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# POLAR DECOMPOSITION OF THE $k$-FOLD PRODUCT OF LEBESGUE MEASURE ON $\mathbb{R}^n$

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**Abstract**

The Blaschke–Petkantschin formula is a geometric measure decomposition of the $q$-fold product of Lebesgue measure on $\mathbb{R}^n$. Here we discuss another decomposition called polar decomposition by considering $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ as $\mathcal{M}_{n\times k}$ and using its polar decomposition. This is a generalisation of the Blaschke–Petkantschin formula and may be useful when one needs to integrate a function $g : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ with rotational symmetry, that is, for each orthogonal transformation $O$, $g(O(x_1), \ldots, O(x_k)) = g(x_1, \ldots x_k)$. As an application we compute the moments of a Gaussian determinant.

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## 1. Introduction

Usually we need to integrate a function which has some symmetries and these symmetries help us to compute the integration more easily or to obtain some qualitative properties about the result. For instance, we use the well-known polar integral formula for functions with radial symmetry. Another example is integration of a function $g : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ invariant under rotations, that is, for each orthogonal transformation $O$, we have $g(x_1, \ldots, x_k) = g(O(x_1), \ldots, O(x_k))$. If $k \leq n$, by a suitable rotation we may assume that $x_1, \ldots, x_k$ lie on a fixed $k$-dimensional subspace of $\mathbb{R}^n$. Therefore it would be possible to integrate $g$ on this $k$-dimensional subspace rather than $\mathbb{R}^n$. The Blaschke–Petkantschin formula, which is a fundamental relation in stereology, helps us to do that [4, 5]. By this formula, to integrate $g$ on $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ we first integrate it with a special weight on $L \times \cdots \times L$ of all $k$-dimensional subspaces $L$ of $\mathbb{R}^n$ and then integrate the result on the homogeneous space of all $k$-dimensional subspaces. Here we give another relation based on polar decomposition of matrices and prove it by using the co-area formula [1, 2].

A similar problem can be posed for complex spaces: can we compute the integration of $g$ on $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$ by integrating on $\mathbb{C}^k \times \cdots \times \mathbb{C}^k$ when $g : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{R}$ is a rotationally symmetric function, that is, for each unitary transformation $U$ we have $g(x_1, \ldots, x_k) = g(U(x_1), \ldots, U(x_n))$? Note that for integration on $(\mathbb{C}^n)^k$ we consider this space as a real vector space endowed with the standard inner product. If we consider $(\mathbb{C}^n)^k$ as the space of $n \times k$ complex matrices, then the standard inner product on it is given by $\langle A, B\rangle = \mathrm{Re}\,\mathrm{tr}(B^*A)$. The steps for solving this problem are the same as for the real case, so we state our results in both cases, but the proofs will be given only for the complex case.

We also give another version of the polar integral theorem and show how the polar integral theorem implies the Blaschke–Petkantschin formula [4]. We apply this theorem to obtain another way of computing the moments of a Gaussian determinant [3].

Throughout this paper we use capital letters $(A, B, \ldots)$ to denote matrices; calligraphic style $(\mathcal{M}, \mathcal{O}, \ldots)$ to denote sets of matrices; and small letters $(u, v, f, l)$ to denote vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$ and maps between sets of matrices. We also use the following notation.

- $\mathcal{M}_{n\times k} : n \times k$ real matrices with the standard inner product $\langle A, B\rangle = \mathrm{tr}(B^tA)$. ($\mathcal{M}^{\mathbb{C}}_{n\times k}$: $n \times k$ complex matrices with the standard real inner product $\langle A, B\rangle = \mathrm{Re}\,\mathrm{tr}(B^*A)$.)
- $\mathcal{P}_k : k \times k$ symmetric matrices. ($\mathcal{H}_k$: Hermitian matrices.)
- $\mathcal{P}^+_k \subset \mathcal{P}_k$: subset of all matrices with nonnegative eigenvalues. ($\mathcal{H}^+_k \subset \mathcal{H}_k$: subset of all Hermitian matrices with nonnegative eigenvalues.)
- $\mathcal{A}_k : k \times k$ antisymmetric matrices. ($\mathcal{A}^{\mathbb{C}}_k$: anti-Hermitian matrices.)
- $\mathcal{O}(n) : n \times n$ orthogonal matrices. ($\mathcal{U}(n)$: unitary matrices.)

We consider the above sets as Riemannian submanifolds of $\mathcal{M}_{k\times k}$ and $\mathcal{M}^{\mathbb{C}}_{k\times k}$.

