitz algebra. Guo and Zheng [7] and Gu [5] have shown that a finite sum of finite products of Toeplitz operators can be written as a finite sum of products of two Toeplitz operators. Conditions that characterize when a finite sum of finite products of a Toeplitz operator is a compact perturbation of a Toeplitz operator were found by Guo and Zheng [7].

Another motivation is the result of Gu [6], which states that if an operator X on H^2 is such that $X - T_\theta^* X T_\theta$ is of finite rank for every inner function θ, then $X = T_\psi + F$ where $\psi \in L^\infty$ and F is a finite-rank operator on H^2. In particular, if we set $X = \sum_{i=1}^n H^*_{\overline{zf_i(z)}} H_{\overline{zg_i(z)}}$, then $X - T_z^* X T_z = \sum_{i=1}^n f_i \otimes g_i$ is of finite rank. However, we do not know under what conditions the finite sum of two Hankel operator is of finite rank for general symbols. It is easy to see that

$$\sum_{i=1}^n T_{f_i} T_{g_i} - T_{\sum_{i=1}^n f_i g_i} = \sum_{i=1}^n H^*_{\overline{f_i}} H_{g_i}.$$

A natural question arises: When is the finite sum of products of two Toeplitz operators a finite-rank perturbation of a single Toeplitz operator?

In Section 2, we will give a necessary condition for the finite sum of products of two Toeplitz operators to be a finite-rank perturbation of a Toeplitz operator.

In Section 3, we will give some interesting consequences.

2. Necessary condition

We need to introduce some notation. For $x, y \in H^2$, $x \otimes y$ is the operator of rank one defined by

$$x \otimes y(f) = \langle f, y \rangle x,$$

for every $f \in H^2$. It is easy to see that $(x \otimes y)^* = y \otimes x$.

Let A be a finite-rank operator on H^2, where A has rank k. Then there are vectors x_j, y_j in H^2 with $\dim\{x_j\} = \dim\{y_j\} = k$ such that $A = \sum_{j=1}^k x_j \otimes y_j$.

Although our main concern is with bounded Toeplitz operators and Hankel operators, since a product of m (≥ 3) Toeplitz operators can be decomposed into the product of two Toeplitz operators with perhaps unbounded symbols, we will need to make use of densely defined unbounded Toeplitz operators and Hankel operators. Given two operators S_1 and S_2 densely defined on H^2, we say that $S_1 = S_2$ if $S_1 P = S_2 P$ for each analytic polynomial P.

Note that $\bigcap_{1<q<\infty} L^q$ is an algebra, that is, both fg and $f+g$ are in $\bigcap_{1<q<\infty} L^q$ if f and g are in $\bigcap_{1<q<\infty} L^q$. In addition, the Hardy projections P and $1-P$ are bounded on L^q for $1 < q < \infty$. Naturally, we consider the symbols of Toeplitz operators in $\bigcap_{1<q<\infty} L^q$. For $f \in \bigcap_{1<q<\infty} L^q$, let $f^+ = Pf$, the analytic part of f, and let $f^- = (1-P)f$, the conjugate analytic part of f. It is well known that $T_{\bar{z}} T_f T_z = T_f$. Our main result is the following theorem.

THEOREM 2.1. *For f_i, g_i, h in $\bigcap_{1<q<\infty} L^q$ $(i = 1, 2, \ldots, n)$, if $\sum_{i=1}^n T_{f_i} T_{g_i} - T_h$ is a finite-rank operator, then there are analytic polynomials $A_i(z)$, $B_i(z)$ with* $\max\{\deg A_i(z)\} = k$ *and* $\max\{\deg B_i(z)\} = k$, *not all of which are zero, such that*

$$\sum_{i=1}^n A_i \bar{f}_i \in H^2$$

or

$$\sum_{i=1}^n B_i g_i \in H^2.$$

PROOF. Let K be the rank of $\sum_{i=1}^n T_{f_i} T_{g_i} - T_h$. We prove the result by induction on the rank K.

Assume that the rank $K = 0$. Then

$$\sum_{i=1}^n T_{f_i} T_{g_i} = T_h.$$

If one of the $\bar{f}_i$ or one of the g_i is in H^2, then obviously there are constants A_i, B_i with $\sum_{i=1}^n |A_i| > 0$ and $\sum_{i=1}^n |B_i| > 0$ such that

$$\sum_{i=1}^n A_i \bar{f}_i \in H^2$$

or

$$\sum_{i=1}^n B_i g_i \in H^2.$$

If none of the $\bar{f}_i \in H^2$ and none of the $g_i \in H^2$, let $K_\lambda(z) = (1/(1 - \bar{\lambda} z))$ be the reproducing kernel at $\lambda \in D$. Noting that $1 - T_z T_{\bar{z}} = 1 \otimes 1$, it follows that

$$\begin{aligned}
(T_{\bar{z} f_1} 1 \otimes T_{\bar{z} \bar{g}_1} 1 + \cdots + T_{\bar{z} f_n} 1 \otimes T_{\bar{z} \bar{g}_n} 1) &= T_{\bar{z}} \sum_{i=1}^n T_{f_i} (1 - T_z T_{\bar{z}}) T_{g_i} T_z \\
&= T_{\bar{z}} \sum_{i=1}^n T_{f_i} T_{g_i} T_z - \sum_{i=1}^n T_{\bar{z}} T_{f_i} T_z T_{\bar{z}} T_{g_i} T_z \\
&= T_{\bar{z}} \sum_{i=1}^n T_{f_i} T_{g_i} T_z - \sum_{i=1}^n T_{f_i} T_{g_i} \\
&= T_{\bar{z}} T_h T_z - T_h = 0.
\end{aligned}$$

