THE PATH FUNCTOR AND FAITHFUL REPRESENTABILITY OF BANACH LIE ALGEBRAS

Dedicated to the memory of Hanna Neumann

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1. Introduction

In this note "vector space" will mean "Banach space" unless otherwise specified. Accordingly "Lie algebra" will stand for "Banach Lie algebra". Morphisms between Lie algebras will be assumed continuous. A Banach algebra B will be always assumed associative, and it will be also viewed as a Lie algebra with product $[X, Y] = XY - YX$. In particular, the Lie algebra $gl(V)$ of endomorphisms of a vector space V will be equipped with the uniform norm. A morphism of Lie algebras $L \to gl(V)$ will be called a *representation* of L in $gl(V)$. Also, if B is a Banach algebra, a morphism of Lie algebras $L \to B$ will be called a representation of L in B. From such one evidently obtains a representation of L in $gl(B)$. A representation will be called *faithful* if it is injective.

A theorem of Ado [1] asserts that each finite dimensional Lie algebra L admits a faithful representation in some finite dimensional Banach algebra (actually $gl(V)$, where V is finite dimensional). Here we shall seek conditions ensuring that a Banach Lie algebra posesses a faithful representation in some Banach algebra. The field will be that of real numbers, however this assumption is not essential and the field of complex numbers could be taken instead.

The Lie algebra $gl(V)$ is enlargeable, that is, it is the Lie algebra of a Banach Lie group, namely $GL(V)$. Since a Lie algebra which admits a monomorphism into an enlargeable Lie algebra is itself enlargeable, we see that enlargeability is a necessary condition for the existence of a faithful representation. The following Lemma shows that the condition is not sufficient.

LEMMA. *Let $\mathscr{L}$ be a simply connected Banach Lie group such that its Lie algebra L contains two elements p, q with the properties*

(i) $0 \neq [p, q] \in$ *centre of* L,

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(ii) *the one-parameter subgroup* $\mathscr{K} \subset \mathscr{L}$ *tangent to* $[p, q]$ *is a circle. Then L is not faithfully representable.*

An example of a Lie algebra with the above properties is the algebra denoted by M in §6.2., [3]

PROOF. Suppose that $\phi: L \to gl(V)$ is an injection. We may suppose (after complexifying, if necessary) that V is a complex space. ϕ induces a locally faithful representation $\Phi: \mathscr{L} \to GL(V)$, and hence the circle group $\mathscr{K}$ is locally faithfully represented. $\mathscr{K}$ being a commutative compact group, the representation of $\mathscr{K}$ splits into scalar representations [6]. Hence there is a character $\chi \neq 1$ of $\mathscr{K}$ and a maximal subspace V_χ of V in which $\mathscr{K}$ acts by scalar multiplications according to χ. $\mathscr{K}$ being central in $\mathscr{L}$ and V_χ being maximal, $\Phi(\mathscr{L})$ leaves V_χ invariant, and hence we have in V_χ by restriction a representation $\phi_\chi: L \to gl(V_\chi)$. Putting

$$\phi_\chi(p) = P, \ \phi_\chi(q) = Q, \ \phi_\chi([p, q]) = \lambda I,$$

we would have $PQ - QP = \lambda I$ with $\lambda \neq 0$. By Wintner's remark [9] (the proof of which carries over to the Banach space situation), or by a result of Wielandt [8], this equation cannot be satisfied by bounded operators P and Q.

For any Lie algebra L, let in the sequel C denote the centre of L and $H = L/C$. The adjoint representation of L factors out by C and represents H faithfully in L. Hence H is enlargeable. The simply connected Lie group belonging to H will be denoted by $\mathscr{H}$. Non enlargeability of L is intimately connected with the topology of $\mathscr{H}$ in that it implies that the second Betti number of $\mathscr{H}$ is non-zero (for a precise statement see [3]). In other words, a sufficient (but not necessary) condition for enlargeability is the vanishing of the second Betti number of $\mathscr{H}$. In [4], the faithful representability of a finite dimensional L is proved by relating it to the vanishing of the second Betti number of $\mathscr{K}$. (A summary of that proof is in the next section). Faithful representability being a stronger property than just enlargeability it is likely to be more easily affected by the topology of $\mathscr{H}$. For instance, in the above example of an enlargeable but not faithfully representable algebra M we have that $\mathscr{K}$ is the entire centre of $\mathscr{L}$ and $\pi_2(\mathscr{H}) = \pi_2(\mathscr{L}/\mathscr{K}) = Z$.

