Proceedings of the Edinburgh Mathematical Society (2023) **66**, 259–287
doi:10.1017/S0013091523000159

PERFECTION FOR SEMIGROUPS

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(Received 27 April 2022)

Abstract We call a semigroup right perfect if every object in the category of unitary right acts over that semigroup has a projective cover. In this paper, we generalize results about right perfect monoids to the case of semigroups. In our main theorem, we will give nine conditions equivalent to right perfectness of a factorizable semigroup. We also prove that right perfectness is a Morita invariant for factorizable semigroups.

Keywords: perfect semigroup; semiperfect semigroup; projective cover; essential epimorphism; unitary act; sequence act; pullback flat act

2020 Mathematics subject classification: Primary 20M30

1. Preliminaries

Perfect and semiperfect rings with identity were introduced in [3] by Bass. Such rings have been studied in numerous articles. Perfect monoids were defined in [9] by Isbell, who proved that a monoid is left perfect if and only if it satisfies two conditions referred to as (A) and (D). Fountain [6] gave more conditions that are equivalent to left perfectness. In particular, he proved that a monoid is left perfect if and only if all of its weakly flat (in the sense of being pullback flat) unitary left acts are projective. Subsequently, a number of other papers related to perfect monoids or pomonoids have appeared (e.g., [2, 8, 10–12, 20]).

The purpose of this paper is to define (right) perfect semigroups and show that many descriptions of right perfect monoids can be transferred to the case of factorizable semigroups. We say that a semigroup is right perfect if the category of all its unitary right acts is perfect in the sense that each of its objects has a projective cover. In our main theorem, we will give nine different conditions that are equivalent to right perfectness of a factorizable semigroup. Some of these are familiar from the monoid case, but some of them are new. For example, a factorizable semigroup is right perfect if and only if all right sequence acts over it are projective.

It is well known that right perfectness and right semiperfectness are Morita invariants for rings with identity (Corollary 27.8 and Corollary 28.6 in [1]). We prove that right perfectness is also a Morita invariant for factorizable semigroups. Our results allow to conclude that the class of perfect semigroups contains all completely (0-)simple semigroups and all nilpotent semigroups.

Throughout this paper, S denotes a nonempty semigroup (if not stated otherwise), and S^1 is the monoid obtained from S by adjoining an external identity 1 (no matter if S already has an identity element or not). All S-acts, subacts and ideals we consider are nonempty, unless stated otherwise explicitly.

An act A_S is called **unitary** if $A = AS$, that is, for every $a \in A$, there exist $a' \in A$ and $s \in S$ such that $a = a's$. The category of unitary right S-acts with S-act homomorphisms as morphisms will be denoted by UAct_S. Any S-act A_S can be made an S^1-act by defining $a1 = a$ for every $a \in A$.

Tensor products of acts over semigroups are defined in the same way as tensor products of acts over monoids (see [13, Construction 2.5.4]). The tensor product of a right act A_S and a left act ${}_SB$ is the quotient set $A \otimes_S B := (A \times B)/\vartheta$, where ϑ is the smallest equivalence relation on $A \times B$ containing the set $\{((as, b), (a, sb)) \mid a \in A, b \in B, s \in S\}$. The ϑ-class of a pair (a, b) is denoted by $a \otimes b$, so $A \otimes_S B = \{a \otimes b \mid a \in A, b \in B\}$. We will often write $A \otimes B$ instead of $A \otimes_S B$. Each right S-act A_S induces in a natural way a tensor functor $A \otimes_S - : {}_S\mathsf{Act} \to \mathsf{Set}$ (cf. [13, Construction 2.5.15]), where ${}_S\mathsf{Act}$ is the category of all left S-acts and Set is the category of sets. We will use the properties of this functor to define different flatness properties of A_S.

An act A_S is called **firm** if the mapping $\mu_A : A \otimes S \to A,\ a \otimes s \mapsto as$ is bijective. The category of firm right S-acts is denoted by FAct_S. The categories UAct_S and FAct_S are full subcategories of Act_S, the category of all right S-acts. In all three categories, coproducts are disjoint unions.

A semigroup S is called **firm** if S_S is a firm right S-act. A semigroup S is called **factorizable** if $S^2 = S$, i.e., any element in S is a product of two elements.

We will need the following result about firm acts.

Lemma 1.1. **[16, Corollary 2.4]** *If A_S is a unitary act over a factorizable semigroup, then $A \otimes_S S$ is a firm right S-act.*

Let A_S be a right S-act. **Finitely generated** subacts of A_S are those of the form $FS^1 := \{fs \mid f \in F, s \in S^1\}$, where $F \subseteq A$ is a finite subset. An act A_S is called **cyclic** if there exists $a \in A$ such that $A = aS^1 = \{a\} \cup aS$. An S-act is **locally cyclic** if all of its finitely generated subacts are cyclic. It can be shown that A_S is locally cyclic if and only if

$$(\forall a, a' \in A)(\exists a'' \in A)(\exists s, s' \in S^1)(a = a''s \wedge a' = a''s').$$

Essential epimorphisms are defined dually to essential monomorphisms.

Definition 1.2. *An epimorphism $f : P \to A$ in a category $\mathcal{C}$ is called an **essential epimorphism** if, for every morphism $g : Q \to P$ in $\mathcal{C}$,*

$$fg \text{ is an epimorphism} \implies g \text{ is an epimorphism.}$$

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It is easy to see that the composite of two essential epimorphisms is an essential epimorphism.

Definition 1.3. *If $f : P \to A$ is an essential epimorphism in a category $\mathcal{C}$, then the object P is called a* ***cover*** *of the object A. If the object P is projective in $\mathcal{C}$, then it is called a* ***projective cover*** *of A.*

Similar to Proposition 1.3 in [17], one can prove that epimorphisms in UAct_S are precisely the surjective morphisms.

Lemma 1.4. *Let S be a semigroup. An epimorphism $f : P_S \to A_S$ in UAct_S is essential if and only if $f|_Q$ is not surjective for every unitary proper subact Q of P_S.*

Proof. Necessity. Assume that $f : P_S \to A_S$ is an essential epimorphism in UAct_S and let Q be a unitary proper subact of P_S. Then the inclusion map $\iota : Q \to P$ is a morphism in UAct_S and $f|_Q = f\iota$. If $f\iota$ was surjective, then ι would be surjective due to essentiality of f.

Sufficiency. Let $f : P_S \to A_S$ and $g : U_S \to P_S$ be morphisms in UAct_S such that f and fg are surjective. It is easy to see that $g(U)$ is a unitary subact of P_S and $f|_{g(U)}$ is surjective. By assumption, $g(U) = P$, so g is surjective and therefore f is essential. □

Definition 1.5. *We say that a semigroup S is:*

- ***right perfect*** *if every object of UAct_S has a projective cover;*
- ***right semiperfect*** *if every unitary cyclic right S-act has a projective cover in UAct_S.*

Dually, one can define left (semi)perfect semigroups. A semigroup is ***(semi)perfect*** *if it is both right and left (semi)perfect.*

Remark 1.6. We have defined right semiperfect semigroups as nonadditive analogues of right semiperfect rings (see [3]).

Remark 1.7. Note that if S is a monoid, then an act A_S is unitary if and only if $a1 = a$ for every $a \in A$. Hence, S is right perfect in the sense of Definition 1.5 if and only if it is right perfect in the sense of [9] and [6]. Thus, the results in those articles are special cases of our more general approach.

Definition 1.8. (cf. [5]) *A subsemigroup T of a semigroup S is called* ***left unitary*** *if, for every $s, t \in S$,*

$$t, ts \in T \implies s \in T.$$

In this paper, we will use the following conditions on a semigroup S.

Definition 1.9. *A semigroup S satisfies* ***Condition****:*

- *(A) if every right S-act satisfies the ascending chain condition (ACC) for cyclic subacts;*

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- *(D) if every left unitary subsemigroup of S contains a minimal right ideal generated by an idempotent;*
- *(M_L) if S satisfies the descending chain condition (DCC) for principal left ideals of S.*

2. Indecomposable and projective acts

An act is called **indecomposable** if it is not a disjoint union of two nonempty subacts.

Proposition 2.1. [4, **Lemma 4**] *Every right S-act is a disjoint union of indecomposable subacts.*

In what follows, we will call the indecomposable subacts of A_S the **indecomposable components of A_S**. It is easy to see that each cyclic act is indecomposable. We will need the following result about projective acts.

Theorem 2.2. [4, **Lemma 2**] *Let S be a semigroup and let $\mathcal{C}$ be a full subcategory of Act_S which is closed under coproducts. Let $P_S = \bigsqcup_{i \in I} P_i$, where $P_i \in \mathcal{C}$ for every $i \in I$. Then P_S is projective in $\mathcal{C}$ if and only if P_i is projective in $\mathcal{C}$ for every $i \in I$.*

Note that coproducts in Act_S are disjoint unions, and both UAct_S and FAct_S are closed under disjoint unions.

Lemma 2.3. *Let e be an idempotent in a semigroup S. Then the right S-act eS is a projective object in Act_S, UAct_S and FAct_S.*

Proof. By the dual of Lemma 3.1(1) in [18], the act eS is firm, thus also unitary. In all these categories, epimorphisms are precisely the surjective morphisms. Now, in any of these categories, we consider a morphism $f : eS \to B_S$ and an epimorphism $\pi : A_S \to B_S$. Denote $b := f(e)$ and choose $a \in A$ such that $f(e) = \pi(a)$. Consider a homomorphism $g : eS \to A_S$ defined by

$$g(es) := aes, \quad s \in S.$$

Then $\pi g = f$ because $(\pi g)(es) = \pi(aes) = \pi(a)es = f(e)es = f(es)$. □

The last two results allow us to prove the following result.