## 2. Polar integral theorem

Since $g$ is invariant under rotations of $\mathbb{R}^n$, we should first obtain a condition on $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ which guarantees the existence of an orthogonal transformation $O \in \mathcal{O}(n)$ such that $O(x_i) = y_i$ for all $i$. This condition is $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle$ for all $i$ and $j$. By considering $n \times k$ matrices $X = [x_1|\cdots|x_k]$ and $Y = [y_1|\cdots|y_k]$, this condition would be $X^tX = Y^tY$. In this terminology we may consider $g$ as a function on $n \times k$ matrices. Therefore $g$ is invariant under rotations of $\mathbb{R}^n$ if and only if it is constant on the level sets of $f : \mathcal{M}_{n\times k} \to \mathcal{P}_k$; $f(X) = X^tX$. Thus in order to integrate $g$ on $\mathcal{M}_{n\times k}$ it is convenient to first integrate it on the level sets of $f$. This can be done by the co-area formula which is a generalisation of Fubini's theorem [1, 2].

By the co-area formula, if $n \geq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function, then for any smooth function $g : R^n \to \mathbb{R}$,

$$\int_{x\in\mathbb{R}^n} g(x)J_f(x)\,dx = \int_{y\in\mathbb{R}^m}\left(\int_{f^{-1}(y)} g\,d\mu_{n-m}\right)dy,$$

where $J_f(x) = \sqrt{\det(D_xf \circ (D_xf)^*)}$ and $d\mu_{n-m}$ is the $(n-m)$-dimensional Hausdorff measure in $\mathbb{R}^n$. Note that when a surface $S \subset \mathbb{R}^n$ is an $l$-dimensional submanifold of $\mathbb{R}^n$, then the $l$-dimensional Hausdorff measure on it is the natural measure of $S$ as a Riemannian submanifold of $\mathbb{R}^n$.

In our problem $f : \mathcal{M}_{n\times k} \to \mathcal{P}_k$; $f(X) = X^t X$. Therefore,

$$\begin{aligned}\int_{X\in\mathcal{M}_{n\times k}} g(X)J_f(X)\,dX &= \int_{P\in\mathcal{P}_k}\int_{f^{-1}(P)} g\,d\mu\,dP\\ &= \int_{P\in\mathcal{P}_k} \mathrm{vol}(f^{-1}(P))g(f^{-1}(P))\,dP.\end{aligned}$$

We need only compute $J_f(X)$ and $\mathrm{vol}(f^{-1}(P))$. In the complex case $f$ should be defined by $f(X) = X^* X$.

Lemma 2.1. *Let $f : \mathcal{M}^{\mathbb{C}}_{n\times k} \to \mathcal{H}_k$ be defined by $f(X) = X^*X$. If $\lambda_1, \ldots, \lambda_k$ are eigenvalues of $P = f(X)$ then*

$$J_f(X)^2 = \det(D_X f \circ (D_X f)^*) = 2^{k(k+1)}\det(P)\prod_{i<j}(\lambda_i+\lambda_j)^2.$$

*In the real case where $f : \mathcal{M}_{n\times k} \to \mathcal{P}_k$; $f(X) = X^t X$, if $\lambda_1, \ldots, \lambda_k$ are eigenvalues of $P = f(X)$ then $J_f(X)^2 = 2^{k(k+1)/2}\prod_{i\le j}(\lambda_i+\lambda_j)$.*

Proof. We observe that

$$\begin{aligned} D_X f(A) &= A^*X + X^*A,\\ \langle (D_X f)^*(\tilde{A}), A\rangle &= \langle \tilde{A}, D_X f(A)\rangle = \langle \tilde{A}, A^*X + X^*A\rangle\\ &= \mathrm{Re}\,\mathrm{tr}(\tilde{A}A^*X + \tilde{A}X^*A) = \mathrm{Re}\,\mathrm{tr}(A^*X\tilde{A} + A^*X\tilde{A}^*)\\ &= 2\,\mathrm{Re}\,\mathrm{tr}(A^*X\tilde{A}) = \langle 2X\tilde{A}, A\rangle.\end{aligned}$$

Let $l_X = D_X f \circ (D_X f)^* : \mathcal{H}_k \to \mathcal{H}_k$. We have $l_X(\tilde{A}) = 2(\tilde{A}P + P\tilde{A})$. If $v_1, \ldots, v_k$ are orthonormal eigenvectors of $P$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_\kappa$, then each of the Hermitian operators $\tilde{A}^{i,j,+} : \mathbb{C}^k \to \mathbb{C}^k$ $(i \le j)$ and $\tilde{A}^{i,j,-} : \mathbb{C}^k \to \mathbb{C}^k$ $(i < j)$ defined by

$$\begin{aligned}&\tilde{A}^{i,j,+}(v_i) = v_j, \quad \tilde{A}^{i,j,+}(v_j) = v_i, \quad \tilde{A}^{i,j,+}(v_s) = \mathbf{0} \quad (s \ne i, j);\\ &\tilde{A}^{i,j,-}(v_i) = iv_j, \quad \tilde{A}^{i,j,-}(v_j) = -iv_i, \quad \tilde{A}^{i,j,-}(v_s) = \mathbf{0} \quad (s \ne i, j),\end{aligned}$$

is an eigenvector of $l_X$ corresponding to the eigenvalue $2(\lambda_i + \lambda_j)$, because

