Synthesis, Sensibility, and Kant's Philosophy of Mathematics[1]

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Kant's philosophy of mathematics presents two fundamental problems of interpretation: (1) Kant claims that mathematical truths or "judgments" are synthetic a priori; and (2) Kant maintains that intuition is required for generating and/or understanding mathematical statements. Both of these problems arise for us because of developments in mathematics since Kant. In particular, the axiomatization of geometry--Kant's paradigm of mathematical thinking--has made it seem to some commentators as, for example, Russell, that both (1) and (2) are false (Russell 1919, p. 145).[2] If virtually all of mathematics, including geometry, is axiomatizable, it would seem that mathematics results in analytic judgments that are totally independent of sensibility, the source, according to Kant, of intuition. In this paper I will address both of these difficulties. I shall argue that Kant's understanding of both "synthetic" and "intuition" make his position immune to these criticisms.

In Section 1, I argue that Kant's analytic/synthetic distinction depends largely upon a conception of the nature of concepts, going back to Aristotle, which holds that all concepts can be arranged in genus/species hierarchies, with higher or genus concepts being described as "contained in" lower or species concepts. In other words, the so-called "containment metaphor", in terms of which Kant presents his analytic/synthetic distinction, is really not a metaphor at all but rather is a quasi-technical and an historically well-established way of describing one kind of relationship between concepts, a way that leads Kant to define an analytic judgment as one whose predicate is "contained in", meaning "is a genus concept of", its subject and a synthetic judgment as one whose predicate is not so-contained or is not genus/species related to the subject. I will argue that given this notion of concepts, Kant's characterization of mathematical judgments as "synthetic" seems quite reasonable.

Section 2 takes up the difficulty raised by Kant's tying mathematics to sensibility. I shall begin by considering his distinction between intuitions and concepts. I will argue that this is based upon a logical division in the concept 'representation' and that Kant's association of intuition with sensibility has little to do with establishing an

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association between intuition and "the senses". Rather, Kant is better interpreted as claiming that singular representations or "intuitions" must be presented or "given", at least to human cognizers, because these representations are of part/whole relationships whose character cannot be obtained, in the first instance, through description. Given this background, I shall explain Kant's claim that mathematics consists of synthetic, a priori judgments that are discovered through the construction of its concepts.

1. The Analytic/Synthetic Distinction

In the Logic Kant defines a concept as a "general" or "reflected" representation, "a [re]presentation of what is common to several objects" (1800, p. 96). The Critique calls concepts "mediate" representations, representations that relate to objects by means of characteristics [Merkmal] that may be common to different objects (Kant 1787, A320/ B377). It would seem, then, that in Kant's view, it is the nature of concepts to "gather up" other concepts. Furthermore, it is this gathering that makes concepts "characteristics" or "grounds of cognition" (Kant 1800, pp. 63-64; 101). In other words, Kant's notion of concepts seems to be that they operate by serving as functions or "unifiers" that combine representations of objects by being themselves representations that are common "marks" or characteristics found in all representations to which the concept in question applies.

In addition, concepts exhibit hierarchical relations: Every concept has both an "extension" [Umfange] consisting of concepts described as "under" the concept and giving the concept its universality, and an "intension" [Inhalt], a "partial concept. . . contained in the [re]presentation of things" (Kant 1800, p. 101). The latter makes a concept "determinate" or specific (Kant 1787, A654-55/B682-83; See also Kant 1800, pp. 101-102). Kant uses the terms 'higher' and 'lower,' as well as 'genus' and 'species,' to describe these relations. "Higher" or "genus" concepts are those with a greater extension and a lesser content or intension. Being "higher," fewer concepts are in them, although they are applicable to a greater number of lower representations. By contrast, "lower" or "species" concepts have a greater content--they fall under more genus concepts--but a smaller extension or "sphere". Lower concepts can be applied to fewer representations than can higher concepts. Higher concepts are thus what Kant calls "partial concepts" of lower, subordinated concepts. But Kant notes these descriptions are relative: The same concept, for example "mammal", is subordinate to the concept "animal" but is still contained in the concept "horse" (Kant 1800, pp. 102-103). Thus species concepts are described as "subordinate" to the whole or higher concept while the whole is described as "in" the parts.[3]

A similar description of this kind of relationship between concepts can be found as far back as Aristotle. And I would submit that it also explains what are sometimes taken to be obscure references to "containment" as found in other writers of the modern period such as Descartes, Spinoza, and Leibniz.