Corollary 2.4. *Every nilpotent semigroup is perfect. Every nilsemigroup is semiperfect.*

Proof. If S is nilpotent, then $S^n = 0$ for some $n \in \mathbb{N}$. If A_S is unitary, then, for any $a \in A$, we can find $a' \in A$ and $s_1, \ldots, s_n \in S$ such that

$$a = a' s_n \ldots s_2 s_1 = a'0 = a'00 = a0.$$

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So $aS^1 = \{a\}$ is a one-element subact, which is isomorphic to the act $0S = \{0\}$. The last is projective in UAct_S by Lemma 2.3. Thus,

$$A_S = \coprod_{a\in A}\{a\} \cong \coprod_{a\in A} 0S$$

is a projective act by Theorem 2.2, which is a projective cover of itself (with the identity mapping). Thus, all objects of UAct_S (and similarly ${}_S\mathsf{UAct}$) have projective covers and S is perfect.

Assume now that S is a nilsemigroup. If aS^1 is a cyclic unitary act, then $a = as$ for some $s \in S$. For s, there exists $n \in \mathbb{N}$ such that $s^n = 0$. It follows that $a = as^n = a0$, and hence, $aS^1 = \{a\} \cong 0S$, where $0\,S$ is projective. Thus, S is right semiperfect and, similarly, it is also left semiperfect. $\square$

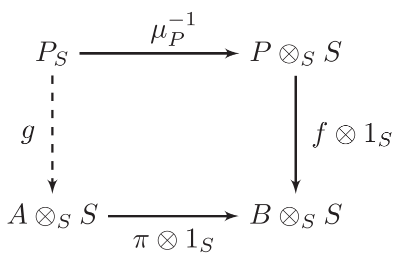
Nilpotent semigroups need not be factorizable in general. For example, any semigroup with more than one element and with zero multiplication is not factorizable.

Over a factorizable semigroup, projective acts can be described as follows. The description relies on Theorem 2 in [4], which generalizes a well-known characterization of projective acts over monoids.

Theorem 2.5. *Let S be a factorizable semigroup. For $P_S \in \mathsf{Act}_S$, the following are equivalent:*

(1) *P_S is a projective object in FAct_S;*
(2) *P_S is a projective object in UAct_S;*
(3) *$P \cong \coprod_{i\in I} e_iS$ for some idempotents $e_i \in S$, $i \in I$.*

Proof. (1) $\Rightarrow$ (2). Assume that P_S is a projective object in FAct_S. Clearly, P_S is an object in UAct_S. We will prove that it is projective in UAct_S. Consider a morphism $f : P_S \to B_S$ and an epimorphism $\pi : A_S \to B_S$ in UAct_S. By Lemma 1.1, the acts in the diagram



are firm and $\pi \otimes 1_S$ is an epimorphism in FAct_S because the functor $-\otimes S$, as a left adjoint, preserves epimorphisms. Using projectivity of P_S, we can find a morphism $g : P_S \to A \otimes_S S$ such that the square commutes. Now $\mu_A g : P_S \to A_S$ is a morphism in UAct_S. Take any $p \in P$. Then there exist $p_0 \in P$, $a \in A$ and $s_0, s \in S$ such that $p = p_0s_0$

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and $g(p) = a \otimes s$. Due to the commutativity of the square, we have

$$\pi(a) \otimes s = (\pi \otimes 1_S)(g(p)) = (f \otimes 1_S)(\mu_P^{-1}(p)) = (f \otimes 1_S)(p_0 \otimes s_0) = f(p_0) \otimes s_0$$

in $B \otimes_S S$. Applying the mapping μ_B to the equality $\pi(a) \otimes s = f(p_0) \otimes s_0$, we obtain $\pi(a)s = f(p_0)s_0$. Consequently,

$$(\pi\mu_A g)(p) = (\pi\mu_A)(a \otimes s) = \pi(as) = \pi(a)s = f(p_0)s_0 = f(p_0 s_0) = f(p).$$

We have shown that $\pi(\mu_A g) = f$, as needed.

(2) $\Rightarrow$ (3). This is a consequence of Theorem 2 in [4] because UAct_S satisfies the assumptions of that theorem.

(3) $\Rightarrow$ (1). By Lemma 2.3, the acts $e_i S$ are projective in FAct_S. A disjoint union of firm acts is firm; thus, P_S is an object of FAct_S. It is projective in FAct_S due to Theorem 2.2. □

A right S-act A_S is called **(finite) limit flat** (see [15, Definition 1.3]) if the functor $A \otimes_S - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves (finite) limits.

Proposition 2.6. *The following are equivalent for an act A_S over a factorizable semigroup S:*

(1) *A_S is an indecomposable projective object in UAct_S;*
(2) *A_S is unitary and limit flat;*
(3) *$A_S \cong eS$ for some idempotent $e \in S$;*
(4) *there exists $a \in A$ and an idempotent $e \in S$ such that $A_S = aS^1$, $a = ae$ and*

$$(\forall s, t \in S^1)(as = at \implies es = et).$$

Proof. (1) $\iff$ (3). This follows from Theorem 2.5 because cyclic acts are indecomposable.

(2) $\iff$ (3). This is Proposition 4.4 in [15].

(3) $\iff$ (4). This is due to Lemma 1.1 and Corollary 1.2 in [7]. □

The properties listed in item (4) may be assumed of any generator of a cyclic projective object in UAct_S.

Corollary 2.7. *An act $A_S = aS^1$ is a projective object in UAct_S if and only if there exists an idempotent $e \in S$ such that $a = ae$ and $as = at$ implies $es = et$ for all $s, t \in S^1$.*

Proof. Necessity. (This proof is similar to the proof of implication (ii) $\implies$ (iii) in [13, Corollary 3.17.9].) By Proposition 2.6, $aS^1 \cong eS$ for some idempotent $e \in S$. Let $f : eS \to aS^1$ be an isomorphism. Then there exist $u \in S^1$ and $v \in S$ such that $f(e) = au$ and $f(ev) = a$. Now $f(e) = au = f(ev)u = f(evu)$ implies $e = evu$ because f is injective. Also, $aue = f(e)e = f(e) = au$. Putting $z := uev \in S$, we have $z^2 = (uev)(uev) = u(evu)ev = ueev = z$, so z is an idempotent.

Suppose that $as = at$, where $s, t \in S^1$. Then $f(evs) = f(ev)s = f(ev)t = f(evt)$, which implies $evs = evt$. Therefore, $zs = uevs = uevt = zt$.

Sufficiency. This follows from Proposition 2.6. □

3. Sequence acts

Let us recall the construction of a sequence act over a semigroup (cf. [15, Construction 3.9]). A similar construction in the case of a monoid appeared already in the proof of Lemma 1 in [6].

Let $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ be a sequence of elements of S. On the set

$$F = \mathbb{N} \times S^1 = \{(k, s) \mid k \in \mathbb{N}, s \in S^1\},$$

we define a right S-action by $(k, s)z := (k, sz)$, $k \in \mathbb{N}$, $s \in S^1$ and $z \in S$. This gives us a right S-act F_S. On the set F, we define a binary relation $\tilde{}$ by

$$(k, s) \sim (k', s') \iff (\exists n \geqslant k, k')(s_n \dots s_k s = s_n \dots s_{k'} s').$$

Then $\sim$ is a congruence of F_S. Form the quotient act

$$M_S = F_S/{\sim} = \{[k, s] \mid k \in \mathbb{N}, s \in S^1\},$$

where $[k, s]$ denotes the $\sim$-class of (k, s). We call such acts **right sequence acts over** S. Note that

$$s_{k+1}s_k \cdot 1 = s_{k+1} \cdot s_k \implies [k, 1] = [k + 1, s_k] = [k + 1, 1]s_k \in [k + 1, 1]S^1$$

for each $k \in \mathbb{N}$, so we see that M_S is unitary and it is a union of a chain of its cyclic subacts

$$[1, 1]S^1 \subseteq [2, 1]S^1 \subseteq [3, 1]S^1 \subseteq \cdots .$$

Lemma 3.1. *Right sequence acts are locally cyclic (and hence indecomposable). A right sequence act M_S is cyclic if and only if $M_S = [k, 1]S^1$ for some $k \in \mathbb{N}$.*

Proof. Given any $[k, s], [\ell, t] \in M_S$, we have that $[k, s], [\ell, t] \in [\max\{k, \ell\}, 1]S^1$. Thus, right sequence acts are locally cyclic. It is well known that locally cyclic acts are indecomposable.

If M_S is cyclic, then $M = [m, u]S^1$ for some $u \in S^1$. By the above,

$$M_S = [m, u]S^1 = [m + 1, s_m u]S^1 \subseteq [m + 1, 1]S^1,$$

so $M_S = [m + 1, 1]S^1$. The converse is obvious. □

Lemma 3.2. *Let S be a semigroup. Each act of the form eS, $e^2 = e \in S$, is isomorphic to a right sequence act.*

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Proof. Let M_S be the right sequence act constructed using the constant sequence $(e, e, \ldots)$. For every $k \in \mathbb{N}$,

$$e \cdot 1 = \underbrace{e \ldots e}_{k} \cdot e \implies (k, 1) \sim (1, e) \implies [k, 1] = [1, 1]e \implies [k, 1]S^1 = [1, 1]S^1,$$

so $M_S = [1, 1]S^1$. Consider the S-act homomorphism

$$f : eS \to [1, 1]S^1, \quad es \mapsto [1, es].$$

It is injective because

$$(1, es) \sim (1, et) \implies e \cdot es = e \cdot et \implies es = et$$

for every $s, t \in S$, and it is surjective because

$$[1, s] = [2, es] = [2, e]es = [1, 1]es = [1, es] = f(es)$$

for every $s \in S^1$. Thus, f is an isomorphism. □

Corollary 3.3. *Let S be a factorizable semigroup. For S-act properties, we have the following implications:*

unitary limit flat $\implies$ isomorphic to a right sequence act $\implies$ unitary finite limit flat.

Proof. The first implication follows from Proposition 2.6 and Lemma 3.2. The second implication holds due to Proposition 4.3 in [15]. □

In our main theorem, we will show (among other things) that factorizable right perfect semigroups S are precisely those for which the converse of the first implication holds for every right S-act. Note also that according to this corollary, 'right sequence act' could be placed between 'eS' and 'firm fin. lim. flat' in the figure on page 90 of [15].