$$\begin{aligned} l_X(\tilde{A}^{i,j,+})(v_i) &= 2[P\tilde{A}^{i,j,+} + \tilde{A}^{i,j,+}P](v_i) = 2(\lambda_j+\lambda_i)v_j = 2(\lambda_j+\lambda_i)\tilde{A}^{i,j,+}(v_i),\\ l_X(\tilde{A}^{i,j,+})(v_j) &= 2[P\tilde{A}^{i,j,+} + \tilde{A}^{i,j,+}P](v_j) = 2(\lambda_i+\lambda_j)v_i = 2(\lambda_i+\lambda_j)\tilde{A}^{i,j,+}(v_j),\\ l_X(\tilde{A}^{i,j,+})(v_s) &= 2[P\tilde{A}^{i,j,+} + \tilde{A}^{i,j,+}P](v_s) = \mathbf{0} = 2(\lambda_i+\lambda_j)\tilde{A}^{i,j,+}(v_s);\\ l_X(\tilde{A}^{i,j,-})(v_i) &= 2[P\tilde{A}^{i,j,-} + \tilde{A}^{i,j,-}P](v_i) = 2i(\lambda_j+\lambda_i)v_j = 2(\lambda_j+\lambda_i)\tilde{A}^{i,j,-}(v_i),\\ l_X(\tilde{A}^{i,j,-})(v_j) &= 2[P\tilde{A}^{i,j,-} + \tilde{A}^{i,j,-}P](v_j) = -2i(\lambda_i+\lambda_j)v_i = 2(\lambda_i+\lambda_j)\tilde{A}^{i,j,-}(v_j),\\ l_X(\tilde{A}^{i,j,-})(v_s) &= 2[P\tilde{A}^{i,j,-} + \tilde{A}^{i,j,-}P](v_s) = \mathbf{0} = 2(\lambda_i+\lambda_j)\tilde{A}^{i,j,-}(v_s).\end{aligned}$$

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These are $k^2$ linearly independent Hermitian operators, so they make a complete set of eigenvectors of $l_X$. Therefore,

$$\det(l_X) = 2^{k^2} \prod_{i \le j} (\lambda_i + \lambda_j) \prod_{i<j} (\lambda_i + \lambda_j) = 2^{k(k+1)} \det(P) \prod_{i<j} (\lambda_i + \lambda_j)^2.$$

This completes the proof. □

Now we need to study the level sets of $f$ and their metrics and measures. Since for every $U \in \mathcal{U}$, $\langle UA, UB\rangle = \operatorname{Re}\operatorname{tr}(A^*U^*UB) = \operatorname{Re}\operatorname{tr}(A^*B) = \langle A, B\rangle$, the left action of $\mathcal{U}(n)$ on $\mathcal{M}^{\mathbb{C}}_{n\times k}$ preserves the inner product and consequently the induced metric on each orbit is invariant under this action. Let $\mathcal{S}_1 = f^{-1}(I_k) \subset \mathcal{M}^{\mathbb{C}}_{n\times k}$, $\tilde{I} = \left[\begin{smallmatrix} I_k \\ \mathbf{0} \end{smallmatrix}\right] \in \mathcal{S}_1$ and define the linear map $l : \mathcal{M}^{\mathbb{C}}_{n\times n} \to \mathcal{M}^{\mathbb{C}}_{n\times k}$ by $l(Z) = Z\tilde{I}$. Clearly $l$ is surjective, $l(\mathcal{U}(n)) = \mathcal{S}_1$ and $l(I_n) = \tilde{I}$. Therefore,

$$\begin{aligned} \mathcal{V}_1 &= \mathrm{T}_{\tilde{I}}\, \mathcal{S}_1 = l(\mathrm{T}_{I_n} \mathcal{U}(n)) = l(\mathcal{A}^{\mathbb{C}}_n) \\ &= \left\{ \begin{bmatrix} A_1 \\ B \end{bmatrix} : A_1 \in \mathcal{A}^{\mathbb{C}}_k, B \in \mathcal{M}^{\mathbb{C}}_{(n-k)\times k} \right\}. \end{aligned}$$

For each $P \in \mathcal{H}^+_k$ there is a simple relation between $\mathcal{S}_1$ and $\mathcal{S}_P = f^{-1}(P)$. Suppose that $Q = \sqrt{P}$. The restriction of the isomorphism $h : \mathcal{M}^{\mathbb{C}}_{n\times k} \to \mathcal{M}^{\mathbb{C}}_{n\times k}$; $h(X) = XQ$ to $\mathcal{S}_1$ is a diffeomorphism from $\mathcal{S}_1$ to $\mathcal{S}_P$ which commutes with the actions of $\mathcal{U}(n)$ on $\mathcal{S}_1$ and $\mathcal{S}_P$, that is, $h(UX) = Uh(X)$.