In Chapter 5 of the Categories, Aristotle describes the category of "substance". He notes that there are two kinds of substances, namely "primary substance" which refers to individuals and "secondary

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substances" which refer to both genera and species (2a11-14). Chapter 2 of the Categories has already explained that "in a subject" does not refer to the kind of "in" which parts bear to wholes but rather means "cannot exist separately from what it is in", that is, from the subject (1a23-25). What this seems to mean, as the subsequent passage suggests, is that all other terms are either predicated of primary substances or are what Kant would call "partial" or "higher" concepts of such predicates. In other words, the reference of all other terms depends upon primary substance. And the significance of these passages for this paper is that here we find Aristotle using the term 'in' in a way that does not connote the sense of 'contained in' as one would expect between parts and wholes or between synonyms and/or partial synonyms. Instead, 'in' means something like "cannot exist without", as the term 'mammal' could not refer were it not for the existence of all the various types of mammals. Thus, for Aristotle, 'mammal' is "in" or "contained in" all of the various species of mammals, and all of the species of mammals, such as 'horse', 'monkey', and 'cow' are "in", or derive their reference from the existence of, all of the individuals that are members of the various species of the class "mammal".[4]

This much is sufficient to show just how closely Kant's theory of concepts draws upon Aristotle. For Kant, as for Aristotle's secondary substances, concepts can always be arranged in hierarchies. Furthermore, those concepts toward the top of the hierarchy have "less content" or a smaller intension, although they have a greater extension than those concepts toward the bottom of the hierarchy which have a smaller extension and a greater intension or content. In addition, anything that is "said of" or "predicated of" a species is also predicated of the corresponding genus. I submit that Aristotle's notion of "in" or "contained in" is closely tied to Kant's sense of "analytic" while what is "synthetic", in Kant's terminology, is similar, but also somewhat distinct from, being "said of" in Aristotle. Let me explain.

When Kant says that in an analytic judgment "the predicate is contained in the subject" he is referring to this conception, derived from Aristotle, of concepts as generalizations or abstractions from either "lower" concepts or intuitions (i.e., representations of primary substances). Using categorical judgment as his model, Kant characterizes an analytic judgment as one whose predicate is what he calls a "partial concept" of the subject. The judgment is obtained by analyzing (in Kant's sense) the subject. That is to say, one establishes the judgment by showing that the predicate is a higher concept, or a genus, of the lower, subject concept. Thus 'Gold is a yellow metal' is an analytic judgment (Kant 1783, p. 14) because the concept 'yellow metal'--a genus and differentia--is contained as a partial concept and as a "ground of cognition" in the concept 'gold'. Kant is not suggesting that 'gold is a yellow metal' is an a priori logical truth but rather is analytic a posteriori.[5] He is saying that our (empirical) concept of gold is subordinate to the concept 'yellow metal'. Kant describes this relation by saying 'yellow metal' is "in" 'gold' in a way not unlike Aristotle's description of a species, in this case, 'gold' as "containing" a genus. In addition, analytic judgments are "known through the Principle of Non-Contradiction" because if the judgment is true, it is contradictory to deny the predicate of the subject for that amounts to denying the genus of the species. (See Kant 1787, A151-52/B190-91 and cf. Kant 1800, pp. 117-118.) In other words, Kant is not claiming that analytic judgments are "known through identity" because some inconsistency regarding synonyms is involved in denying the judgment. Rather the inconsistency arises from disregarding this hierarchical arrangement of concepts, a hierarchy that anyone who understands the concepts in question would share.[6]

By contrast, a synthetic categorical judgment is one where the predicate is not a partial concept of the subject. It is a judgment in which the predicate is "said of" the subject rather than being "in" it, to use Aristotle's terminology. For Kant, a synthetic judgment derives either from an empirically based conjunction of the subject with the predicate concept, as Kant claims for the concepts 'body' and 'weight' in the judgment 'All bodies are heavy', or because some rational fabrication requires their union. 'A straight line is the shortest distance between two points' is synthetic, in Kant's view, not because when we "intuit" a priori a straight line, we "see" that it is always the shortest distance between two points. The latter is not abstracted from the former. Instead, the judgment results from the formalization and fabrication or synthesis that is mathematics.