4. Condition (A)

A subset X of an act A_S is a **set of generators** if $A = XS^1$. A set X of generators is called **independent** ([6, p. 90]) if

$$(\forall x, x' \in X) \left(x \in x'S^1 \implies x = x' \right).$$

Lemma 4.1. Cf. [6, Lemma 2] *Let A_S be an act over a semigroup S satisfying (ACC) for cyclic subacts. Then A_S has an independent set of generators.*

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Proof. Denote

$$X := \left\{x \in A \mid (\forall x' \in A)(xS^1 \subseteq x'S^1 \implies xS^1 = x'S^1)\right\}.$$

We first show that X is a set of generators for A_S. It suffices to prove that $A \subseteq XS^1$. Take arbitrary $a \in A$ and consider the set

$$P_a = \left\{a'S^1 \mid a' \in A, aS^1 \subseteq a'S^1\right\}$$

as a poset with respect to inclusion. Since $aS^1 \in P_a$, by assumption, this poset must have a maximal element a_0S^1. In particular, $aS^1 \subseteq a_0S^1$. We show that $a_0 \in X$.

If $x' \in A$ and $a_0S^1 \subseteq x'S^1$, then also $aS^1 \subseteq x'S^1$, so $x'S^1 \in P_a$. Due to maximality of a_0S^1, we have $a_0S^1 = x'S^1$. Hence, $a_0 \in X$ and $a \in a_0S^1 \subseteq XS^1$, as needed.

Define a relation $\approx$ on X by

$$x \approx x' \iff xS^1 \subseteq x'S^1.$$

It is clear by the choice of X that we have an equivalence relation.

From every $\approx$-class, we choose a representative and form a set $X^{'}$ of those elements. We show that $X^{'}$ is an independent set of generators for A_S. By the definition of $\approx$, for every $x \in X$, there exist $x' \in X'$ and $s \in S^1$ such that $x = x's$. Since X is a set of generators, $X^{'}$ is also a set of generators.

To prove that $X^{'}$ is independent, we suppose that $x \in x'S^1$, where $x, x' \in X' \subseteq X$. Then $xS^1 \subseteq x'S^1$, which means that $x \approx x'$. Now $x = x'$ because each $\approx$-class contains precisely one element from $X^{'}$. □

Definition 4.2. Cf. [2, Definition 4.1] *We say that a semigroup S is **right $\mathcal{IC}$-perfect** if every unitary act A_S has a cover $f : B_S \to A_S$, where B_S is unitary, and indecomposable components of B_S are cyclic. Such a cover will be called an **$\mathcal{IC}$-cover**.*

Note that if $b = b's$, where $b, b' \in B$ and $s \in S$, then b and $b^{'}$ must be in the same indecomposable component. Therefore, indecomposable components of B_S must be unitary. Hence,

$$B_S = \bigsqcup_{i \in I} b_iS^1,$$

where each b_iS^1 is a cyclic unitary act.

The following result is inspired by Result 1.2 in [9], by Lemma 1.3 in [10] and also by Lemma 2.2 and Theorem 5.2 in [2]. Among other things, it shows that Condition (A) can be given a description that does not refer to acts but only uses the elements of S.

Theorem 4.3. *For a semigroup S, the following are equivalent:*

(1) *S satisfies Condition (A);*
(2) *every locally cyclic right S-act is cyclic;*
(3) *every right sequence act over S is cyclic;*

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(4) *for every sequence* $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$,

$$(\exists k, m \in \mathbb{N})(\exists u \in S^1)(k \geqslant m+1 \text{ and } s_k \ldots s_{m+1} = s_k \ldots s_{m+1} s_m u);$$

(5) *for every act* A_S*, there exists a set* $\{A_i \mid i \in I\}$ *of cyclic subacts of* A_S *such that*

$$A = \bigcup_{i\in I} A_i \quad \text{and} \quad (\forall j \in I)\left(A_j \not\subseteq \bigcup_{i\neq j} A_i\right); \tag{4.1}$$

(6) S *is right* $\mathcal{IC}$*-perfect.*

Proof. (1) $\Longrightarrow$ (2). Suppose that A_S is a locally cyclic act which is not cyclic. Choose $a_1 \in A$. Since A_S is not cyclic, there exists $b_1 \in A \setminus a_1 S^1$. Using that A_S is locally cyclic, we can find $a_2 \in A$ and $u, v \in S^1$ such that $a_1 = a_2 u$ and $b_1 = a_2 v$. Note that $a_2 \notin a_1 S^1$ because otherwise $b_1 \in a_1 S^1$. Thus, $a_1 S^1 \subset a_2 S^1$.

Again, $a_2 S^1 \neq A$, and we can find an element $b_2 \in A \setminus a_2 S^1$. Continuing in this manner, we can construct a strictly increasing sequence

$$a_1 S^1 \subset a_2 S^1 \subset a_3 S^1 \subset \cdots$$

of cyclic subacts of A_S, which contradicts Condition (A).

(2) $\Longrightarrow$ (3). Every right sequence act is locally cyclic because it is the union of an ascending chain of its cyclic subacts.

(3) $\Longrightarrow$ (4). Assume that all right sequence acts over S are cyclic. Consider a sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ and the right sequence act

$$M_S = \{[k, s] \mid k \in \mathbb{N},\ s \in S^1\}$$

determined by it. By assumption, M_S must be cyclic. By Lemma 3.1, there exists $m \in \mathbb{N}$ such that $M = [m, 1]S^1$. Then $(m+1, 1) \sim (m, u)$ for some $u \in S^1$, which means that there exists $k \geqslant m+1$ such that $s_k \ldots s_{m+1} = s_k \ldots s_{m+1} s_m u$.

(4) $\Longrightarrow$ (1). Suppose that A_S does not satisfy (ACC) for cyclic subacts. Then there exists a sequence $(a_i)_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that

$$a_1 S^1 \subset a_2 S^1 \subset a_3 S^1 \subset \cdots.$$

Consequently, for every $i \in \mathbb{N}$, there exists $s_i \in S$ such that $a_i = a_{i+1} s_i$. For the sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$, there exist $k, m \in \mathbb{N}$, $k \geqslant m+1$ and $u \in S^1$ such that $s_k \ldots s_{m+1} = s_k \ldots s_{m+1} s_m u$. Then

$$\begin{aligned} a_{m+1} = a_{m+2} s_{m+1} = a_{m+3} s_{m+2} s_{m+1} &= a_{k+1} s_k \ldots s_{m+1} = a_{k+1} s_k \ldots s_{m+1} s_m u \\ &= a_k s_{k-1} \ldots s_{m+1} s_m u = a_{m+1} s_m u = a_m u \in a_m S^1. \end{aligned}$$

It follows that $a_{m+1} S^1 = a_m S^1$, a contradiction.

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(1) $\implies$ (5). Assume that S satisfies Condition (A). Consider an act A_S. By Lemma 4.1, A_S has an independent set of generators X. Then, clearly, $A = \bigcup_{x\in X} xS^1$. Take $y \in X$ and suppose that $yS^1 \subseteq \bigcup_{x\neq y} xS^1$. Then there exists $z \in X \setminus \{y\}$ such that $y \in zS^1$. Independence of X implies $y = z$, a contradiction. Hence,

$$yS^1 \nsubseteq \bigcup_{x\neq y} xS^1.$$

(5) $\implies$ (6). Let A_S be a unitary act. By (5), there exist cyclic subacts $A_i = a_iS^1$, $i \in I$, such that Equation (4.1) holds. It is easy to see that each a_iS^1 is unitary.

Now $B_S := \bigsqcup_{i\in I} a_iS^1$ is an act whose indecomposable components are cyclic and unitary. Consider the S-act homomorphism

$$f : B_S \to A_S, \qquad a_is \mapsto a_is.$$

Suppose that f is not essential. Then there exists $j \in I$ and (possibly empty) proper subact C of a_jS^1 such that $f : C \sqcup \bigsqcup_{i\neq j} a_iS^1 \to A$ is surjective. In particular, there exists $b \in C \sqcup \bigsqcup_{i\neq j} a_iS^1$ such that $f(b) = a_j$. We have two possibilities.

1) $b \in C$. Then $b = a_js$ for some $s \in S^1$. Hence, $a_j = f(b) = f(a_js) = a_js = b \in C$, which implies $a_jS^1 = C$, a contradiction.
2) $b \in \bigsqcup_{i\neq j} a_iS^1$. Then $a_jS^1 = f(b)S^1 = f(bS^1) \subseteq \bigcup_{i\neq j} a_iS^1$, a contradiction.

(6) $\implies$ (3). Assume that S is right $\mathcal{IC}$-perfect and consider a right sequence act M_S determined by a sequence $(s_i)_{i\in\mathbb{N}}$. Since M_S is unitary by the comment preceding Lemma 3.1, it has a cover $f\colon B_S \to M_S$, where

$$B_S = \bigsqcup_{i\in I} b_iS^1.$$

Fix an index $j \in I$. Let $f(b_j) = [k, s] \in M_S$. Then $f(b_jS^1) \subseteq [k, 1]S^1$. Consider a number $\ell > k$. Suppose $[\ell, 1]$ has an f-preimage in b_iS^1 for some $i \neq j$. Then $[\ell, 1]S^1 \subseteq f(b_iS^1)$, which implies that f restricted to $B \setminus b_jS^1$ is surjective. But that contradicts essentiality of f. Thus, $[\ell, 1]$ must have a preimage in b_jS^1. Let $[\ell, 1] = f(b_ju)$ for some $u \in S^1$. Now

$$[\ell, 1] = f(b_ju) = f(b_j)u = [k, s]u = [k, 1]su \in [k, 1]S^1,$$

which implies $[\ell, 1]S^1 = [k, 1]S^1$. Since this equality holds for every $\ell > k$, we have shown that $M_S = [k, 1]S^1$, so M_S is cyclic. □

Remark 4.4. Condition (5) in Theorem 4.3 appears first in Lemma 2.2 of [2], but a similar condition was used in [10]. It is kind of interesting that this condition can be formulated in topological terms as follows.