Lemma 2.2. *If $\lambda_1, \ldots, \lambda_k$ are eigenvalues of $P \in \mathcal{H}^+_k$ and $\omega_1$ and $\omega_P$ are the measures on $\mathcal{S}_1$ and $\mathcal{S}_P$ corresponding to their induced metrics, then $h^*(\omega_P) = C_p\omega_1$ where $C_P$ is a constant that depends only on $P$. Moreover,*

$$C_P^2 = 2^{-k(k-1)} \det(P)^{2(n-k)+1} \prod_{i<j} (\lambda_i + \lambda_j)^2.$$

*In the real case where $P \in \mathcal{P}_k$,*

$$C_P^2 = 2^{-k(k+1)/2} \det(P)^{n-k-1} \prod_{i \leqslant j} (\lambda_i + \lambda_j).$$

Proof. Suppose that $\langle \cdot, \cdot \rangle_P$ is the pullback of the induced metric on $\mathcal{S}_P$ by $h$. Since $h$ commutes with the action of $\mathcal{U}(n)$ on $\mathcal{S}_1$ and $\mathcal{S}_P$, $\langle \cdot, \cdot \rangle_P$ and its corresponding measure $h^*(\omega_P)$ are also invariant under this action. Because this action is transitive each invariant measure should be a constant multiple of $\omega_1$. Thus $h^*(\omega_P) = C_p\omega_1$ where $C_P$ is constant. To compute $C_P$ we compare $\omega_1$ and $h^*(\omega_P)$ at the tangent space of $\tilde{I}$. For this reason we find a base for this space which is orthogonal with respect to both metrics. Note that for each $U, V \in \mathcal{V}_1 = \mathrm{T}_{\tilde{I}}\, \mathcal{S}_1$:

$$\begin{aligned} \langle U, V\rangle_P &= \langle h(U), h(V)\rangle = \langle UQ, VQ\rangle \\ &= \operatorname{Re}\operatorname{tr}(VQQ^tU^t) = \operatorname{Re}\operatorname{tr}(VPU^t) = \langle U, VP\rangle \\ &= \langle U, \operatorname{Proj}_{\mathcal{V}_1}(VP)\rangle. \end{aligned}$$

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The eigenvectors of the positive definite map $\mathbf{r}:\mathcal{V}_1\to\mathcal{V}_1$ defined by $\mathbf{r}(V)=\mathrm{Proj}_{\mathcal{V}_1}(VP)$ are orthogonal with respect to both metrics $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_P$. Therefore $C_P^2$ is equal to the product of the eigenvalues of $\mathbf{r}$. By definition of $\mathbf{r}$:

$$\forall \begin{bmatrix}A\\B\end{bmatrix}\in\mathcal{V}_1 : \mathbf{r}\left(\begin{bmatrix}A\\B\end{bmatrix}\right)=\mathrm{Proj}_{\mathcal{V}_1}\left(\begin{bmatrix}AP\\BP\end{bmatrix}\right)=\begin{bmatrix}\dfrac{AP+PA}{2}\\BP\end{bmatrix}.$$

Thus $\mathbf{r}$ is the direct sum of two operators $\mathbf{r}_1:\mathcal{A}_k\to\mathcal{A}_k$, $\mathbf{r}_1(A)=(PA+AP)/2$, and $\mathbf{r}_2:\mathcal{M}^{\mathbb{C}}_{(n-k)\times k}\to\mathcal{M}^{\mathbb{C}}_{(n-k)\times k}$, $\mathbf{r}_2(B)=BP$. If $v_1,\ldots,v_k$ are orthonormal eigenvectors of $P$ corresponding to eigenvalues $\lambda_1,\ldots,\lambda_\kappa$ , then, as in Lemma 2.1, each of the anti-Hermitian operators $\tilde{A}^{i,j,+}$ $(i\le j)$, $\tilde{A}^{i,j,-}$ $(i<j)$: $\mathbb{C}^k\to\mathbb{C}^k$ defined by

$$\begin{aligned}&\tilde{A}^{i,j,+}(v_i)=iv_j,\quad \tilde{A}^{i,j,+}(v_j)=iv_i,\quad \tilde{A}^{i,j,+}(v_s)=\mathbf{0}\quad (s\ne i,j),\\ &\tilde{A}^{i,j,-}(v_i)=v_j,\quad \tilde{A}^{i,j,-}(v_j)=-v_i,\quad \tilde{A}^{i,j,-}(v_s)=\mathbf{0}\quad (s\ne i,j),\end{aligned}$$

is an eigenvector of $\mathbf{r}_1$ corresponding to the eigenvalue $(\lambda_i+\lambda_j)/2$. Eigenvectors of $\mathbf{r}_2$ are the maps $\tilde{B}^{i,l,\pm}:\mathbb{C}^k\to\mathbb{C}^{n-k}$ which are defined by $\tilde{B}^{i,l,\pm}(v_i)=i^{(1\pm1)/2}e_l$, $\tilde{B}^{i,l,\pm}(v_j)=0$, and the eigenvalue corresponding to $\tilde{B}^{i,l,\pm}$ is $\lambda_i$. Therefore,