Thus although it is correct in some sense to say, on the one hand, that for Kant an analytic judgment is true in virtue of the so-called "meaning" of its terms, Kant is referring not to some relation of synonymy between the terms of the judgment but rather to the hierarchical arrangement which he views as obtaining between its concepts.[7] 'Analytic judgment' is also associated with the process for discerning this arrangement. In this, one is said to "separate" the characteristics of the subject. On the other hand, one could likewise call synthetic judgments "true" on account of their so-called "meaning", not because the predicate bears a genus relation to the subject but because these judgments issue from synthesis. 'Bodies are heavy' or 'Two lines cannot enclose a space' both illustrate synthetic judgments because their subject concepts do not "contain" their predicate concepts.[8] In all of these cases, the description "analytic" or "synthetic" derives from the procedure followed in formulating the judgment and not from an inspection of "meanings".

2. The Nature of Kantian Sensibility

Section 1 has argued that there is an understanding of 'synthetic' that is independent of 'contingent' by showing that Kant's sense of 'analytic' is not intended to convey a sense of logical a priority or "truth by reduction to synonyms". If this is the case, then this is sufficient to disarm one of the standard criticisms leveled against Kant's description of mathematics as "synthetic", namely that the axiomatization of mathematics shows that mathematics is "analytic".[9] Although mathematics might be "analytic" if one assumes, say, Quine's sense of 'analytic', Kant's analytic/synthetic distinction is distinct from Quine's and leaves room for a description of mathematics as "synthetic". When Kant claims that mathematics is "synthetic" what he intends to say, I think, is that mathematics proceeds in terms of a process of rational fabrication where the subjects and predicates of the resulting propositions of the science are not genus/species related.

But there remains the second major difficulty with explaining Kant's philosophy of mathematics, namely, Kant's view that mathematics requires what he calls "intuition" and hence is associated with "sensibility". I shall now address this second problem. I shall argue that the intuition/concept distinction arises from the application of what Kant calls "dichotomous logical division" to the concept "representation". Kantian intuitions, I want to claim, are representations of Aristotelian primary substances or individuals. They have a fundamentally different structure from Kantian concepts which, as we have seen, exhibit hierarchical or genus/species relations. I shall argue that intuitions or singular representations exhibit a mereological or part/whole structure and that the standard association of intuitions with the output of what the empiricist calls "the senses" misdirects Kant's philosophy of both experience and mathematics. In other words, Kant postulates "sensibility" as a "faculty" or "ability" not to account for the output the senses (although the senses do fit into sensibility) but rather to serve as the source of these radically different type of representations, and to contrast this with what might be termed "intelligibility"--that is, understanding, judgment, and reason--which is the source, for Kant, of the conceptual component of experience.

Kant describes "logical division" as a method for clarifying concepts (1800, pp. 147-148). It is a technique for helping one find the species which belong to a given genus. In logical division the understanding starts with a given concept and, by finding its species, gathers a sphere for that concept which, in turn, becomes a "characteristic" of those collected species. Since logical division enables the understanding to bring together various concepts while yet noting their respective differences, this technique describes one way the understanding gives concepts their universality.

Kant goes on to describe two types of logical division. The first, called "dichotomy", is said to be an a priori division of a concept which results in two members, while the second, termed "polytomy", involves the empirical division of a concept and always yields more than two members:

> A division into two members is called dichotomy; if, however, it has more than two members, it is called polytomy. Note 1. All polytomy is empirical; dichotomy is the only division out of principles a priori--thus the only primary division. For the members of the division shall be opposed to one another, and the opposite of every A is indeed nothing more than a non A. Note 2. Polytomy cannot be taught in logic, for cognition of the object belongs to it. Dichotomy needs only the proposition of contradiction, without knowing the concept one wants to divide as to content. Polytomy needs intuition, either a priori, as in mathematics (e.g., the division of conic sections), or empirical intuition, in describing nature. . . . (1800, p. 148).

Hence, dichotomy is the a priori division of a concept that can be performed by the understanding alone, providing the understanding knows at least one way to "divide" the concept. Polytomy, by contrast, requires both intuition and concepts, that is, "cognition of the object", to be performed, either a priori intuition, as in mathematics, or empirical intuition, as in the natural sciences. I would like to suggest that dichotomy is really the source of the Kantian distinction between intuitions and concepts, although experience is, of course, required to give that division content.