If A_S is an act, then the set of its subacts (here including the empty subact) is a topology on the set A because any union or intersection of subacts is a subact. The set of all cyclic subacts is a basis for this topology. Now condition (5) says that this basis contains a subset which is a minimal cover (in the sense of topology) for A.

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5. Condition (D)

In this section, we will prove that a factorizable semigroup satisfies Condition (D) if and only if it is right semiperfect. The following result can be proved precisely as Lemma 4.4 in [8].

Lemma 5.1. *Let T be a left unitary subsemigroup of a semigroup S. Then*

$$(\forall u, v \in T)\left(uT^1 \subseteq vT^1 \iff uS^1 \subseteq vS^1\right).$$

Lemma 5.2. *If a subsemigroup T of a semigroup S is left unitary, then it is a ρ-class of some right congruence ρ on S.*

Proof. Assume that T is a left unitary subsemigroup of S. Let ρ be the right congruence on S generated by the set $T \times T$. Take $t \in T$. Then $T \subseteq [t]_\rho$. We will prove that $[t]_\rho \subseteq T$. If $s \in [t]_\rho$, then $t \,\rho\, s$ and

$$\begin{array}{ccccccccccc} t & = & t_1 s_1 & & t_2' s_2 & = & t_3 s_3 \;\ldots\; t_{n-1}' s_{n-1} & = & t_n s_n & & \\ & & t_1' s_1 & = & t_2 s_2 & & & & t_n' s_n & = & s \end{array}$$

for some $t_i, t_i' \in T$ and $s_i \in S^1$. If $s_1 \neq 1$, then $s_1 \in T$ because T is left unitary. Hence, $t_1' s_1 \in T$, both when $s_1 \neq 1$ and when $s_1 = 1$. If $s_2 \neq 1$, then $s_2 \in T$, and so on. Thus, for every $i \in \{1, \ldots, n\}$, either $s_i = 1$ or $s_i \in T$. It follows that $s = t_n' s_n \in T$. □

Lemma 5.3. *If S is a semigroup and aS^1 is a unitary cyclic S-act, then*

$$T = \{s \in S \mid as = a\}$$

is a left unitary subsemigroup of S.

Proof. Since aS^1 is unitary, there exist $u \in S^1$ and $s \in S$ such that $a = (au)s = a(us)$. We see that $us \in T$, so T is nonempty, and it is clearly a subsemigroup of S. If $t, ts \in T$, then $as = (at)s = a(ts) = a$, so $s \in S$. Thus, T is left unitary. □

Theorem 5.4. Cf. [9], Result 1.5 and Theorem 5.2 in [20] *For a factorizable semigroup S, the following assertions are equivalent:*

(1) *S is right semiperfect;*
(2) *S satisfies Condition (D).*

Proof. (1) $\implies$ (2). (This is inspired by the proof of Proposition 4.6 in [8]. Differently from that proof, we cannot assume that R is the congruence class of the identity element, but with appropriate modifications, the proof will work.)

Let R be a left unitary subsemigroup of S. By Lemma 5.2, it is a ρ-class for some right congruence ρ on S, say $R = [t]$, $t \in R$. Then $[t]S^1$ is a cyclic subact of the right S-act S/ρ. It is a unitary S-act because $t, t^2 \in R$ implies $[t] = [t]t$.

By assumption, $[t]S^1$ has a projective cover in UAct_S. This must be a unitary cyclic projective act (see the proof of Lemma 4.1 in [8]). Due to Theorem 2.5, it must be of

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the form eS, where $e \in S$ is an idempotent. Also, there exists an essential epimorphism $f : eS \to [t]S^1$ in UAct_S. Because of surjectivity of f, there exists $eu \in eS$ such that $[t] = f(eu)$. Since S is factorizable, euS is a unitary subact of eS. Now $eut \in euS$ and $[t] = [tt] = [t]t = f(eu)t = f(eut)$ imply that $f|_{euS}$ is surjective. Due to essentiality, we have the equality $euS = eS$. It follows that $e = eus'$ for some $s' \in S$. Putting $s := s'e \in S$, we have $e = eus$ and $s = se$. Now

$$(su)(su) = (seu)(su) = s(eus)u = seu = su,$$

so su is an idempotent. In addition,

$$[t] = f(eu) = f(eusu) = f(eu)su = [t]su = [tsu].$$

Since R is left unitary, $t, tsu \in R$ implies $su \in R$. Let us prove that suR^1 is a minimal right ideal in R.

For this, we show that for every $r \in R$, $rR^1 \subseteq suR^1$ implies $rR^1 = suR^1$. Let $r \in R$. If $rR^1 \subseteq suR^1$, then $r = sur'$ for some $r' \in R^1$. Hence, $r = (su)^2r' = (su)(sur') = sur$ and

$$f(eurt) = f(eu)rt = [t]rt = [trt] = [t]$$

because $trt \in R$. Since $eurS$ is a unitary subact of eS and $f|_{eurS}$ is surjective, $eurS = eS = eS^1$. Now

$$sS = seS = seurS = surS = rS \subseteq rS^1 = surS^1 \subseteq suS^1 \subseteq sS,$$

which implies $rS^1 = suS^1$. Since $r, su \in R$, we conclude that $rR^1 = suR^1$ by Lemma 5.1.

(2) $\implies$ (1). Assume that S satisfies Condition (D) and consider a unitary cyclic S-act aS^1. The fact that aS^1 is unitary means that $a = at_0$ for some $t_0 \in S$. Now

$$T := \{t \in S \mid at = a\}$$

is a left unitary subsemigroup of S by Lemma 5.3. Since S satisfies Condition (D), there exists an idempotent $e \in T$ (so $ae = a$) such that eT is a minimal right ideal in T. We will show that the unitary S-act eS is a projective cover for aS^1 in UAct_S.

For this, we consider the surjective S-act homomorphism

$$f : eS \to aS^1, \qquad es \mapsto aes = as.$$

Suppose that B is a unitary subact of eS such that $f|_B$ is surjective. Then there exists $b \in B$ such that $f(b) = a$. Since $B \subseteq eS$, there exists $s \in S$ such that $b = es$. Observe that $a = f(es) = as$. Consequently, $s \in T$ and also $es \in T$. Now, using minimality of eT and Lemma 5.1,

$$esT^1 \subseteq eT \implies esT^1 = eT \implies esS^1 = eS \implies eS = bS^1 \subseteq B \subseteq eS \implies B = eS.$$

$\square$

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6. Condition (M_L)

Similar to Condition (A), Condition (M_L) has a description in terms of sequences of elements of S.

Proposition 6.1. *A semigroup S satisfies Condition (M_L) if and only if*

$$(\forall (s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}})(\exists n \in \mathbb{N})(\forall m > n)(\exists u \in S^1)(us_m \dots s_1 = s_n \dots s_1).$$

Proof. Necessity. Assume that S satisfies Condition (M_L). Consider a sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ and denote $t_i := s_i s_{i-1} \dots s_1$ for each $i \in \mathbb{N}$. Since the descending chain

$$S^1t_1 \supseteq S^1t_2 \supseteq S^1t_3 \supseteq \cdots$$

stabilises, there exists $n \in \mathbb{N}$ such that $S^1t_n = S^1t_m$ for every $m > n$. Hence, $t_n \in S^1t_m$, which means that $us_m \dots s_1 = s_n \dots s_1$ for some $u \in S^1$.

Sufficiency. Suppose to the contrary that S does not satisfy Condition (M_L). Then we have a strictly decreasing chain of principal left ideals

$$S^1t_1 \supset S^1t_2 \supset S^1t_3 \supset \cdots .$$

For every $i \in \mathbb{N}$, there exists $s_{i+1} \in S^1$ such that $t_{i+1} = s_{i+1}t_i$. Since we assumed that $t_i \neq t_{i+1}$, we have $s_{i+1} \in S$. Also, put $s_1 := t_1$. By our assumption, there exists $n \in \mathbb{N}$ such that for every $m > n$, there exists $u \in S^1$ such that $us_m \dots s_1 = s_n \dots s_1$. If $n=1$, then there exists $u \in S^1$ such that $us_2t_1 = t_1$, which implies $ut_2 = t_1$. Consequently, $S^1t_1 = S^1t_2$, a contradiction. Otherwise, let $m > n > 1$. Then $s_n \dots s_1 = u's_m \dots s_1$ for some $u' \in S^1$ and

$$t_n = s_nt_{n-1} = s_ns_{n-1}t_{n-2} = \cdots = s_ns_{n-1} \dots s_1 = u's_ms_{m-1} \dots s_1 = u't_m,$$

whence $S^1t_n \subseteq S^1t_m$ and therefore $S^1t_n = S^1t_m$, a contradiction. □

We will also need the following lemma.

Lemma 6.2. Cf. [6, Lemma 1] *If all right sequence acts over a semigroup S are projective, then S satisfies Condition (M_L).*

Proof. We will employ Proposition 6.1. Consider a sequence $(s_i)_{i\in\mathbb{N}}$ and the right sequence act M_S determined by it. By assumption, M_S is projective. Hence, for the surjective homomorphism

$$\pi : \mathbb{N} \times S^1 = F_S \to M_S, \qquad (k, s) \mapsto [k, s],$$

there exists a homomorphism $\mu : M_S \to F_S$ such that $\pi\mu = 1_M$.

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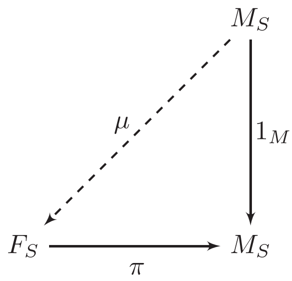

Let $\mu([1,1]) = (k,s) \in \mathbb{N} \times S^1$. Then

$$[1,1] = (\pi\mu)([1,1]) = \pi(k,s) = [k,s].$$

Since $(1,1) \sim (k,s)$, there exists $n \geqslant k$ such that $s_n \dots s_1 \cdot 1 = s_n \dots s_k \cdot s$. Let $m > n$ and put $\mu([m+1,1]) = (r,c)$ for some $r \in \mathbb{N}$ and $c \in S^1$. Note that $(1,1) \sim (m+1, s_m \dots s_1)$ because

$$s_{m+1} \dots s_1 \cdot 1 = s_{m+1} \cdot s_m \dots s_1.$$

Now

$$\begin{aligned}(k,s) = \mu([1,1]) = \mu([m+1, s_m \dots s_1]) &= \mu([m+1,1])s_m \dots s_1 \\ &= (r,c)s_m \dots s_1 = (r, cs_m \dots s_1),\end{aligned}$$

whence $k = r$ and $s = cs_m \dots s_1$. Putting $u := s_n \dots s_k c \in S$, we have

$$s_n \dots s_1 = s_n \dots s_k s = s_n \dots s_k c s_m \dots s_1 = u s_m \dots s_1,$$

as needed. □

7. Pullback flatness

Definition 7.1. *A right S-act A_S is called* ***pullback flat*** *if the functor $A \otimes_S - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves pullbacks.*

Since pullbacks are finite limits, every finite limit flat act is pullback flat.