Then

$$\sum_{i=1}^n \overline{T_{\bar{z}} \bar{g}_i 1(\lambda)} T_{\bar{z} f_i} 1 = \sum_{i=1}^n T_{\bar{z} f_i} 1 \otimes T_{\bar{z} \bar{g}_i} 1 (K_\lambda) = 0.$$

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It is easy to see that $T_{\bar{z}\bar{g}_i}1 = 0$ if and only if $g_i \in H^2$ and $T_{\bar{z}f_i}1 = 0$ if and only if $\bar{f}_i \in H^2$. Hence there is a $\lambda_0 \in D$ such that $A_i = T_{\bar{z}\bar{g}_i}1(\lambda_0) \neq 0$ for all $1 \leq i \leq n$. Thus

$$\sum_{i=1}^{n} \bar{A}_i T_{\bar{z}f_i}1 = 0$$

implies that

$$\sum_{i=1}^{n} A_i \bar{f}_i \in H^2.$$

Next we assume that the result is true if the rank k is less than K. We need to show that the result is true for $k = K$.

We write

$$\sum_{i=1}^{n} T_{f_i} T_{g_i} - T_h = \sum_{j=1}^{k} x_j \otimes y_j,$$

where x_j, y_j are in H^2 and $\dim\{x_j\} = \dim\{y_j\} = K$.

We have

$$\begin{aligned}\sum_{i=1}^{n} T_{\bar{z}f_i}1 \otimes T_{\bar{z}\bar{g}_i}1 &= T_{\bar{z}}\left\{\sum_{i=1}^{n} T_{f_i}(1 - T_z T_{\bar{z}})T_{g_i}\right\}T_z \\ &= T_{\bar{z}} \sum_{i=1}^{n} T_{f_i} T_{g_i} T_z - \sum_{i=1}^{n} T_{f_i} T_{g_i} \\ &= T_{\bar{z}}\left(T_h + \sum_{j=1}^{k} x_j \otimes y_j\right)T_z - \left(T_h + \sum_{j=1}^{k} x_j \otimes y_j\right) \\ &= \sum_{j=1}^{k} T_{\bar{z}}x_j \otimes T_{\bar{z}}y_j - \sum_{j=1}^{k} x_j \otimes y_j.\end{aligned}$$

That is,

$$\sum_{i=1}^{n} T_{\bar{z}f_i}1 \otimes T_{\overline{zg_i}}1 = \sum_{j=1}^{k} T_{\bar{z}}x_j \otimes T_{\bar{z}}y_j - \sum_{j=1}^{k} x_j \otimes y_j. \tag{2.1}$$

If $T_{\bar{z}}y_1, \ldots, T_{\bar{z}}y_k$ are linearly dependent, without loss of generality, we may assume that

$$T_{\bar{z}}y_k = c_1 T_{\bar{z}}y_1 + \cdots + c_{k-1}T_{\bar{z}}y_{k-1},$$

for some constants $c_1, \ldots, c_{k-1}$. Then

$$\sum_{i=1}^{n} T_{\bar{z}f_i} T_{zg_i} = T_h + \sum_{j=1}^{k-1} T_{\bar{z}}(x_j + \bar{c}_j x_k) \otimes T_{\bar{z}}y_j.$$

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Thus the rank of $\sum_{i=1}^{n} T_{\bar{z} f_i} T_{z g_i} - T_h$ is at most $K - 1$. So, by the induction hypothesis, there exist analytic polynomials $a_i(z)$ and $b_i(z)$ with $\max\{\deg a_i(z)\} \le K - 1$, $\max\{\deg b_i(z)\} \le K - 1$, and $\sum_{i=1}^{n} |a_i| \sum_{i=1}^{n} |b_i| > 0$ such that

$$\sum_{i=1}^{n} a_i(z) z \bar{f}_i \in H^2$$

or

$$\sum_{i=1}^{n} b_i(z) z g_i \in H^2.$$

Let $l = \max\{\deg a_i(z)\}$ or $l = \max\{\deg b_i(z)\}$. Then $A_i = z^{k-l} a_i(z)$ and $B_i = z^{k-l} b_i(z)$ are both analytic polynomials with $\max\{\deg A_i\} = \max\{\deg B_i\} = k$, $\sum_{i=1}^{n} |A_i| \sum_{i=1}^{n} |B_i| \neq 0$ such that

$$\sum_{i=1}^{n} A_i \bar{f}_i \in H^2$$

or

$$\sum_{i=1}^{n} B_i g_i \in H^2.$$

Thus the result is true in this case.

If $T_{\bar{z}} x_1, \ldots, T_{\bar{z}} x_k$ are linearly dependent, by the same argument as above, we obtain that the result is true.

To finish the proof, we may assume that $T_{\bar{z}} y_1, \ldots, T_{\bar{z}} y_k$ are linearly independent and $T_{\bar{z}} x_1, \ldots, T_{\bar{z}} x_k$ are linearly independent.