$$C_P^2=2^{-k(k-1)}\det(P)^{2(n-k)+1}\prod_{i<j}(\lambda_i+\lambda_j)^2.$$

This completes the proof. □

Consider again the map $l:\mathcal{U}(n)\to\mathcal{S}_1$ defined by $l(U)=U\tilde{I}$. One can apply the co-area formula for this map to convert the integration on $\mathcal{S}_1$ to integration on $\mathcal{U}(n)$. Note that the level sets of $l$ are the left cosets of $\mathcal{G}$ where $\mathcal{G}=l^{-1}(\tilde{I})$ is the stabiliser of $\tilde{I}$:

$$\mathcal{G}=\begin{bmatrix}I_k & \mathbf{0}\\ \mathbf{0} & \mathcal{U}(n-k)\end{bmatrix},$$

and therefore its volume as a Riemannian submanifold of $\mathcal{U}(n)$ is equal to $\mathrm{vol}(\mathcal{U}(n-k))$.

Lemma 2.3. *Suppose that $\omega$ is the measure on $\mathcal{U}(n)$ corresponding to the induced metric on $\mathcal{U}(n)$ as a Riemannian submanifold of $\mathcal{M}^{\mathbb{C}}_{n\times n}$. For each function $g:\mathcal{S}_1\to\mathbb{R}$,*

$$\int_{\mathcal{S}_1} g(X)\omega_1=\frac{2^{-k(n-k)}}{\mathrm{vol}(\mathcal{U}(n-k))}\int_{\mathcal{U}(n)} g(l(U))\omega.$$

*In the real case, if $\omega$ is the standard measure on $O(n)$, then*

$$\int_{\mathcal{S}_1} g(X)\omega_1=\frac{2^{-k(n-k)/2}}{\mathrm{vol}(O(n-k))}\int_{O(n)} g(l(O))\omega.$$

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Proof. By the co-area formula,

$$\int_{\mathcal{U}(n)} g(l(U))J_l(U)\omega = \int_{\mathcal{S}_1}\left(\int_{l^{-1}(X)} g(X)\omega_X\right)\omega_1 = \mathrm{vol}(\mathcal{G})\int_{\mathcal{S}_1} g(X)\omega_1,$$

where $J_l(U)^2 = \det(D_U l \circ (D_U l)^*)$. Since the left action of $\mathcal{U}(n)$ on $\mathcal{U}(n)$ and $\mathcal{S}_1$ preserves their metrics and $l$ commutes with these actions, $J_l$ is constant. We compute it at the point $U = I_n$. Note that $l = D_{I_n} l : T_{I_n}\mathcal{U}(n) = \mathcal{A}_n \to T_{\tilde{I}}\mathcal{S}_1 = \mathcal{V}_1$, so for each $V \in \mathcal{A}_n$ and $W \in \mathcal{V}_1$,

$$\begin{aligned}\langle V, (D_{I_n} l)^*(W)\rangle &= \langle D_{I_n} l(V), W\rangle = \langle V\tilde{I}, W\rangle = \mathrm{Re}\,\mathrm{tr}(W\tilde{I}^*V^*)\\ &= \langle V, W\tilde{I}^*\rangle = \langle V, \mathrm{Proj}_{\mathcal{A}_n}(W\tilde{I}^*)\rangle = \left\langle V, \frac{W\tilde{I}^* - \tilde{I}W^*}{2}\right\rangle.\end{aligned}$$

Therefore, letting $\mathbf{r} = D_{I_n} l \circ (D_{I_n} l)^* : \mathcal{V}_1 \to \mathcal{V}_1$,

$$\forall \begin{bmatrix} A \\ B \end{bmatrix} \in \mathcal{V}_1 : \mathbf{r}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = \begin{bmatrix} A \\ \frac{1}{2}B \end{bmatrix}.$$

Clearly, $\det(\mathbf{r}) = 2^{-2k(n-k)}$. This completes the proof of the lemma. □

Corollary 2.4. *Consider $\mathcal{U}(n)$ and $S^{2n-1} \subset \mathbb{C}^n$ with their standard metrics and define* $\mathrm{vol}(\mathcal{U}(0)) = 1$. *Then*

$$\mathrm{vol}(\mathcal{S}_1) = 2^{-k(n-k)}\frac{\mathrm{vol}(\mathcal{U}(n))}{\mathrm{vol}(\mathcal{U}(n-k))}, \qquad \frac{\mathrm{vol}(\mathcal{U}(n))}{\mathrm{vol}(\mathcal{U}(n-1))} = 2^{n-1}\,\mathrm{vol}(S^{2n-1}).$$

*In the real case, by defining* $\mathrm{vol}(O(1)) = 2$, $\mathrm{vol}(O(0)) = 1$ *and* $\mathrm{vol}(S^0) = 2$,

$$\mathrm{vol}(\mathcal{S}_1) = 2^{-k(n-k)/2}\frac{\mathrm{vol}(O(n))}{\mathrm{vol}(O(n-k))}, \qquad \frac{\mathrm{vol}(O(n))}{\mathrm{vol}(O(n-1))} = 2^{(n-1)/2}\,\mathrm{vol}(S^{n-1}).$$

Proof. The first equation is the direct result of the previous lemma. For the second, let $k = 1$ in the first and note that $\mathcal{M}^{\mathbb{C}}_{n\times 1}$ is $\mathbb{C}^n$ with the standard metric and $\mathcal{S}_1 = S^{2n-1}$. □

Now we have enough tools to prove the polar integral theorem.