Let us consider again Kant's description of concepts and contrast it with his description of intuitions. Recall firstly that "representation" is Kant's most general term for so-called "mental contents"; as the preceding passage just indicated, Kant can postulate "representation" without knowing precisely either its content or its extension. Logical division provides a way of making a concept such as 'representation' more precise by determining its extension. Recall secondly that Kant associates intuitions with immediacy, receptivity, and singularity while concepts are associated with mediacy, spontaneity, and generality. He says that intuitions are singular representations that are "given" to us while concepts are general representations that are "made". Concepts, Kant states, are "reflected" representations "of what is common to several objects" (1800, p. 96). However, "singularity" is just another way of saying "nongeneral". Likewise the contradictories of "spontaneity" and "mediacy" are "nonspontaneity" or "receptivity" and "nonmediacy" or "immediacy". Thus it would seem that rather than arising from some kind of introspection on the nature of experience, the intuition/concept distinction in Kant comes from logical considerations as opposed to experiential ones. Nevertheless, logic cannot justify this a priori division of 'representation' as a real division. To do the latter we must consider Kant's cognitive or epistemological, as well as his metaphysical, reasons for this division.

Kant's cognitive reasons for maintaining the intuition/concept distinction are presented primarily in the "Transcendental Aesthetic", and in those passages of the Inaugural Dissertation and the Prolegomena which correspond to the Transcendental Aesthetic. In this first part of the "Doctrine of Elements", Kant describes the nature of intuitions by describing the nature of space and of time. These are always associated with individuality and are said to be differentiated on the basis of boundaries or limits introduced into a prior whole (1787, A25/B39; A32-33/B47-48). With respect to time, Kant says that its representation cannot be a conceptual one because concepts contain "partial representations" (1787, A/32/B48). But space and time have no such intension or "characteristics" that enable their application.[10] Different times and different spaces can only be conceived as parts of the whole of space and time, parts that are distinguished on the basis of limitation (1787, A/25/B39; A31-32/B46-47). As Kirk Dallas Wilson (1975) points out, Kant is here differentiating intuitions from concepts on the basis of patterns of organization. By emphasizing that intuitions are distinguished through the introduction of limits, Kant is, in effect, describing them as the source of mereological relations. Intuitions are wholes that can be divided into parts only by discerning or adding boundaries within them. Thus, Kant has a reason for distinguishing intuitions and concepts that derives from a cognitive difference between them. But Kant also has metaphysical reasons for maintaining this distinction. To see these, we must consider his reaction against Leibniz.

As Part 1 of the "General Observations on Transcendental Aesthetic" and "The Amphiboly of the Concepts of Reflection" make clear, Kant thought that the fundamental error of the Leibnizian system was its assimilation of what he calls "sensible representation" to intellectual

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or conceptual representation. If all representations are conceptual, all differences must be conceptual differences (1787, A270/B326) and individuals must be identified with essences, ("*infima species*") as Leibniz held. They cannot be identified in the primary instance with individual substances, as we saw Aristotle wanting to maintain. But if individuals are essences, the phenomenal world--the world of experimental science and the world that was the central concern of Aristotle--is illusory. The way out, Kant thought, was to postulate two kinds of representations and to establish an absolute chasm between them. In the *Logic* it is clear that Kant conceives just such a chasm to obtain between intuitions and concepts.

Kantian concepts, as we've seen, are mediate and general representations that are "in" as a "ground of cognition", a plurality of other representations. Kant declares that there is a "highest" or most abstract concept, a concept that is never a species (1800, p. 103). This is the concept "something" (1800, p. 101). However, he is adamant, in contrast to both Aristotle and Leibniz, there is no lowest or "singular concept" (1800, p. 103-104), what those writers call an *infima species*. The reason for this is that a concept, by definition, must be general: It must be possible for it to be "contained in" or "abstracted from" other representations (cf. p. 96). The representation of an individual, by contrast, that is, an intuition, is a representation of a whole made up of parts. The parts are not related to the whole as species in hierarchical relations but rather as proper parts in mereological relations. This means, Kant thinks, that representations of individuals must originate in a distinct "faculty" or "ability". They cannot be the product of understanding because understanding operates through the structure provided by genus/species or hierarchical relations.

We are now ready to understand the connection Kant sees between mathematics and sensibility. Kant defines mathematical cognition as "the cognition of reason from the construction of concepts" (1787, A713/B741). Kant describes construction as the activity of exhibiting a priori in imagination the nonempirical intuition which corresponds to a concept. Construction must provide a way of establishing a kind or type of mereological relationship without compromising the "universality [*Allgemeinheit*]" or generality of the constructed concept. Mathematical concepts must be able to express in a general way the part/whole relationships which characterize intuitive or singular representations. We can see this if we consider, for example, the nature of a triangle.