2. The representation theorem

Given a Banach space V, denote by $\Gamma^r V$; $(r = 0, 1, 2, \cdots)$ the vector space of all r times continuously differentiable paths $\gamma(t)$; $0 < t < 1$ in V such that γ and the derivatives $\gamma', \cdots, \gamma^{(r)}$ can be extended to continuous functions on $\{t \mid 0 \leqq t \leqq 1\}$. Denoting these extensions also by $\gamma, \gamma', \cdots, \gamma^{(r)}$ define

$$\|\gamma\| = \max_t \|\gamma(t)\| + \max \|\gamma'(t)\| + \cdots + \max_t \|\gamma^{(r)}(t)\|.$$

Then $\Gamma^r V$ is again a Banach space. The closed subspace of $\Gamma^r V$ consisting of the

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paths which satisfy $\gamma(0) = 0$ will be denoted by $\Lambda^r V$. If L is a Lie algebra, then so are $\Gamma^r L$ and $\Lambda^r L$, when the bracket is defined pointwise:

$$[\gamma_1, \gamma_2](t) = [\gamma_1(t), \gamma_2(t)].$$

It was shown recently that the Lie algebra $\Lambda^0 L$ is always enlargeable (see [7], where two proofs are presented). It is therefore natural to ask for faithful representations of $\Lambda^0 L$. This paper goes somewhat in that direction by constructing, under some additional assumptions, a faithful representation for $\Lambda^1 L$.

To state the main theorem, let

$$(*) \qquad 0 \to E \subset M \underset{\phi}{\to} H \to 0$$

be a topologically split central extension of Lie algebras. By saying that (*) is topologically split, we mean that the map ϕ admits a continuous linear section $H \to M$, or, equivalently, that there exists a continuous linear projection $\omega_M: M \to E$. To (*) there corresponds in the usual way a cohomology class $\Sigma \in H^2(H, E)$. Each cocycle $\sigma_H \in \Sigma$ is obtained from some projection $\omega_M: M \to E$ by setting $\sigma_H(\phi X, \phi Y) = \omega_M([X, Y])$ for any $X, Y \in M$. Consider a representation $\rho: H \to B$, where B is a Banach algebra. We shall say that a cocycle $\sigma_H \in \Sigma$ *harmonizes* with ρ if there is a bounded bilinear map $\sigma_B: B \times B \to E$ such that $\sigma_H = \sigma_B \rho$.

THEOREM. *If* $(*)\colon 0 \to E \subset M \to H \to 0$ *is a topologically split central extension of Banach Lie algebras and there is a cocycle in the cohomology class* Σ *of* (*) *which harmonizes with some faithful representation* $\rho: H \to B$ *in a Banach algebra* B, *then* $\Lambda^1 M$ *is faithfully representable.**)

The assumptions of this Theorem were dictated by the demands of the proof; we obtained the latter by trying to generalize to Banach Lie algebras the proof of Ado's theorem for a finite dimensional Lie algebra L given in [4]. The latter proceeded as follows.

Let $\mathscr{L}$ denote a simply connected local Lie group with Lie algebra L, let C be the centre of L and let $H = L/C$. Denote by ρ a faithful representation of H in $gl(V)$, where V is finite dimensional. For example, we can take for ρ the representation in $gl(L)$ induced by the adjoint representation of L. Denote by $\mathscr{H}$ the subgroup of $GL(V) \subset gl(V)$ generated by $\rho(H)$ and let $\pi: \mathscr{L} \to \mathscr{H}$ be the morphism of Lie groups corresponding to ρ. We select any right invariant C-valued differential 1-form ω on $\mathscr{L}$ such that the restriction of ω to L, where L is viewed as the tangent space at the unit element, is a projection $\omega_L: L \to C$. It is easily checked that $d\omega = \sigma\pi$, where σ is a right invariant closed 2-form on $\mathscr{H}$. ($\sigma\pi$ denotes the image of σ under the map $\pi: \mathscr{L} \to \mathscr{H}$). *It was shown in* [4] *that* $\sigma = d\tau$, *where* τ *is a form whose right translates span a finite dimensional vector space.* The

*) Any Banach Lie algebra L may be embedded into a topologically split central extension (*) where $H = L/C$ and such that C in embedded into E (see next footnote).