Theorem 2.5 (Polar integral theorem). *Denoting the volume of $\mathcal{S}^i$ by $\sigma_i$ and induced measures on $\mathcal{U}(n) \subset \mathcal{M}^{\mathbb{C}}_{n\times n}$ and $\mathcal{S}_1 \subset \mathcal{M}^{\mathbb{C}}_{n\times k}$ by $\omega$ and $\omega_1$, for each integrable function $g : \mathcal{M}^{\mathbb{C}}_{n\times k} \to \mathbb{R}$ and $\psi : \mathcal{H}_k \to \mathcal{H}_k$:*

(1)

$$\begin{aligned}\int_{\mathcal{M}^{\mathbb{C}}_{n\times k}} g(X)\,dX &= 2^{-k^2}\int_{\mathcal{H}_k^+}\left(\int_{\mathcal{S}_1} g(X\sqrt{P})\omega_1\right)\det(P)^{n-k}\,dP\\ &= C_c\int_{\mathcal{H}_k^+}\left(\int_{\mathcal{U}(n)} g(U\tilde{I}\sqrt{P})\omega\right)\det(P)^{n-k}\,dP,\end{aligned}$$

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*where*

$$C_c = \frac{2^{-kn}}{\mathrm{vol}(\mathcal{U}(n-k))} = (2^{(n(n+1)+k(k-1))/2}\sigma_{2(n-k)-1}\cdots\sigma_1)^{-1};$$

(2)

$$\int_{\mathcal{M}^{\mathbb{C}}_{n\times k}} \psi(X^*X)\,dX = 2^{k(k-1)/2}\sigma_{2n-1}\cdots\sigma_{2(n-k)+1}\int_{\mathcal{H}_k^+}\psi(P)\det(P)^{n-k}\,dP$$
$$= \frac{\sigma_{2n-1}\cdots\sigma_{2(n-k)+1}}{\sigma_{2k-1}\cdots\sigma_1}\int_{\mathcal{M}^{\mathbb{C}}_{k\times k}}\psi(Y^*Y)\det(Y^*Y)^{n-k}\,dY.$$

*In the real case, where $\omega$ and $\omega_1$ are measures on $O(n)\subset\mathcal{M}_{n\times n}$ and $\mathcal{S}_1\subset\mathcal{M}_{n\times k}$:*

(1)

$$\int_{\mathcal{M}_{n\times k}} g(X)\,dX = 2^{-k(k+1)/2}\int_{\mathcal{P}_k^+}\left(\int_{\mathcal{S}_1} g(X\sqrt{P})\omega_1\right)\det(P)^{(n-k-1)/2}\,dP$$
$$= C_r\int_{\mathcal{P}_k^+}\left(\int_{O(n)} g(U\tilde{I}\sqrt{P})\omega\right)\det(P)^{(n-k-1)/2}\,dP,$$

*where*

$$C_r = \frac{2^{-k(n+1)/2}}{\mathrm{vol}(O(n-k))} = (2^{(n(n-1)+k(k+3))/4}\sigma_{n-k-1}\cdots\sigma_0)^{-1};$$

(2)

$$\int_{\mathcal{M}_{n\times k}}\psi(X^tX)\,dX = 2^{-k(k+3)/4}\sigma_{n-1}\cdots\sigma_{n-k}\int_{\mathcal{P}_k^+}\psi(P)\det(P)^{(n-k-1)/2}\,dP$$
$$= \frac{\sigma_{n-1}\cdots\sigma_{n-k}}{\sigma_{k-1}\cdots\sigma_0}\int_{\mathcal{M}_{k\times k}}\psi(Y^tY)\det(Y^tY)^{(n-k)/2}\,dY.$$

Proof. Using the above results, by applying the co-area formula to the function $g(X)/J_f(X)$ instead of $g$,

$$\int_{\mathcal{M}^{\mathbb{C}}_{n\times k}} g(X)\,dX = \int_{\mathcal{H}_k^+}\left(\int_{\mathcal{S}_p}\frac{g(X)}{J_f(X)}\omega_P\right)dP$$
$$= \int_{\mathcal{H}_k^+}\left(\int_{\mathcal{S}_1}\frac{g(X\sqrt{P})}{J_f(X\sqrt{P})}C_p\omega_1\right)dP$$
$$= 2^{-k^2}\int_{\mathcal{H}_k^+}\left(\int_{\mathcal{S}_1} g(X\sqrt{P})\det(P)^{n-k}\omega_1\right)dP$$
$$= \frac{2^{-kn}}{\mathrm{vol}(\mathcal{U}(n-k))}\int_{\mathcal{H}_k^+}\left(\int_{\mathcal{U}(n)} g(U\tilde{I}\sqrt{P})\omega\right)\det(P)^{n-k}\,dP.$$