Applying $T_{\bar{z}} y_l$ to both sides of (2.1) gives that

$$\sum_{i=1}^{n} \langle T_{\bar{z}} y_l, T_{\overline{z g_i}} 1 \rangle T_{\bar{z} f_i} 1 = \sum_{j=1}^{k} \langle T_{\bar{z}} y_l, T_{\bar{z}} y_j \rangle T_{\bar{z}} x_j - \sum_{j=1}^{k} \langle T_{\bar{z}} y_l, y_j \rangle x_j,$$

for $l = 1, 2, \ldots, k$.

Let $a_{lj} = \langle T_{\bar{z}} y_l, T_{\bar{z}} y_j \rangle$, $b_{lj} = \bar{z} a_{lj} - \langle T_{\bar{z}} y_l, y_j \rangle$, $c_{lj} = \langle T_{\bar{z}} y_l, T_{\bar{z} \bar{g}_i} 1 \rangle$. Since $T_{\bar{z}} x_j = \bar{z} x_j - \bar{z} x_j(0)$,

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{k1} & c_{k2} & \cdots & c_{kn} \end{pmatrix} \begin{pmatrix} T_{\bar{z} f_1} 1 \\ T_{\bar{z} f_2} 1 \\ \vdots \\ T_{\bar{z} f_n} 1 \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_k(0) \end{pmatrix} \bar{z}.$$

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That is,

$$CT_{\bar{z}f}1 = BX - \bar{z}AX(0),$$

where $C = (c_{lj})$, $B = (b_{lj})$, $A = (a_{lj})$, $T_{\bar{z}f}1 = (T_{\bar{z}f_1}1, \ldots, T_{\bar{z}f_k}1)^{\mathrm{T}}$, $X = (x_1, \ldots, x_k)^{\mathrm{T}}$, $X(0) = (x_1(0), \ldots, x_k(0))^{\mathrm{T}}$. The determinant of the matrix $B = (b_{lj})_{k\times k}$ is

$$D(z) = \det(b_{lj}) = a\bar{z}^k + a_1\bar{z}^{k-1} + \cdots + a_k,$$

where $a = \det(a_{lj})$ is the Gram determinant of vectors $T_{\bar{z}}y_1, T_{\bar{z}}y_2, \ldots, T_{\bar{z}}y_k$. Since $T_{\bar{z}}y_1, T_{\bar{z}}y_2, \ldots, T_{\bar{z}}y_k$ are linearly independent, $a = \det(a_{lj}) \neq 0$, and a_i are constants. Hence $\deg D(z) = k$, and $D(z)$ is a co-analytic polynomial in z.

The adjoint of the matrix B is

$$\operatorname{adj} B = \begin{pmatrix} B_{11} & B_{21} & \cdots & B_{k1} \\ B_{12} & B_{22} & \cdots & B_{k2} \\ \cdots & \cdots & \cdots & \cdots \\ B_{1k} & B_{2k} & \cdots & B_{kk} \end{pmatrix},$$

where B_{lj} denotes the cofactor of b_{lj} and it is a co-analytic polynomial in z with degree at most $k-1$.

So

$$(\operatorname{adj} B)CT_{\bar{z}f}1 = D(z)X - (\operatorname{adj} B)AX(0)\bar{z}.$$

Let

$$(C_{li}(z)) = (\operatorname{adj} B)C,$$

where $C_{li}(z)$ are co-analytic polynomials in z with degree at most $k-1$.

Applying the projection P to both sides of the above equation gives that

$$P[(C_{li}(z))T_{\bar{z}f}1] = PD(z)X.$$

That is,

$$\begin{pmatrix} T_{\bar{z}\sum_{i=1}^n c_{1i}(z)f_i}1 \\ \vdots \\ T_{\bar{z}\sum_{i=1}^n c_{ki}(z)f_i}1 \end{pmatrix} = \begin{pmatrix} T_{D(z)}x_1 \\ \vdots \\ T_{D(z)}x_k \end{pmatrix}.$$

By the same argument, we also have

$$\begin{pmatrix} T_{\bar{z}\sum_{i=1}^n u_{1i}(z)\overline{g_i}}1 \\ \vdots \\ T_{\bar{z}\sum_{i=1}^n u_{ki}(z)\overline{g_i}}1 \end{pmatrix} = \begin{pmatrix} T_{E(z)}y_1 \\ \vdots \\ T_{E(z)}y_k \end{pmatrix},$$

where $u_{li}(z)$ are co-analytic polynomials in z with degree at most $k-1$ and $E(z)$ is a co-analytic polynomial in z with degree k.

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If $T_{\overline{zg_1}}1, T_{\overline{zg_2}}1, \ldots, T_{\overline{zg_n}}1, y_1, y_2, \ldots, y_k$ are linearly dependent, then there exist constants a_i, b_i, not all zero, such that

$$\sum_{i=1}^{n} a_i T_{\overline{zg_i}}1 + \sum_{j=1}^{k} b_j y_j = 0.$$

One of the $a_1, a_2, \ldots, a_n$ must be nonzero since $y_1, \ldots, y_k$ are linearly independent. Without loss of generality, assume that

$$T_{\overline{zg_n}}1 = a_1 T_{\overline{zg_1}}1 + \cdots + a_{n-1} T_{\overline{zg_{n-1}}}1 + b_1 y_1 + \cdots b_k y_k.$$