We will need the following conditions (introduced first for acts over monoids in [21]) and the following theorem.

(P) If $as = a's'$ for some $a, a' \in A$ and $s, s' \in S$, then there exist $a'' \in A$ and $u, v \in S$ such that

$$a = a''u, \quad a' = a''v, \quad us = vs'.$$

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(E) If $as = as'$ for some $a \in A$ and $s, s' \in S$, then there exist $a' \in A$ and $u \in S$ such that

$$a = a'u, \quad us = us'.$$

Theorem 7.2. [**15**, **Theorem 3.5**] *Let A_S be a unitary act over a semigroup S. Then A_S is pullback flat if and only if it satisfies both Condition (P) and Condition (E).*

Proposition 7.3. *A cyclic S-act aS^1 is unitary and pullback flat if and only if*

$$(\forall s, t \in S^1)(as = at \implies (\exists u \in S)(a = au \wedge us = ut)).$$

Proof. Necessity. Let $A = aS^1$ be unitary and pullback flat. Take $s, t \in S^1$ such that $as = at$. Since aS^1 is unitary, $a = av$ for some $v \in S$. Then $a(vs) = a(vt)$, where $vs, vt \in S$. By Theorem 7.2, aS^1 satisfies Condition (E), so there exists $w \in S$ such that $a = aw$ and $w(vs) = w(vt)$. Putting $u := wv$, we have $a = au$ and $us = ut$.

Sufficiency. By assumption, there exists $u \in S$ such that $a = au$, thus aS^1 is unitary. That aS^1 satisfies Condition (E) follows immediately from the assumption. To prove Condition (P), suppose that $(as_1)s = (as_2)s'$, where $s_1, s_2 \in S^1$ and $s, s' \in S$. By assumption, there exists $u \in S$ such that $a = au$ and $us_1s = us_2s'$. Now $as_1 = a(us_1)$ and $as_2 = a(us_2)$, where $us_1, us_2 \in S$. □

Lemma 7.4. Cf. [**6**, **Lemma 3**] *Let A_S be a unitary pullback flat act which satisfies (ACC) for cyclic subacts. If A_S is indecomposable, then it is cyclic.*

Proof. Take any $x \in A$ and consider the set

$$P_x := \{bS^1 \mid b \in A,\ xS^1 \subseteq bS^1\}$$

as a poset with respect to inclusion. Since $xS^1 \in P_x$, we have $P_x \neq \emptyset$, so by assumption, there exists a maximal element $aS^1 \in P_x$. In particular, $xS^1 \subseteq aS^1$. Consider the set $Y := A \setminus aS^1$. Assume for a contradiction that $Y \neq \emptyset$. Now, A_S is a union of its subacts aS^1 and YS^1. Since A_S is indecomposable,

$$aS^1 \cap YS^1 \neq \emptyset,$$

which means that there exist $y \in Y$ and $s,\ t \in S^1$ such that $as = yt$.

On the one hand, if $t = 1$, then $y = as \in aS^1$, a contradiction. On the other hand, suppose $s = 1$. The equality $a = yt$ implies that $xS^1 \subseteq aS^1 \subseteq yS^1$. It follows that $yS^1 \in P_x$ and $aS^1 = yS^1$ due to maximality. So again, we must have $y \in aS^1$, which is impossible. Thus, we must have $s, t \in S$.

Since A_S is a unitary pullback flat act, by Theorem 7.2, it satisfies Condition (P). Hence, there exist $a' \in A$ and $u, v \in S$ such that $a = a'u$, $y = a'v$ and $us = vt$. We conclude that $aS^1 \subseteq a'S^1$. It follows that $a'S^1 \in P_x$, so $aS^1 = a'S^1$ due to maximality. Now, for some $z \in S^1$, $a' = az$ and therefore $y = a'v = azv$, whence $y \in aS^1$, a contradiction. Thus, $Y = \emptyset$ and $A = aS^1$ is cyclic, as required. □

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Corollary 7.5. *If a semigroup S satisfies Condition (A), then every unitary pullback flat S-act is a disjoint union of cyclic unitary pullback flat subacts.*

Proof. Assume that S satisfies Condition (A) and let A_S be unitary and pullback flat. By Proposition 2.1,

$$A = \bigsqcup_{i\in I} A_i,$$

where A_i is an indecomposable subact of A_S for every $i \in I$. It is clear by Theorem 7.2 that the A_i are pullback flat. Since S satisfies Condition (A), each A_i satisfies (ACC) for cyclic subacts. We see that the A_i satisfy all assumptions of Lemma 7.4, so they must be cyclic. □

Lemma 7.6. Cf. [6, Lemma 5 and Lemma 4] *If a factorizable semigroup S satisfies either Condition (D) or Condition (M_L), then every unitary cyclic pullback flat right S-act is projective.*

Proof. Let $A_S = aS^1$ be a unitary cyclic pullback flat S-act. By Lemma 5.3, the set $T = \{s \in S \mid as = a\}$ is a left unitary subsemigroup of S. If S satisfies Condition (D), then T contains a minimal right ideal eT, where $e \in T$ is an idempotent. If S satisfies Condition (M_L), then the set

$$\mathcal{I} = \{S^1 t \mid t \in T\}$$

of principal left ideals of S contains a minimal element $S^1 e$ for some element $e \in T$. In both cases, it suffices to prove that condition (4) of Proposition 2.6 is satisfied.

Assume that $as = at$ for some $s, t \in S^1$. Since aS^1 is pullback flat, by Proposition 7.3, there exists $u \in S$ such that $a = au$ and $us = ut$. The equality $au = ae$ implies that $a = av$ and $vu = ve$ for some $v \in S$. Note that $u, v \in T$.

First, let S satisfy Condition (D). By Lemma 8.12 in [5], Te is a minimal left ideal of T. It follows that

$$Tvu \subseteq Tu \cap Te \subseteq Te,$$

whence $Te = Tvu \subseteq Tu$ due to minimality. Therefore, $e = wu$ for some $w \in T$. We conclude that $es = wus = wut = et$, as needed.

Second, assume S satisfies Condition (M_L). Since T is a subsemigroup, $vu \in T$ and $S^1 vu \in \mathcal{I}$. The equality $vu = ve$ implies $S^1 vu \subseteq S^1 e$. We have $S^1 vu = S^1 e$ due to minimality and $e = wvu$ for some $w \in S^1$. This implies $es = wvus = wvut = et$. In particular, $a1 = ae$ implies that $e1 = ee$ by the previous argument, so e is an idempotent. □

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8. Condition (K)

Definition 8.1. Cf. [11, Definition 1.6] *We say that a subsemigroup T of S is **left collapsible** if T^1 is a left collapsible submonoid of S^1, that is,*

$$(\forall t, t' \in T^1)(\exists u \in T^1)(ut = ut').$$

Condition (K) for monoids was introduced by Kilp in [11] and used later in [12]. We use it for semigroups in the following form.

Definition 8.2. *A semigroup S satisfies **Condition** (K) if every left collapsible subsemigroup of S has a left zero.*

Proposition 8.3. Cf. [11, Theorem 2.3] *A factorizable semigroup S satisfies Condition (K) if and only if every unitary cyclic pullback flat right S-act is projective.*

Proof. Necessity. Suppose S satisfies Condition (K). Let aS^1 be unitary pullback flat and let $T = \{s \in S \mid as = a\}$. If $t, t' \in T^1$, then $at = at'$. By Proposition 7.3, there exists $u \in S$ such that $a = au$ and $ut = ut'$. In particular, $u \in T$. We have shown that T^1 is a left collapsible submonoid of S^1.

By assumption, T has a left zero e. We will check condition (4) of Proposition 2.6. We know that $a = ae$. Suppose that $as = at$, $s, t \in S^1$. Then there exists $u \in S$ such that $a = au$ and $us = ut$. Then $u \in T$, $eu = e$ and $es = eus = eut = et$.

Sufficiency. Assume that every unitary cyclic pullback flat right S-act is projective. Let $T \subseteq S$ be a left collapsible subsemigroup of S. Then by definition $P := T^1$ is a left collapsible submonoid of S^1. By Lemma 2.1 in [11], the relation ρ, defined by

$$s\,\rho\,t \iff (\exists p, q \in P)(ps = qt),$$

is a right congruence on the monoid S^1. In particular, $1\,\rho\,p$ for all $p \in P$. The cyclic S^1-act $S^1/\rho = [1]_\rho S^1$ is pullback flat.

It turns out that $S^1/\rho = [1]_\rho S^1$ is also a cyclic unitary pullback flat S-act. Since T is nonempty, we can choose $t_0 \in T \subseteq S$. If $[1]_\rho\, s = [1]_\rho\, t$, $s, t \in S^1$, then Condition (E) implies that there exists $u \in S^1$ such that $[1]_\rho = [1]_\rho\, u$ and $us = ut$. If it happens that $u = 1$, then still $[1]_\rho = [1]_\rho\, t_0$ and $t_0 s = t_0 t$. By Proposition 7.3, S^1/ρ is a unitary pullback flat S-act.