Part (2) is the direct result of part (1). □

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## 3. Some remarks and applications

**3.1. Another version of the polar integral theorem** The polar integral theorem for $k = 1$ says that

$$\int_{\mathbb{R}^n} g(x)\, dx = 2^{-1} \int_{y\in S^{n-1}} \int_{p\in\mathbb{R}^+} g(y\sqrt{p})p^{(n-1)/2}\, dp\omega_1.$$

This is exactly the well-known polar integral but for variable $p = r^2$. By change of variable,

$$2^{-1} \int_{y\in S^{n-1}} \int_{p\in\mathbb{R}^+} g(y\sqrt{p})p^{(n-1)/2}\, dp\omega_1 = \int_{y\in S^{n-1}} \int_{r\in\mathbb{R}^+} g(yr)r^{n-1}\, dr\omega_1.$$

One can write the polar integral theorem for the variable $Q = \sqrt{P}$. If $l : \mathcal{H}_k^+ \to \mathcal{H}_k^+$ is defined by $l(Q) = Q^2$, then $dP = \det(D_Q l)\, dQ$. But $D_Q l(A) = QA + AQ$. Thus, as in the proof of Lemma 2.1,

$$\det(D_Q l) = 2^k \det(Q) \prod_{i<j} (\mu_i + \mu_j)^2,$$

where $\mu_1, \ldots, \mu_k$ are eigenvalues of $Q$. So

$$\begin{aligned}\int_{\mathcal{M}_{n\times k}^{\mathbb{C}}} g(X)\, dX &= 2^{-k^2+k} \int_{\mathcal{H}_k^+} \biggl(\int_{\mathcal{S}_1} g(XQ)\omega_1\biggr) \det(Q)^{2(n-k)+1} \prod_{i<j} (\mu_i + \mu_j)^2\, dQ\\ &= 2^k C_c \int_{\mathcal{H}_k^+} \biggl(\int_{\mathcal{U}(n)} g(U\tilde{I}Q)\omega\biggr) \det(Q)^{2(n-k)+1} \prod_{i<j} (\mu_i + \mu_j)^2\, dQ.\end{aligned}$$

In the real case,

$$\begin{aligned}\int_{\mathcal{M}_{n\times k}} g(X)\, dX &= 2^{-k(k-1)/2} \int_{\mathcal{P}_k^+} \biggl(\int_{\mathcal{S}_1} g(XQ)\omega_1\biggr) \det(Q)^{n-k} \prod_{i<j} (\mu_i + \mu_j)\, dP\\ &= 2^k C_r \int_{\mathcal{P}_k^+} \biggl(\int_{\mathcal{O}(n)} g(U\tilde{I}Q)\omega\biggr) \det(Q)^{n-k} \prod_{i,j} (\mu_i + \mu_j)\, dP.\end{aligned}$$

**3.2. Blaschke–Petkantschin formula** The polar integral theorem in the real case includes the Blaschke–Petkantschin formula as follows. The left action of $\mathcal{O}(n)$ on $\mathcal{S}_1$ is transitive and the right action of $\mathcal{O}(k)$ on $\mathcal{S}_1$ is proper and free. These actions commute with each other and preserve the metric of $\mathcal{S}_1$. Therefore $\mathcal{S}_1/\mathcal{O}(k)$ is a homogeneous space and its induced metric and measure are invariant under the natural action of $\mathcal{O}(n)$. We denote this measure by $\alpha$. This is the unique $\mathcal{O}(n)$-invariant measure on $\mathcal{S}_1/\mathcal{O}(k)$ such that $\mathrm{vol}(\mathcal{S}_1/\mathcal{O}(k)) = \mathrm{vol}(\mathcal{S}_1)/\mathrm{vol}(\mathcal{O}(k))$. Considering each element of $\mathcal{S}_1$ as $k$ orthonormal vectors in $\mathbb{R}^n$, each $\mathcal{O}(k)$-orbit consists of all $k$ orthonormal vectors spanning the same $k$-dimensional subspace of $\mathbb{R}^n$. So $\mathcal{S}_1/\mathcal{O}(k)$ is the space