What, exactly, is a triangle? I suggest that in the last analysis it is a particular way of organizing part/whole structures. It is something that is defined by introducing into space a particular set of limits or boundaries such that a three-sided figure in one dimension results. Such a representation is a representation of an individual. It is an a priori representation of a particular set of mereological relations. Nevertheless, because imagination has not concerned itself with generating a particular (i.e., completely determined) individual but rather only with the activity of generating a set of part/whole relationships (which are, of course, applicable to more than one individual), the construction of the triangle retains the universality characteristic of concepts.[11] But since, for Kant, all representations of individuals--all representations of part/whole relations--must be

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"given", at least to humans, the concept 'triangle', like all concepts, remains rooted with respect to its content in sensibility. This is not to say that sensibility must provide an image for the concept 'triangle' but rather that imagination, which Kant postulates as the bridge between understanding and sensibility, must be able to generate a "schema", or a procedure for generating an image (1787, A140/B179) for that concept. Thus mathematics, although a priori, must involve sensibility as it develops because sensibility is defined as the source of non-conceptual or mereologically-structured representations.

In conclusion, I have tried to show why Kant characterized mathematics as involving synthetic judgments which are based upon a priori intuition. Mathematical judgments are synthetic, for Kant, because the predicates involved in such judgments do not derive from discerning the genus/ species hierarchy in which all concepts partake. Furthermore, mathematical concepts must derive their content from a priori intuition because they "unify" or generalize over types of mereological relations. These latter must be "given" in sensibility because human understanding, Kant thinks, is constrained to work in terms of the universality or generality associated with the conceptual hierarchy. Thus Kant postulates two sources for human cognition--a conceptual or hierarchical source and an intuitive or mereological source. Mathematical cognition bridges the fundamental chasm between these two cognitive sources by constructing synthetic, a priori concepts of given, part/whole relations.

Notes

[1]I gratefully acknowledge helpful discussion from my colleagues in the Philosophy Department at Ohio University and also from the members of the Philosophy Department at Michigan State University. The comments made by Dr. Cynthia Hampton and Dr. Rhoda Kotzin were especially useful.

[2]Axiomatization _per se_ is not the problem to which Russell and others have pointed but if one devises a system where the primitive terms and axioms are all "logical" as in Russell's _Principia Mathematica_ and if one thinks that (a) no nonlogical terms or principles were smuggled in, and (b) logic is "analytic" in the sense of true by definition, which is the sort of argument Russell makes against Kant, then it might look at first as if mathematics cannot be synthetic. Various commentators, for example, Jaakko Hintikka (1967) and Charles Parsons (1969), have tried to make sense of Kant's claims about the syntheticity of mathematics in what I believe are rather contorted ways. Section 1 of my paper could be read as an attempt to show that we do not need "contortion" to understand why Kant said mathematics is synthetic.

[3]This dual aspect of concepts relates to the first _Critique_'s "logical maxims" of reason by which reason directs the understanding to both seek unity and diversity in cognition. (See Kant 1787, A645-64/B673 92.) In this connection we should recall that understanding, as the source of "spontaneity", makes all concepts.

[4]At _Categories_ 3a7-20, Aristotle argues that primary substance is never present _in_ a subject. What I take this to mean is that subjects

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are always abstractions from individuals or primary substance for Aristotle but individuals are never abstractions from subjects.

[5]The implicit and perhaps somewhat surprising claim that there are analytic a posteriori judgments is based upon Kant's discussion of "definition" in the Logic in which he allows for "given a posteriori" definitions (1800, pp. 141-142).

[6]Needless to say, Kant is not relying on some psychologistic theory of concepts as Bennett (1966), for example, would have us believe.

[7]I say "so-called 'meaning'" because 1) this term has a wide variety of connotations that have been variously analyzed since Frege, and 2) there is no extensive and exact discussion of 'meaning' in Kant so far as I know.

[8]I am not denying that there have been analyses of the analytic/synthetic distinction that would allow the description "synthetic" to be compatible with the axiomatization of arithmetic and geometry. But what I would like to point out is that Kant's writings on logic suggest a straightforward understanding of 'analytic' and 'synthetic' such that axiomatization is not an issue at all.

[9]As mentioned in note 2, others, including Hintikka (1967), Parsons (1969), and J. Michael Young (1982), have presented interpretations that would make intelligible Kant's claim that mathematics is synthetic. However, none of these writers has considered the question by considering Kant's theory of concepts.

[10]Of course insofar as we are able to form concepts of space and of time, these would have intensions and could be arranged in the conceptual hierarchy. However, Kant's point is that the representations which serve as the basis for these concepts are not themselves organizable into hierarchical patterns.

[11]J. Michael Young (1982) offers an extended analysis of the way in which construction relies on the procedures or "schema" generated by imagination.

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