By assumption, $[1]_\rho S^1$ is projective. Using Corollary 2.7, we know that there exists $e \in E(S)$ such that $[1]_\rho = [1]_\rho e$ and $[1]_\rho s = [1]_\rho t$ implies $es = et$ for all $s, t \in S^1$. In other words, $1\,\rho\,e$ and $s\,\rho\,t$ implies $es = et$ for all $s, t \in S^1$. From $1\,\rho\,e$, we obtain $p, q \in P$ such that $p = qe$. Let $r \in P$ be arbitrary. Then $1\,\rho\,r$, which implies $e = er$, and therefore $pr = qer = qe = p$. Thus, p is a left zero for P. If $p = 1$, then $1 = qe = qee = e$, which contradicts the fact that $e \in S$. Thus, $p \in T$ and p is a left zero for T. □

Corollary 8.4. *If S is factorizable, then Condition (M_L) implies Condition (K).*

Proof. This follows from Proposition 8.3 and Lemma 7.6. □

9. The main theorem

Our main theorem is the following.

Theorem 9.1. *For a factorizable semigroup S, the following are equivalent:*

(1) *S is right perfect;*
(2) *S is right $\mathcal{IC}$-perfect and right semiperfect;*
(3) *S satisfies both Condition (A) and Condition (D);*
(4) *S satisfies both Condition (A) and Condition (M_L);*
(5) *S satisfies both Condition (A) and Condition (K);*
(6) *every right sequence act over S is projective in UAct_S;*
(7) *every finite limit flat right S-act is projective in UAct_S;*
(8) *every unitary pullback flat right S-act is projective in UAct_S;*
(9) *every finite limit flat right S-act is limit flat;*
(10) *every right sequence act over S is limit flat.*

Proof. (1) ⇒ (2). If a semigroup S is right perfect, then it is clearly right semiperfect. If $f : \bigsqcup_{i\in I} e_iS \to A_S$ is a projective cover for a unitary act A_S, then its indecomposable components e_iS, $i \in I$, are cyclic and unitary. Hence, S is also right $\mathcal{IC}$-perfect.

(2) ⇒ (1) Let A_S be a unitary S-act. Since S is $\mathcal{IC}$-perfect, there exists a cover $g : B_S \to A_S$, where $B_S = \bigsqcup_{i\in I} b_iS^1$ is a disjoint union of unitary cyclic subacts. Semiperfectness of S gives a unitary projective cover $f_i : P_i \to b_iS^1$ for each $i \in I$.

Then $P_S := \bigsqcup_{i\in I} P_i$ is a unitary projective act by Theorem 2.5 and

$$f : P_S \to B_S, \qquad x \mapsto f_i(x) \quad \text{if } x \in P_i$$

is a surjective homomorphism of right S-acts. We will show that f is essential.

Suppose that Q is a proper subact of P_S. Then there exists $i \in I$ such that $Q_i := Q \cap P_i$ is a proper (possibly empty) subact of P_i. Hence, $f_i|_{Q_i} : Q_i \to b_iS^1$ is not surjective because f_i is essential. But then also $f|_Q$ is not surjective because the preimages of elements of b_iS^1 can only come from P_i. Thus, f is an essential epimorphism. Hence, $gf : P_S \to A_S$ is an essential epimorphism and P_S is a projective cover for A_S.

(2) ⇔ (3). This follows from Theorem 5.4 and Theorem 4.3.

(6) ⇒ (3). Assume that all right sequence acts are projective. Since they are indecomposable (see Lemma 3.1), unitary and projective, they must be cyclic. Hence, S satisfies Condition (A) by Theorem 4.3.

We will prove that S satisfies Condition (D). The proof of this implication is inspired by the proof of Theorem 6.2 in [8].

Let T be a left unitary subsemigroup of S. Take any $t \in T$. We will prove that the principal right ideal tT^1 of T contains an idempotent. By the assumption, the right sequence act M_S, determined by the constant sequence $(t) \in S^{\mathbb{N}}$, is projective. Since M_S is indecomposable, it must be cyclic. From Lemma 3.1, we conclude that there exists $i \in \mathbb{N}$ such that $M = [i,1]S^1$. By Corollary 2.7, there exists $e^2 = e \in S$ such that $[i,1] = [i,e]$ and

$$(\forall x, y \in S^1)([i,x] = [i,y] \implies ex = ey). \tag{9.1}$$

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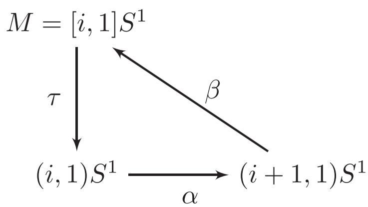
From $[i,1]S^1 \subseteq [i+1,1]S^1 \subseteq M_S = [i,1]S^1$, we conclude that $[i,1]S^1 = [i+1,1]S^1$. Hence, there exists $z \in S^1$ such that $[i+1,1] = [i,z]$. We have the diagram



with 1-generated free S-acts $(i,1)S^1$ and $(i+1,1)S^1$ and with right S-act homomorphisms defined by

$$\begin{aligned} \alpha(i,s) &:= (i+1,ts),\\ \beta(i+1,s) &:= [i+1,s] = [i,zs],\\ \tau([i,s]) &:= (i,es),\\ \psi &:= \alpha\tau\beta : (i+1,1)S^1 \to (i+1,1)S^1. \end{aligned}$$

Note that τ is well defined due to the implication (9.1).

Denoting $h := tez$, since $[i+1,t] = [i,1]$, we have that

$$\begin{aligned} (\beta\alpha\tau)([i,z]) &= (\beta\alpha)(i,ez) = \beta(i+1,tez) = [i+1,tez] = [i+1,t]ez = [i,1]ez\\ &= [i,e]z = [i,1]z = [i,z] \end{aligned}$$

and

$$\begin{aligned} \psi^2(i+1,1) &= (\psi\alpha\tau\beta)(i+1,1) = (\psi\alpha\tau)([i+1,1]) = (\alpha\tau\beta\alpha\tau)([i,z])\\ &= (\alpha\tau)([i,z]) = (\alpha\tau\beta)(i+1,1) = \psi(i+1,1) = (\alpha\tau)([i,z])\\ &= \alpha(i,ez) = (i+1,tez) = (i+1,h). \end{aligned}$$

Hence,

$$(i+1,h) = \psi^2(i+1,1) = \psi(i+1,h) = \psi(i+1,1)h = (i+1,h)h = (i+1,h^2),$$

which yields $h^2 = h$. The equalities $[i+1,1] = [i,z]$ and $[i,1] = [i,e]$ imply that $t^n = t^{n+1}z$ and $t^m = t^m e$ for some $m,n \in \mathbb{N}$. Since T is a left unitary subsemigroup, $z,e \in T$ and we have an idempotent $h \in tT^1$.

Now consider a chain

$$e_1T^1 \supseteq e_2T^1 \supseteq e_3T^1 \supseteq \cdots$$

of principal right ideals of T generated by idempotents $e_i \in T$. By Lemma 5.1, we have a chain

$$e_1S^1 \supseteq e_2S^1 \supseteq e_3S^1 \supseteq \cdots.$$

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Similar to the proof of Theorem 6.2 in [8], we can find idempotents $g_i \in S$ such that

$$S^1 g_1 \supseteq S^1 g_2 \supseteq S^1 g_3 \supseteq \cdots ,$$

$e_i S^1 = g_i S^1$ and $g_{i+1} g_i = g_{i+1} = g_i g_{i+1}$ for every $i \in \mathbb{N}$.

By Lemma 6.2, S satisfies Condition (M_L). Thus, there exists $n \in \mathbb{N}$ such that $S^1 g_i = S^1 g_{i+1}$ for every $i \geqslant n$. Now $g_i = g_i g_{i+1} = g_{i+1}$; therefore, $g_i = g_{i+1}$ and $e_i S^1 = e_{i+1} S^1$ for every $i \geqslant n$. By Lemma 5.1, it follows that $e_i T^1 = e_{i+1} T^1$ for every $i \geqslant n$. We have shown that T satisfies the DCC for principal right ideals generated by idempotents.

Suppose that T does not have a minimal principal right ideal. Take any $b_1 \in T$. Then $b_1 T^1$ contains an idempotent e_1 and $b_1 T^1 \supseteq e_1 T^1$. Since $e_1 T^1$ is not a minimal ideal, there exists $b_2 \in T$ such that $e_1 T^1 \supset b_2 T^1$. In this way, we get an infinite descending chain

$$b_1 T^1 \supseteq e_1 T^1 \supset b_2 T^1 \supseteq e_2 T^1 \supset b_3 T^1 \supseteq e_3 T^1 \supset \cdots$$

of principal right ideals of T, where $e_i \in E(T)$. Thus, we also have a chain

$$e_1 T^1 \supset e_2 T^1 \supset e_3 T^1 \supset \cdots ,$$

a contradiction. It follows that T has a minimal principal right ideal bT^1. As it contains an idempotent e, we have $bT^1 = eT^1$ due to minimality.

(4) $\Rightarrow$ (5). By Corollary 8.4.

(5) $\Rightarrow$ (6). Assume that S satisfies both Condition (A) and Condition (K). By Theorem 4.3, every right sequence act M_S is cyclic. Due to Corollary 3.3, every M_S is unitary and finite limit flat, hence also pullback flat. By Proposition 8.3, M_S is projective.

(6) $\Rightarrow$ (4). By Lemma 6.2, S satisfies Condition (M_L). We show that S satisfies Condition (A) using Theorem 4.3. Take any sequence $(s_i)_{i \in \mathbb{N}} \in S^{\mathbb{N}}$ and consider the right sequence act M_S determined by it. By assumption, it is projective. It is also indecomposable; hence, by Proposition 2.6, there exists an S-isomorphism $f : M_S \to eS$, where $e \in E(S)$ is some idempotent. Then there exists $[m, s] \in M$ such that $f([m, s]) = e$. Also, there exists $s' \in S$ such that $f([m + 1, 1]) = es'$. Now

$$f([m + 1, 1]) = f([m, s])s' = f([m, ss']) \implies [m + 1, 1] = [m, ss'].$$

Thus, there exists $k \geqslant m + 1$ such that

$$s_k \ldots s_{m+1} = s_k \ldots s_{m+1} s_m s s'.$$

(3) $\vee$ (4) $\Rightarrow$ (8). Let A_S be a unitary pullback flat act. Using Condition (A), by Corollary 7.5, we conclude that $A_S = \bigsqcup_{i \in I} A_i$, where A_i is a cyclic unitary pullback flat act for every $i \in I$. We have by Lemma 7.6 that each A_i is projective when S satisfies either Condition (D) or Condition (M_L). Thus, A_S is projective due to Theorem 2.2.