Then

$$\begin{aligned} T_{E(z)} T_{\overline{zg_n}}1 &= T_{E(z)\overline{zg_n}}1 \\ &= \sum_{i=1}^{n-1} a_i T_{\bar{z}E(z)\overline{g_i}}1 + \sum_{j=1}^{k} b_j T_{E(z)} y_j \\ &= \sum_{i=1}^{n-1} a_i T_{\bar{z}E(z)\overline{g_i}}1 + \sum_{j=1}^{k} b_j T_{\sum_{i=1}^{n} u_{ji}(z)\overline{zg_i}}1 \\ &= \sum_{i=1}^{n-1} a_i T_{\bar{z}E(z)\overline{g_i}}1 + \sum_{i=1}^{n} T_{\sum_{j=1}^{k} b_j u_{ji}(z)\overline{zg_i}}1. \end{aligned}$$

Therefore,

$$T_{\bar{z}}\left\{\left[E(z) - \sum_{j=1}^{k} b_j u_{jn}(z)\right]\overline{g_n} - \sum_{i=1}^{n-1}\left[a_i E(z) + \sum_{j=1}^{k} b_j u_{ji}(z)\right]\overline{g_i}(z)\right\}1 = 0.$$

From this equation, it follows that

$$\overline{\left[E(z) - \sum_{j=1}^{k} b_j u_{jn}(z)\right]} g_n - \sum_{i=1}^{n-1} \overline{\left[a_i E(z) + \sum_{j=1}^{k} b_j u_{ji}(z)\right]} g_i \in H^2.$$

Let

$$B_n(z) = \overline{\left[E(z) - \sum_{j=1}^{k} b_j u_{jn}(z)\right]},$$

$$B_i(z) = -\overline{\left[a_i E(z) + \sum_{j=1}^{k} b_j u_{ji}(z)\right]}, \quad 1 \le i \le n-1.$$

Then $B_l(z)$ are analytic polynomials in z with degree $B_n(z) = k$, $\deg B_i(z) \le k$, $1 \le i \le n-1$, and $\sum_{i=1}^{n} B_i g_i \in H^2$. This is the result as desired.

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By the same argument, if $T_{\bar{z}f_1}1, T_{\bar{z}f_2}1, \ldots, T_{\bar{z}f_n}1,\ x_1, \ldots, x_n$ are linearly dependent, we also have that there exist analytic polynomials $A_i(z)$ with $\max\{\deg A_i(z)\} = k$ such that

$$\sum_{i=1}^{n} A_i \bar{f}_i \in H^2.$$

Next we assume that $T_{\bar{z}f_1}1, \ldots, T_{\bar{z}f_n}1,\ x_1, \ldots, x_k$ are linearly independent and $T_{\overline{zg_1}}1, \ldots, T_{\overline{zg_n}}1,\ y_1, \ldots, y_k$ are also linearly independent. We will derive a contradiction.

First we claim that

$$\dim \operatorname{span}\{x_1, \ldots, x_k, T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\} \geq k + n.$$

In fact, since $T_{\overline{zg_1}}1, \ldots, T_{\overline{zg_n}}1$ are linearly independent, there is a vector $\xi \in H^2$ such that $\langle \xi, T_{\overline{zg_i}}1\rangle = 1$ and $\langle \xi, T_{\overline{zg_j}}1\rangle = 0$ for all $j \neq i$.

Hence

$$T_{\bar{z}f_i}1 = \sum_{j=1}^{k} \langle \xi, T_{\bar{z}}y_j\rangle T_{\bar{z}}x_j - \sum_{j=1}^{k} \langle \xi, y_j\rangle x_j,$$

by (2.1). This implies that $T_{\bar{z}f_i}1 \in \operatorname{span}\{x_1, \ldots, x_k, T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\}$.

This gives

$$\operatorname{span}\{T_{\bar{z}f_1}1, \ldots, T_{\bar{z}f_n}1, x_1, \ldots, x_k\} \subseteq \operatorname{span}\{x_1, \ldots, x_k, T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\}.$$

Thus

$$\begin{aligned}\dim \operatorname{span}\{x_1, \ldots, x_k, T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\} &\geq \dim \operatorname{span}\{T_{\bar{z}f_1}1, \ldots, T_{\bar{z}f_n}1, x_1, \ldots, x_k\}\\ &= k + n.\end{aligned}$$

Since

$$\dim \operatorname{span}\{T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\} = k < k + n,$$

there is a nonzero vector ξ in $\operatorname{span}\{x_1, \ldots, x_k, T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\}$ such that

$$\xi \perp \{T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\}.$$

By (2.1),

$$\sum_{i=1}^{n} \langle \xi, T_{\bar{z}f_i}1\rangle T_{\overline{zg_i}}1 = -\sum_{j=1}^{k} \langle \xi, x_j\rangle y_j.$$

Not all of $\{\langle \xi, x_j\rangle\}_{j=1}^{k}$ are zero since

$$\xi \in \operatorname{span}\{x_1, \ldots, x_k, T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\}.$$

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Otherwise ξ is orthogonal to $\{x_1, \ldots, x_k, T_{\bar{z}}x_1, \ldots, T_{\bar{z}}x_k\}$, which would imply that $\xi = 0$. This is a contradiction. Also

$$\sum_{i=1}^{n} \langle \xi, T_{\bar{z} f_i} 1 \rangle T_{\bar{z} \bar{g}_i} 1 + \sum_{i=1}^{n} \langle \xi, x_j \rangle y_j = 0,$$

where at least one coefficient $\langle \xi, x_j \rangle$ is different from zero. Thus, the vectors

$$T_{\bar{z} g_1} 1, \ldots, T_{\bar{z} g_n} 1, y_1, \ldots, y_k,$$

are linearly dependent. We have obtained a contradiction to complete the proof. □