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of $k$-dimensional subspaces of $\mathbb{R}^n$, which is denoted by $\mathcal{L}_k$. For each $k$-plane $L \in \mathcal{L}_k$, fix an element $S_L \in \mathcal{S}_1$ with columns in $L$. Note that $l : O(k) \to \mathcal{S}_1$, $X \to S_L X$ is a metric-preserving map, because it is a restriction of a metric-preserving linear map $X \to S_L X$ from $\mathcal{M}_{k\times k}$ to $\mathcal{M}_{n\times k}$. Thus

$$\int_{\mathcal{S}_1} g(X)\omega_1 = \int_{L\in\mathcal{L}_k} \left(\int_{O\in O(k)} g(S_L O)\omega\right)\tilde{\alpha},$$

where $\omega$ is the standard measure on $O(k)$ and $\tilde{\alpha}$ is a unique $O(k)$-invariant measure on $\mathcal{L}_k$ such that $\mathrm{vol}(\mathcal{L}_k) = \mathrm{vol}(\mathcal{S}_1)/\mathrm{vol}(O(k))$. Therefore, $\tilde{\alpha} = \alpha$. The polar integral theorem for $n = k$ says that

$$\int_{\mathcal{M}_{k\times k}} \tilde{g}(X)\,dx = 2^{-k(k+1)/2} \int_{O(k)} \left(\int_{\mathcal{P}_k} \tilde{g}(O\sqrt{P})\det(P)^{-1/2}\,dP\right)\omega.$$

Letting $\tilde{g}_L(X) = g(S_L X)\det(X)^{n-k}$ in the above equation,

$$\begin{aligned}
\int_{\mathcal{M}_{n\times k}} g(X)\,dx &= 2^{-k(k+1)/2} \int_{\mathcal{S}_1} \left(\int_{\mathcal{P}_k} g(Y\sqrt{P})\det(P)^{(n-k-1)/2}\,dP\right)\omega_1 \\
&= \int_{L\in\mathcal{L}_k} 2^{-k(k+1)/2} \int_{O\in O(k)} \int_{\mathcal{P}_k} \tilde{g}_L(O\sqrt{P})\det(P)^{-1/2}\,dP\omega\alpha \\
&= \int_{L\in\mathcal{L}_k} \left(\int_{\mathcal{M}_{k\times k}} \tilde{g}_L(X)\,dX\right)\alpha \\
&= \int_{L\in\mathcal{L}_k} \left(\int_{\substack{X=[x_1|\cdots|x_k]\\ x_1,\ldots,x_k\in L}} g(X)\det(X^tX)^{(n-k)/2}\,dx_1\cdots dx_k\right)\alpha.
\end{aligned}$$

Since $\det(X^tX)^{1/2}$ is $k!$ times the $k$-dimensional volume of the $k$-simplex with vertices $O, x_1, \ldots, x_k$, the above relation is the Blaschke–Petkantschin formula.

**3.3. Moments of the Gaussian determinant** As an application of the polar integral theorem we will compute the moments of the Gaussian determinant.

Theorem 3.1. *If $\Delta_k$ is the determinant of a $k \times k$ random matrix with independent Gaussian entries, then, for each $l \geq 0$,*

$$E(|\Delta_k|^l) = (2\pi)^{kl/2} \prod_{i=1}^{k} \frac{\mathrm{vol}(S^{i-1})}{\mathrm{vol}(S^{l+i-1})} = 2^{kl/2} \prod_{i=1}^{k} \frac{\Gamma((l+i)/2)}{\Gamma(i/2)}.$$

Proof. Let $\psi(P) = e^{-\mathrm{tr}(P)/2}$. By Theorem 2.5,

$$\begin{aligned}
\int_{\mathcal{M}_{n\times k}} e^{-(\sum_{i,j=1}^k x_{ij}^2)/2}\,dX &= \int_{\mathcal{M}_{n\times k}} \psi(X^tX)\,dX = C\int_{\mathcal{M}_{k\times k}} \psi(Y^tY)|\det(Y)|^{n-k}\,dY \\
&= C\int_{\mathcal{M}_{k\times k}} e^{-(\sum_{i,j=1}^k y_{ij}^2)/2}|\det(Y)|^{n-k}\,dY,
\end{aligned}$$

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where

$$C = \frac{\mathrm{vol}(S^{n-1}) \cdots \mathrm{vol}(S^{n-k})}{\mathrm{vol}(S^{k-1}) \cdots \mathrm{vol}(S^{0})}.$$

Since

$$\int_{\mathcal{M}_{n\times k}} e^{-(\sum_{i,j=1}^{k} x_{ij}^2)/2}\, dX = \left(\int_{\mathbb{R}} e^{-x^2/2}\right)^{nk} = (2\pi)^{nk/2},$$

we obtain

$$C^{-1}(2\pi)^{(nk-k^2)/2} = (2\pi)^{-k^2/2} \int_{\mathcal{M}_{k\times k}} e^{-(\sum_{i,j=1}^{k} y_{ij}^2)/2} |\det(Y)|^{n-k}\, dY = \mathrm{E}(|\Delta_k|^{n-k}).$$

Considering the well-known relation $\mathrm{vol}(S^{n-1}) = 2\pi^{n/2}\Gamma(n/2)^{-1}$, the proof of the theorem is complete. □

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