(8) $\Rightarrow$ (7). Since pullbacks are finite limits, each finite limit act is pullback flat.

(7) $\Rightarrow$ (6). Every right sequence act M_S satisfies Condition (E) by Lemma 3.10 in [15]. It is easy to see that M_S satisfies Condition (LC): if $a, a' \in M$, then there exist $a'' \in M$

and $u, v \in S$ such that $a = a''u$ and $a' = a''v$. By Theorem 4.2 in [15], right sequence acts are finite limit flat. We also know that they are unitary. By assumption, they are projective.

(7) $\Leftrightarrow$ (9). A finite limit flat act A_S must satisfy Condition (LC) by Theorem 4.2 in [15]. This implies that A_S is indecomposable. By Proposition 2.6, A_S is an indecomposable projective in UAct_S if and only if it is a unitary limit flat act.

(6) $\Leftrightarrow$ (10). Let M_S be a right sequence act. Then M_S is limit flat if and only if $M_S \cong eS_S$ for some idempotent $e \in S$ by Proposition 4.4 in [15]. In other words, a right sequence act over a factorizable semigroup is limit flat if and only if it is projective. □

Remark 9.2. 1. Condition (4) in Theorem 9.1 is important because together with Theorem 4.3 and Proposition 6.1, it shows that perfectness of a factorizable semigroup S can be verified by checking two conditions that are formulated in terms of sequences of elements of S. These conditions are internal to S and do not refer to any categories.

2. If S is factorizable and right perfect, then any right sequence act over S is projective. In particular, S must contain at least one idempotent.

It is easy to check that Rees matrix semigroups (with zero) over a group satisfy both Condition (A) and Condition (M_L) (and their duals). Thus, we have the following result.

Corollary 9.3. Cf. [2, Corollary 3.13] *Every completely* (0-)*simple semigroup is perfect.*

10. Morita invariance

Recall that semigroups S and T are called **Morita equivalent** if the categories FAct_S and FAct_T are equivalent (see [18]). A **Morita invariant** is a property that is shared by all Morita equivalent semigroups. In this section, we will prove that right semiperfectness, right $\mathcal{IC}$-perfectness and right perfectness are Morita invariants on the class of factorizable semigroups. For this, we need to examine essential epimorphisms in UAct_S and FAct_S.

Proposition 10.1. *Let S be a semigroup and $A_S \in \mathsf{UAct}_S$. Then μ_A is an essential epimorphism in UAct_S.*

Proof. Since A_S is unitary, μ_A is surjective, and it is easy to see that $A \otimes_S S$ is unitary. Let $U \subseteq A \otimes S$ be a unitary subact and consider the diagram

$$U_S \xrightarrow{\iota} A \otimes_S S \xrightarrow{\mu_A} A_S$$

in UAct_S, where ι is the embedding. Suppose $\mu_A \iota$ is surjective. By Lemma 1.4, it suffices to show that ι is surjective, that is, $U = A \otimes S$. Take $a \otimes s \in A \otimes S$. By surjectivity of $\mu_A \iota$, there exist $a' \in A$ and $s' \in S$ such that $a' \otimes s' \in U$ and $a = (\mu_A \iota)(a' \otimes s') = a's'$. Now

$$a \otimes s = a's' \otimes s = a' \otimes s's = (a' \otimes s')s \in US \subseteq U.$$

Thus, ι is surjective, as required. □

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Proposition 10.2. *Let S be a factorizable semigroup. For a morphism $f : P_S \to A_S$ in FAct_S, the following are equivalent:*

(1) *f is an essential epimorphism in UAct_S;*
(2) *f is an essential epimorphism in FAct_S.*

Proof. (1) $\Rightarrow$ (2). This is clear.

(2) $\Rightarrow$ (1). Let $U \subseteq P_S$ be a unitary subact and let $\iota : U \to P$ be the embedding. Suppose $f\iota$ is surjective. By Lemma 1.4, it suffices to show that ι is surjective, that is, $U = P$. According to Lemma 1.1, $U \otimes_S S$ is a firm right S-act. Consider the diagram

$$U \otimes_S S \xrightarrow{\mu_U} U_S \xrightarrow{\iota} P_S \xrightarrow{f} A_S.$$

Since μ_U is surjective, $f\iota\mu_U$ is a surjective morphism in FAct_S. Also, $\iota\mu_U$ is a morphism in FAct_S. Then $\iota\mu_U$ is surjective due to essentiality of f. It follows that ι is surjective. □

Proposition 10.3. *The following are equivalent for a factorizable semigroup S:*

(1) *every cyclic act in UAct_S has a projective cover;*
(2) *every cyclic act in FAct_S has a projective cover.*

Proof. (1) $\Rightarrow$ (2). Take a cyclic act $aS^1 \in \mathsf{FAct}_S$. Since aS^1 is unitary, it has a cyclic projective cover $f : eS \to aS^1$ in UAct_S (note that a cover of a cyclic act must be cyclic). Recall that eS is firm (see Lemma 2.3). Consider a diagram

$$B_S \xrightarrow{g} eS \xrightarrow{f} aS^1$$

in FAct_S. If fg is surjective, then g must be surjective because this diagram is also in UAct_S and f is an essential epimorphism in UAct_S. Thus, f is an essential epimorphism in FAct_S.

(2) $\Rightarrow$ (1). Take a cyclic act $aS^1 \in \mathsf{UAct}_S$. By Lemma 1.1, $aS^1 \otimes_S S$ is firm. Since aS^1 is unitary, there exist $s \in S$ and $u \in S^1$ such that $a = (au)s$. Then

$$aS^1 \otimes_S S = (au \otimes s)S^1.$$

The inclusion $\supseteq$ is clear. On the other hand, if $v \in S^1$ and $t \in S$, then

$$av \otimes t = (au)sv \otimes t = au \otimes svt = (au \otimes s)vt \in (au \otimes s)S^1.$$

Therefore, $aS^1 \otimes_S S$ is a cyclic right S-act.

By assumption, there exists a projective cover $f : eS \to aS^1 \otimes S$ in FAct_S. By Proposition 10.2, f is also an essential epimorphism in UAct_S. Proposition 10.1 implies that $\mu_{aS^1} : aS^1 \otimes S \to aS^1$ is an essential epimorphism in UAct_S. Therefore,

$$eS \xrightarrow{\mu_{aS^1} f} aS^1$$

is a projective cover in UAct_S. □

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Having a projective cover is a purely categorical property; thus, it is preserved by equivalence functors. Also, indecomposability is a categorical property: an act is indecomposable if it is not a coproduct of two noninitial objects (the initial object is just the empty act). However, for cyclicity, there is no obvious categorical description. Still, cyclic acts are preserved under tensor multiplication functors coming from certain Morita contexts.

Definition 10.4. [22] *A **Morita context** is a six-tuple* $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, *where* S *and* T *are semigroups,* ${}_SP_T \in {}_S\mathsf{Act}_T$ *and* ${}_TQ_S \in {}_T\mathsf{Act}_S$ *are biacts and*

$$\theta : {}_S(P \otimes Q)_S \to {}_SS_S, \qquad \phi : {}_T(Q \otimes P)_T \to {}_TT_T$$

are biact homomorphisms such that for every $p, p' \in P$ *and* $q, q' \in Q$, *we have*

$$\theta(p \otimes q)p' = p\phi(q \otimes p') \quad \textit{and} \quad q\theta(p \otimes q') = \phi(q \otimes p)q'.$$

*This context is called **unitary** if* ${}_SP_T$ *and* ${}_TQ_S$ *are unitary biacts, meaning that* $SP = P = PT$ *and* $TQ = Q = QS$.

By Theorem 5.9 in [14], two firm semigroups S and T are Morita equivalent if and only if they are connected by a unitary Morita context with surjective mappings θ and ϕ.

Proposition 10.5. *Let* S *and* T *be firm semigroups connected by a unitary Morita context* $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ *with surjective mappings. Then the functors*

$$- \otimes {}_SP_T : \mathsf{FAct}_S \to \mathsf{FAct}_T \quad \textit{and} \quad - \otimes_T Q_S : \mathsf{FAct}_T \to \mathsf{FAct}_S$$

take cyclic acts to cyclic acts.

Proof. As in Proposition 3.16 of [18], one can prove that $- \otimes {}_SP_T : \mathsf{FAct}_S \to \mathsf{FAct}_T$ and $- \otimes {}_TQ_S : \mathsf{FAct}_T \to \mathsf{FAct}_S$ are equivalence functors inverse to each other (see also Theorem 5.9 in [14]). We will prove that $-\otimes_S P_T$ takes cyclic acts to cyclic acts. The same is true for $- \otimes {}_TQ_S$, so cyclic acts will correspond to each other under these functors.

Consider a cyclic act $aS^1 \in \mathsf{FAct}_S$. We will prove that $aS^1 \otimes_S P_T$ is cyclic. Since aS^1 is unitary, there exists $s \in S$ such that $as = a$. Using surjectivity of θ, we can find $p_s \in P$ and $q_s \in Q$ such that $s = \theta(p_s \otimes q_s)$. We will prove that

$$aS^1 \otimes_S P_T = (a \otimes p_s)T^1.$$

The inclusion $(a \otimes p_s)T^1 \subseteq aS^1 \otimes_S P_T$ is clear. To prove the converse, we note that

$$\begin{aligned} au \otimes p = asu \otimes p = a \otimes sup &= a \otimes \theta(p_s \otimes q_s)up = a \otimes p_s\phi(q_s \otimes up) \\ &= (a \otimes p_s)\phi(q_s \otimes up) \in (a \otimes p_s)T^1 \end{aligned}$$

for every $u \in S^1$ and $p \in P$. □

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Proposition 10.6. *Let S be a factorizable semigroup. Then the functors*

$$-\otimes_S S_{S\otimes S} : \mathsf{FAct}_S \to \mathsf{FAct}_{S\otimes S} \quad \textit{and} \quad -\otimes_{S\otimes S} S_S : \mathsf{FAct}_{S\otimes S} \to \mathsf{FAct}_S$$

are inverse equivalence functors which take cyclic acts to cyclic acts.