REMARK. Let f_i and g_i all be in L^∞. Then the result of Theorem 2.1 is that there are no all-zero analytic polynomials $A_i(z)$, $B_i(z)$ with $\max\{\deg A_i(z)\} = k$ and $\max\{\deg B_i(z)\} = k$ such that

$$\sum_{i=1}^{n} A_i \bar{f}_i \in H^\infty$$

or

$$\sum_{i=1}^{n} B_i g_i \in H^\infty.$$

3. Consequences

In this section, we will obtain some necessary and sufficient conditions. For convenience, we write $A = B \bmod(F)$ to denote that the operator $A - B$ is a finite-rank operator. One of the first results about Hankel matrices was Kronecker's theorem that describes the Hankel matrices of finite rank. Kronecker's theorem states that, for $f \in L^\infty$, H_f is of finite rank if and only if f is the sum of an analytic function h and a rational function $r(z)$ whose poles are not on the unit circle. The following theorem is another form [9] of Kronecker's theorem, which we will use often in this section.

THEOREM 3.1 (Kronecker's theorem). *Suppose that $f \in L^\infty$. Then H_f has finite rank if and only if there exists a nonzero analytic polynomial $p(z)$ such that $pf \in H^\infty$.*

As is well known, for $f, g \in L^\infty$, Brown and Halmos [2] have shown that the product of two Toeplitz operators T_f and T_g is also a Toeplitz operator if and only if $\bar{f} \in H^\infty$ or $g \in H^\infty$. Axler *et al.* [1] have shown that the product $T_f T_g$ is a finite-rank perturbation of a Toeplitz operator if and only if one of the operators $H_{\bar{f}}$ or H_g has finite rank, where f, g both are in L^∞. We need the following lemma which may have been known before.

LEMMA 3.2. *Let A be a bounded linear operator on H^2. Suppose that $p(z)$ and $q(z)$ are nonzero analytic polynomials. If $T_{\bar{p}} A T_q$ has finite rank, then A has finite rank.*

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PROOF. Factorize $q(z)$ as the product $q(z) = B(z)F(z)$ of a finite Blaschke product $B(z)$ and an outer function $F(z)$. Let $M = T_{\bar{p}}AT_q H^2$. Since $T_p A T_q$ has finite rank, M is a finite dimension subspace of H^2. Since $F(z)$ is an outer function, $\text{closure}\{T_F H^2\} = H^2$. Thus

$$\begin{aligned}\text{closure}\{T_{\bar{p}}AT_q H^2\} &= \text{closure}\{T_{\bar{p}}AT_B T_F H^2\} \\ &= \text{closure}\{T_{\bar{p}}AT_B H^2\} = M.\end{aligned}$$

This gives that $T_{\bar{p}}AT_B$ has finite rank and then

$$T_{\bar{p}}A = T_{\bar{p}}AT_{\bar{B}}T_B = T_{\bar{p}}AT_B T_{\bar{B}} \quad \text{mod}(F).$$

So $T_{\bar{p}}A$ has finite rank. By the same argument, we have that A^* has finite rank. Hence A has finite rank also. This completes the proof. □

For $f \in L^2$, the Toeplitz operator T_f and the Hankel operator H_f are densely defined on H^2. For $f, g \in L^2$, Zheng [10] has given some conditions for the boundedness of the product of two Hankel operators. In the following theorem we will assume that the symbols of the Toeplitz operators lie in $\bigcap_{1<q<\infty} L^q$, and hence we extend Axler, Chang and Sarason's theorem.

THEOREM 3.3. *For f, g, h in $\bigcap_{1<q<\infty} L^q$ where $T_f T_g - T_h$ is a bounded operator on H^2, then*

$$T_f T_g = T_h \quad \text{mod}(F),$$

if and only if $h = fg$ and there is a nonzero analytic polynomial $A(z)$ such that

$$A\bar{f} \in H^2 \quad \textit{or} \quad Ag \in H^2.$$

PROOF. First we prove the necessary part.

Let k_z be the normalized reproducing kernel of H^2 at the point $z \in D$. We know that k_z weakly converges to zero in H^2 as z tends to the boundary of D. For $\xi \in \partial D$, $0 < r < 1$, by the hypothesis that $T_f T_g = T_h$ mod(F),

$$\lim_{r\to 1}\langle T_f T_g k_{r\xi}, k_{r\xi}\rangle = \lim_{r\to 1}\langle T_h k_{r\xi}, k_{r\xi}\rangle.$$

It follows that $h = fg$ on ∂D. By Theorem 2.1, there is a nonzero analytic polynomial $A(z)$ such that $A\bar{f} \in H^2$ or $Ag \in H^2$.

Next we prove the sufficient part.

If $h = fg$, there is a nonzero analytic polynomial $A(z)$ such that $A\bar{f} \in H^2$ or $Ag \in H^2$.

Assume that $Ag \in H^2$, then

$$(T_f T_g - T_h)T_A = T_f T_{Ag} - T_{Ah} = T_{fgA-hA} = 0.$$

Thus $T_f T_g = T_h$ mod(F) by Lemma 3.2.

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Assume that $A\bar{f} \in H^2$, then

$$T_{\bar{A}}(T_f T_g - T_h) = T_{\bar{A}f} T_g - T_{\bar{A}h} = T_{\bar{A}(fg-h)} = 0.$$

Hence $T_f T_g = T_h \mod(F)$ by Lemma 3.2.

This completes the proof of the theorem. □

THEOREM 3.4. *For f_1, f_2, g_1, g_2 in L^∞ with $f_1 g_1 = f_2 g_2$, then*

$$T_{f_1} T_{g_1} = T_{f_2} T_{g_2} \quad \mod(F),$$

if and only if one of the following conditions holds:

(1) *$H^*_{\bar{f}_1} H_{g_1}$ and $H^*_{\bar{f}_2} H_{g_2}$ are both finite-rank operators;*

(2) *there exist nonzero analytic polynomials $A_1(z)$, $A_2(z)$, $B_1(z)$ and $B_2(z)$ with $A_1 \bar{B}_1 = A_2 \bar{B}_2$ such that*

$$A_1 \bar{f}_1 + A_2 \bar{f}_2 \in H^\infty \quad \text{and} \quad B_1 g_1 + B_2 g_2 \in H^\infty.$$

PROOF. First we prove the 'only if' part. As

$$T_{f_1} T_{g_1} - T_{f_2} T_{g_2} = H^*_{\bar{f}_2} H_{g_2} - H^*_{\bar{f}_1} H_{g_1},$$

hence

$$T_{f_1} T_{g_1} = T_{f_2} T_{g_2} \quad \mod(F).$$

Equivalently,

$$H^*_{\bar{f}_2} H_{g_2} - H^*_{\bar{f}_1} H_{g_1} = 0 \quad \mod(F).$$

Either $H^*_{\bar{f}_2} H_{g_2}$ and $H^*_{\bar{f}_1} H_{g_1}$ are both finite-rank operators or none of $H^*_{\bar{f}_2} H_{g_2}$ and $H^*_{\bar{f}_1} H_{g_1}$ are finite-rank operators. If none of $H^*_{\bar{f}_2} H_{g_2}$ and $H^*_{\bar{f}_1} H_{g_1}$ are finite-rank operators, by Theorem 2.1, there exist nonzero analytic polynomials A_1, A_2, B_1 and B_2 such that

$$A_1 \bar{f}_1 + A_2 \bar{f}_2 = h_1 \in H^\infty$$

or

$$B_1 g_1 + B_2 g_2 = h_2 \in H^\infty.$$

Assume that $A_1 \bar{f}_1 + A_2 \bar{f}_2 = h_1 \in H^\infty$, then $\bar{A}_1 f_1 + \bar{A}_2 f_2 = \bar{h}_1$.

Since $T_{f_1} T_{g_1} = T_{f_2} T_{g_2} \quad \mod(F)$,

$$T_{\bar{A}_1 f_1} T_{g_1} = T_{\bar{A}_1 f_2} T_{g_2} \quad \mod(F).$$

This implies that

$$T_{f_2} T_{(\bar{A}_1 g_2 + \bar{A}_2 g_1)} = T_{\bar{h}_1 g_1} \quad \mod(F).$$

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By Theorem 3.3, there is nonzero analytic polynomial $p(z)$ such that $p\overline{f_2} \in H^\infty$ or $p(\bar{A}_1 g_2 + \bar{A}_2 g_1) \in H^\infty$. But $H_{\overline{f_2}}$ is not a finite-rank operator by the hypothesis, so $p(\bar{A}_1 g_2 + \bar{A}_2 g_1) \in H^\infty$. Let $l = \max\{\deg \overline{A_1}, \deg \overline{A_2}\}$, $B_1 = z^l p(z)\overline{A_2}$, $B_2 = z^l p(z)\overline{A_1}$, so B_1 and B_2 are analytic polynomials such that

$$B_1 g_1 + B_2 g_2 \in H^\infty$$

and

$$A_1\overline{B_1} = A_2\overline{B_2}.$$

If $B_1 g_1 + B_2 g_2 = h_2 \in H^\infty$, by a similar argument, we obtain the same result.

Now we prove the 'if' part. Obviously, condition (1) is sufficient. Assume that there exist nonzero analytic polynomials A_1, A_2, B_1 and B_2 with $A_1\overline{B_1} = A_2\overline{B_2}$ such that $A_1\overline{f_1} + A_2\overline{f_2} = h_1 \in H^\infty$ and $B_1 g_1 + B_2 g_2 = h_2 \in H^\infty$.

Therefore,

$$\begin{aligned} T_{\bar{A}_1}(T_{f_1}T_{g_1} - T_{f_2}T_{g_2})T_{B_1} &= T_{\bar{A}_1 f_1}T_{g_1 B_1} - T_{\bar{A}_1 f_2}T_{g_2 B_1} \\ &= (T_{\bar{h}_1} - T_{\bar{A}_2 f_2})T_{g_1 B_1} - T_{\bar{A_1} f_2}T_{g_2 B_1} \\ &= T_{\bar{h}_1 g_1 B_1} - T_{\bar{A}_2 f_2}(T_{h_2} - T_{B_2 g_2}) - T_{\bar{A}_1 f_2}T_{g_2 B_1} \\ &= T_{(\bar{h}_1 g_1 B_1 - \bar{A}_2 h_2 f_2)} + T_{\bar{A}_2 f_2}T_{B_2 g_2} - T_{\bar{A}_1 f_2}T_{g_2 B_1}. \end{aligned}$$