Proof. By Proposition 4.9 in [16], these functors are inverse equivalence functors. We also recall that $S\otimes S$ is considered as a semigroup with the multiplication $(s\otimes t)(u\otimes v) = st\otimes uv$, and the action of $S_{S\otimes S}$ is $s(u\otimes v) = suv$.

Let $aS^1 \in \mathsf{FAct}_S$ be cyclic. We want to show that $aS^1 \otimes_S S_{S\otimes S}$ is a cyclic $S\otimes S$-act. Suppose $a = (au)s$ for some $u \in S^1$ and $s \in S$. We will prove that

$$aS^1 \otimes_S S_{S\otimes S} = (au\otimes s)(S\otimes S)^1.$$

On the one hand,

$$(au\otimes s)(S\otimes S)^1 \subseteq (aS^1 \otimes_S S)(S\otimes S)^1 = aS^1 \otimes_S S.$$

Conversely, suppose that $av\otimes t \in aS^1 \otimes_S S_{S\otimes S}$. Since S is factorizable, $t = s_1s_2$ for some $s_1, s_2 \in S$. Hence,

$$\begin{aligned} av\otimes t &= ausv\otimes t = au\otimes svt = au\otimes svs_1s_2 = au\otimes (s(vs_1\otimes s_2)) \\ &= (au\otimes s)(vs_1\otimes s_2) \in (au\otimes s)(S\otimes S)^1. \end{aligned}$$

Let now $a(S\otimes S)^1 \in \mathsf{FAct}_{S\otimes S}$ be cyclic. We want to show that $a(S\otimes S)^1 \otimes_{S\otimes S} S_S$ is a cyclic S-act. Since $a(S\otimes S)^1$ is unitary, $a = a(s_1\otimes s_2)$ for some $s_1, s_2 \in S$. We will prove that

$$a(S\otimes S)^1 \otimes_{S\otimes S} S_S = (a\otimes s_1s_2)S^1.$$

The inclusion $\supseteq$ is clear. Conversely, let $a(u_1\otimes u_2)\otimes s \in a(S\otimes S)^1 \otimes_{S\otimes S} S_S$, where $u_1, u_2, s \in S$. Then

$$\begin{aligned} a(u_1\otimes u_2)\otimes s &= a(s_1\otimes s_2)(u_1\otimes u_2)\otimes s = a(s_1s_2\otimes u_1u_2)\otimes s = a\otimes (s_1s_2\otimes u_1u_2)s \\ &= a\otimes s_1s_2u_1u_2s = (a\otimes s_1s_2)u_1u_2s \in (a\otimes s_1s_2)S^1. \end{aligned}$$

□

Our first theorem about invariants is the following.

Theorem 10.7. *Right semiperfectness is a Morita invariant for factorizable semigroups.*

Proof. Suppose S and T are Morita equivalent factorizable semigroups. Equivalently, by Proposition 4.9 in [16], $S\otimes S$ and $T\otimes T$ are Morita equivalent firm semigroups. According to Theorem 5.9 in [14], there exists a unitary Morita context with bijective mappings connecting $S\otimes S$ and $T\otimes T$. Consequently,

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$$
\begin{aligned}
S \text{ is right semiperfect} &\iff \text{all cyclic objects in } UAct_S \text{ have a projective cover}\\
\text{(by Proposition 10.3)} &\iff \text{all cyclic objects in } FAct_S \text{ have a projective cover}\\
\text{(by Proposition 10.6)} &\iff \text{all cyclic objects in } FAct_{S\otimes S} \text{ have a projective cover}\\
\text{(by Proposition 10.5)} &\iff \text{all cyclic objects in } FAct_{T\otimes T} \text{ have a projective cover}\\
\text{(by Proposition 10.6)} &\iff \text{all cyclic objects in } FAct_T \text{ have a projective cover}\\
\text{(by Proposition 10.3)} &\iff \text{all cyclic objects in } UAct_T \text{ have a projective cover}\\
&\iff T \text{ is right semiperfect.}
\end{aligned}
$$

□

Proposition 10.8. *The following are equivalent for a factorizable semigroup S:*

(1) *every act in* UAct_S *has an* $\mathcal{IC}$*-cover;*
(2) *every act in* FAct_S *has an* $\mathcal{IC}$*-cover.*

Proof. (1) ⇒ (2) Take $A_S \in \mathsf{FAct}_S$ and let

$$B_S := \coprod_{i\in I} b_i S^1 \xrightarrow{f} A_S$$

be an $\mathcal{IC}$-cover in UAct_S. By Lemma 1.1, $B \otimes_S S$ is firm. The indecomposable components of B are unitary. Hence, for every $i \in I$, there exist $u_i \in S^1$ and $s_i \in S$ such that $b_i = (b_i u_i)s_i$. Then

$$B \otimes_S S = \left(\coprod_{i\in I} b_i S^1\right) \otimes_S S = \coprod_{i\in I}(b_i S^1 \otimes_S S) = \coprod_{i\in I}(b_i u_i \otimes s_i)S^1.$$

The acts $(b_i u_i \otimes s_i)S^1$ are indecomposable because they are cyclic.
Consider the diagram

$$B \otimes S \xrightarrow{\mu_B} B_S \xrightarrow{f} A_S.$$

By Proposition 10.1, μ_B is essential in UAct_S. Thus, the composition $f\mu_B$ is an essential epimorphism in UAct_S. Since $f\mu_B$ is also a morphism in FAct_S, it is essential in FAct_S by Proposition 10.2.

(2) ⇒ (1) Take $A_S \in \mathsf{UAct}_S$. Then $A \otimes_S S$ is firm by Lemma 1.1. By assumption, there exists an $\mathcal{IC}$-cover

$$C_S := \coprod_{i\in I} c_i S^1 \xrightarrow{f} A \otimes_S S$$

in FAct_S. Then f is an essential epimorphism in UAct_S by Proposition 10.2. By Proposition 10.1, μ_A is essential in UAct_S. Therefore, $\mu_A f : C_S \to A_S$ is an $\mathcal{IC}$-cover in UAct_S. □

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Theorem 10.9. *Right $\mathcal{IC}$-perfectness is a Morita invariant for factorizable semigroups.*

Proof. Let S and T be factorizable semigroups and $F : \mathsf{FAct}_S \to \mathsf{FAct}_T$ an equivalence functor that takes cyclic acts to cyclic acts. Suppose that $A_S \in \mathsf{FAct}_S$ has an $\mathcal{IC}$-cover

$$f : \bigsqcup_{i \in I} b_i S^1 \to A_S.$$

Since F preserves coproducts and essential epimorphisms,

$$F(f) : \bigsqcup_{i \in I} F(b_i S^1) \cong F\left(\bigsqcup_{i \in I} b_i S^1\right) \to F(A_S)$$

is an $\mathcal{IC}$-cover of $F(A_S)$, where $F(b_i S^1) \in \mathsf{FAct}_S$ are cyclic acts (and hence indecomposable). So $F(A_S)$ also has an $\mathcal{IC}$-cover. If F has an inverse equivalence functor G, which also takes cyclic acts to cyclic acts, then we can prove that A_S has an $\mathcal{IC}$-cover if and only if $F(A_S)$ has an $\mathcal{IC}$-cover.

Let now S and T be as in the proof of Theorem 10.7. Then

$$\begin{aligned}
S \text{ is right } \mathcal{IC}\text{-perfect} &\iff \text{all objects in } \mathsf{UAct}_S \text{ have an } \mathcal{IC}\text{-cover} \\
\text{(by Proposition 10.8)} &\iff \text{all objects in } \mathsf{FAct}_S \text{ have an } \mathcal{IC}\text{-cover} \\
\text{(by Proposition 10.6)} &\iff \text{all objects in } \mathsf{FAct}_{S \otimes S} \text{ have an } \mathcal{IC}\text{-cover} \\
\text{(by Proposition 10.5)} &\iff \text{all objects in } \mathsf{FAct}_{T \otimes T} \text{ have an } \mathcal{IC}\text{-cover} \\
\text{(by Proposition 10.6)} &\iff \text{all objects in } \mathsf{FAct}_T \text{ have an } \mathcal{IC}\text{-cover} \\
\text{(by Proposition 10.8)} &\iff \text{all objects in } \mathsf{UAct}_T \text{ have an } \mathcal{IC}\text{-cover} \\
&\iff T \text{ is right } \mathcal{IC}\text{-perfect.}
\end{aligned}$$

□

Corollary 10.10. *Right perfectness and perfectness are Morita invariants for factorizable semigroups.*

Proof. Let S and T be Morita equivalent factorizable semigroups. Then FAct_S and FAct_T are equivalent categories. By Theorem 10.7, Theorem 10.9 and Theorem 9.1, it follows that right perfectness is a Morita invariant.

From Remark 4.12 in [16] we know that also ${}_S\mathsf{FAct}$ and ${}_T\mathsf{FAct}$ are equivalent categories. If now S is left perfect, then the duals of the proofs of Theorem 10.7 and Theorem 10.9 yield that T is also left perfect. It follows that perfectness is a Morita invariant. □

We saw that completely simple semigroups are perfect. This fact can also be concluded from Morita invariance of perfectness in case of factorizable semigroups.

Corollary 10.11. *Completely simple semigroups are perfect.*

https://doi.org/10.1017/S0013091523000159 Published online by Cambridge University Press

Proof. Clearly, groups are perfect semigroups. Factorizable semigroups Morita equivalent to a given group are precisely Rees matrix semigroups over that group [19]. □

Acknowledgement. We would like to thank the anonymous referee for careful reading and numerous useful suggestions.

Funding. This work was partially supported by the Estonian Research Council grant PRG1204.

Competing interests. None.

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