Note that, for any analytic polynomial $p(z)$, any $f \in L^\infty$, $T_p T_f = T_f T_p \operatorname{mod}(F)$. Hence $T_{\bar{A}_2 f_2}T_{B_2 g_2} - T_{\bar{A}_1 f_2}T_{g_2 B_1} = 0 \operatorname{mod}(F)$. It is easy to calculate that $\bar{h}_1 g_1 B_1 - \bar{A}_2 h_2 f_2 = 0$. Thus,

$$T_{\bar{A}_1}(T_{f_1}T_{g_1} - T_{f_2}T_{g_2})T_{B_1} = 0 \quad \operatorname{mod}(F).$$

By Lemma 3.2, $T_{f_1}T_{g_1} = T_{f_2}T_{g_2} \operatorname{mod}(F)$. This completes the proof. □

COROLLARY 3.5. *For f and g in L^∞, then*

$$T_f T_g = T_g T_f \quad \operatorname{mod}(F),$$

if and only if one of the following conditions holds:

(1) *$H_{\bar{f}}^* H_g$ and $H_{\bar{g}}^* H_f$ are both finite-rank operators;*

(2) *there exist nonzero analytic polynomials $A_1(z)$, $A_2(z)$, $B_1(z)$ and $B_2(z)$ with $A_1\bar{B}_1 = A_2\bar{B}_2$ such that*

$$A_1\bar{f} + A_2\bar{g} \in H^\infty \quad \text{and} \quad B_1 g + B_2 f \in H^\infty.$$

THEOREM 3.6. *For f in L^∞, the self-commutator*

$$T_f^* T_f - T_f T_f^*$$

has finite rank if and only if one of the following conditions holds:

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(1) *there is a nonzero analytic polynomial $p(z)$ such that $pf \in H^\infty$ and $p\bar{f} \in H^\infty$;*
(2) *there exist nonzero analytic polynomials A and B with $|A|^2 = |B|^2$ such that*

$$Af + B\bar{f} \in H^\infty.$$

PROOF. First we prove the 'only if' part. Suppose that

$$T_f^* T_f - T_f T_f^* = H_{\bar{f}}^* H_{\bar{f}} - H_f^* H_f = 0 \mod(F).$$

Therefore, H_f has finite rank if and only if $H_{\bar{f}}$ has finite rank.

If H_f and $H_{\bar{f}}$ have finite rank, by Kronecker's theorem, there exist nonzero polynomials p_1 and p_2 such that $p_1\bar{f} \in H^\infty$ and $p_2 f \in H^\infty$. Let $p(z) = p_1 p_2$, so pf and $p\bar{f}$ are both in H^∞.

If none of $H_{\bar{f}}$ and H_f have finite rank, by Theorem 3.4, there exist nonzero analytic polynomials A and B such that

$$Af + B\bar{f} = h \in H^\infty.$$

Therefore,

$$\begin{aligned} T_{\bar{A}}(T_{\bar{f}}T_f - T_f T_{\bar{f}})T_B &= T_{\overline{Af}} T_{fB} - T_{\bar{A}f} T_{\bar{f}B} \\ &= (T_{\bar{h}} - T_{\bar{B}f}) T_{fB} - T_{\bar{A}f}(T_h - T_{Af}) \\ &= T_{(\bar{h}fB - \bar{A}fh)} - T_{\bar{B}f} T_{fB} + T_{\bar{A}f} T_{fA} \\ &= T_{(|B|^2 - |A|^2)f^2} - T_{\bar{B}f} T_{fB} + T_{\bar{A}f} T_{fA} \\ &= T_{(|B|^2 - |A|^2)f^2} + T_f T_{f(|A|^2 - |B|^2)} \mod(F). \end{aligned}$$

Since $T_{\bar{f}}T_f - T_f T_{\bar{f}} = 0 \mod(F)$, $T_f T_{f(|A|^2-|B|^2)} = T_{(|A|^2-|B|^2)f^2} \mod(F)$. This implies that $|A|^2 = |B|^2$ by Theorem 3.3 and the hypothesis follows. This completes the proof of the 'only if' part.

Next we prove the 'if' part. Obviously the condition 1 is sufficient. Suppose that condition (2) holds. That is, $|A|^2 = |B|^2$ such that $Af + B\bar{f} = h \in H^\infty$, where A and B are nonzero analytic polynomials. Thus,

$$\begin{aligned} T_{\bar{A}}(T_{\bar{f}}T_f - T_f T_{\bar{f}})T_B &= (T_{\bar{h}} - T_{\bar{B}f}) T_{fB} - T_{\bar{A}f}(T_h - T_{Af}) \\ &= T_{f(\bar{h}B - \bar{A}h)} + T_f T_{f(|A|^2 - |B|^2)} \mod(F) \\ &= 0. \end{aligned}$$

Hence

$$T_{\bar{f}}T_f - T_f T_{\bar{f}} = 0 \mod(F),$$

since Lemma 3.2. This completes the proof of the theorem. □

THEOREM 3.7. *For f_1, f_2, f_3 and h in L^∞, then*

$$T_{f_1} T_{f_2} T_{f_3} = T_h \mod(F),$$

if and only if one of the following conditions holds:

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(1) $H^*_{\overline{f_1 f_2^+}} H_{f_3}$ *and* $H^*_{\bar{f_1}} H_{f_2^- f_3}$ *are both finite-rank operators;*

(2) *there exist nonzero analytic polynomials* A_1, A_2, B_1 *and* B_2 *with* $A_1\overline{B_1} + A_2\overline{B_2} = 0$ *such that*

$$A_1\overline{f_1 f_2^+} + A_2\overline{f_1} = r_1 \in H^2$$

and

$$B_1 f_3 + B_2 f_2^- f_3 = r_2 \in H^2.$$

PROOF. We first prove the 'if' part. Since $h = f_1 f_2 f_3$,

$$\begin{aligned} T_{f_1} T_{f_2} T_{f_3} - T_h &= T_{f_1 f_2^+} - T_{f_1 f_2^+ f_3} + T_{f_1} T_{f_2^- f_3} - T_{f_1 f_2^- f_3} \\ &= -\Big(H^*_{\overline{f_1 f_2^+}} H_{f_3} + H^*_{\bar{f}_1} H_{\bar{f}_2 f_3}\Big). \end{aligned}$$

Thus condition (1) implies that

$$T_{f_1} T_{f_2} T_{f_3} = T_h \quad \mathrm{mod}(F).$$

Next we suppose that condition (2) is true. We have

$$\begin{aligned} T_{\overline{A_1}}(T_{f_1} T_{f_2} T_{f_3} - T_h) T_{B_2} &= T_{\overline{A_1} f_1 f_2^+} T_{f_3 B_2} + T_{\overline{A_1} f_1} T_{f_2^- f_3 B_2} - T_{\overline{A_1} h B_2} \\ &= (T_{\overline{r_1}} - T_{\overline{A_2} f_1}) T_{f_3 B_2} + T_{\overline{A_1} f_1}(T_{r_2} - T_{B_1 f_3}) - T_{\overline{A_1} h B_2} \\ &= T_{\bar{r}_1 B_2 f_3 + r_2 \bar{A}_1 f_1 - \bar{A}_1 B_2 h} - T_{f_1 f_2 (\bar{A}_2 B_2 + \bar{A}_1 B_1)} \quad \mathrm{mod}(F) \\ &= 0. \end{aligned}$$

Hence

$$T_{f_1} T_{f_2} T_{f_3} = T_h \quad \mathrm{mod}(F),$$

from Lemma 3.2.

Next we prove the 'only if' part. If $T_{f_1} T_{f_2} T_{f_3} = T_h \ \mathrm{mod}(F)$, by [3, Douglas theorem], $h = f_1 f_2 f_3$. Hence

$$\begin{aligned} T_{f_1} T_{f_2} T_{f_3} - T_h &= T_{f_1 f_2^*} T_{f_3} + T_{f_1} T_{f_2^- f_3} - T_{f_1 f_2 f_3} \\ &= -\Big(H^*_{\overline{f_1 f_2^+}} H_{f_3} + H^*_{\bar{f}_1} H_{\bar{f}_2 f_3}\Big). \end{aligned}$$

If none of $H^*_{\overline{f_1 f_2^+}} H_{f_3}$ and $H^*_{\bar{f}_1} H_{f_2^- f_3}$ have finite rank, by Theorem 2.1, there exist nonzero analytic polynomials A_1, A_2, B_1 and B_2 such that

$$A_1\overline{f_1 f_2^+} + A_2\overline{f_1} = r_1 \in H^2$$

or

$$B_1 f_3 + B_2 f_2^- f_3 = r_2 \in H^2.$$

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Assume that

$$A_1\overline{f_1 f_2^+} + A_2\overline{f_1} = r_1 \in H^2,$$

then

$$\begin{aligned} T_{\bar{A}_1}(T_{f_1}T_{f_2}T_{f_3} - T_h) &= T_{\bar{A}_1}(T_{f_1 f_2^+}T_{f_3} + T_{f_1}T_{f_2^- f_3} - T_{\bar{A}_1 h}) \\ &= (T_{\bar{r}_1} - T_{\bar{A}_2 f_1})T_{f_3} + T_{\bar{A}_1 f_1}T_{f_2^- f_3} - T_{\bar{A}_1 h} \\ &= T_{f_1}T_{(\bar{A}_1 f_2^- f_3 - \bar{A}_2 f_3)} + T_{\bar{r}_1 f_3 - \bar{A}_1 h} \quad \operatorname{mod}(F). \end{aligned}$$

This implies that

$$T_{f_1}T_{(\bar{A}_1 f_2^- f_3 - \bar{A}_2 f_3)} = T_{\bar{A}_1 h - \bar{r}_1 f_3} \quad \operatorname{mod}(F).$$

By Theorem 3.3 and the hypothesis, there is a nonzero analytic polynomial p such that

$$p(\bar{A}_1 f_2^- f_3 - \bar{A}_2 f_3) \in H^2.$$

Let $l = \max\{\deg \bar{A}_1, \deg \bar{A}_2\}$, $B_1 = -z^l p(z)\bar{A}_2(z)$ and $B_2 = z^l p(z)\bar{A}_1(z)$. Then B_1 and B_2 are both nonzero analytic polynomials with

$$A_1\overline{B_1} + A_2\overline{B_2} = 0,$$

such that

$$B_1 f_3 + B_2 f_2^- f_3 \in H^2.$$

This complete the proof. □

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