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existence of such τ was deduced from a Hopf theorem on rational differential forms on an algebraic group, and it is in this disguise that the vanishing of the second Betti number of $\mathscr{H}$ came in. Now $v = \omega - \tau\pi$ is a closed 1-form on $\mathscr{L}$ whose translates span a finite dimensional vector space, and in the tangent space at the unit element $v|C = \omega|C$ is the identity $C \to C$. Let F denote the space of functions f on $\mathscr{L}$ whose differentials df are right translates of v. Then F is finite dimensional. For $X \in L, f \in F$, let $\theta(X)f$ denote the derivative of f in the direction of the left invariant vector field on $\mathscr{L}$ defined by X. Then $\theta(X)f \in F$ and θ is a representation of L in F. The representation is faithful on C since the function f for which $df = v$ satisfies

$$(\theta(X)f)(1) = (df)_1(X) = \omega(X) = \omega_L(X) = X \neq 0 \text{ when } 0 \neq X \in C.$$

Combining this representation with the adjoint representation yields a faithful representation of L.

In the above proof, the existence of a form ω with the required properties posed no problems: Any projection $\omega_L: L \to C$ defines a right invariant form ω on $\mathscr{L}$ whose restriction to the tangent space L at $1 \in \mathscr{L}$ is ω_L. The corresponding form σ on $\mathscr{H}$ is then defined by letting its restriction σ_H to the tangent space H at $1 \in \mathscr{H}$ satisfy $\sigma_H(\bar{X}, \bar{Y}) = \omega_L([X, Y])$ for any cosets $\bar{X}, \bar{Y} \in H$ corresponding to $X, Y \in L$. The injection $\rho: H \to gl(V)$ maps the space of skew-symmetric forms on $gl(V)$ onto the space of skew-symmetric forms on H. Hence $\sigma_H = \sigma_B \rho$ for some form σ_B on $gl(V)$. Such a form was used to construct τ.

For an arbitrary Banach Lie algebra L we still have a faithful representation $\rho: H \to B$, where B is a Banach algebra, e.g. the representation in $B = gl(L)$ induced by the adjoint representation of L. But in this case the existence of ω_L and σ_B have to be assumed. The Theorem above implies that these assumptions suffice for obtaining the faithful representability of $\Lambda^1 L$. More precisely, let us prove the following.

COROLLARY. *Let L be a Banach Lie algebra with centre C, let $H = L/C$ and let $p: L \to H$ be the quotient map. Suppose there exists a bounded linear map $\omega_L: L \to E$, where E is a Banach space, a faithful representation $\rho: H \to B$, where B is a Banach algebra, and a bilinear map $\sigma_B: B \times B \to E$ such that*

(i) $\omega_L(X) \neq 0$ *whenever* $0 \neq X \in C$,

(ii) $\omega_L([X, Y]) = \sigma_B(\rho p(X), \rho p(Y))$ *for all* $X, Y \in L$.

Then L admits an injection into a Lie algebra M which satisfies the assumption of the Theorem. Consequently $\Lambda^1 L$ is faithfully representable.

For the proof, we construct (*) from the data in the Lemma. Thus let $\sigma_H: H \times H \to E$ be the form satisfying $\sigma_H(pX, pY) = \omega_L([X, Y])$ for all $X, Y \in L$. Then σ_H is a cocycle determining a cohomology class Σ in $H^2(H, E)$, and by (ii), σ_H harmonizes with $\rho: H \to B$. To obtain a corresponding extension (*), take $M = E \oplus H$ and define in M the Lie product by

$$[(e_1, pX_1),\ (e_2, pX_2)] = (\omega_L([X_1, X_2]),\ [pX_1, pX_2])$$

for any $(e_i, pX_i) \in E \oplus H$; $X_i \in L$, $i = 1, 2$. Then M is a Lie algebra, E is central in M and $H = M/E$. Moreover the map $X \mapsto (\omega_L(X), pX)$ defines an injection $L \to M$.*)

3. Representation by Lie derivatives

Consider an exact, topologically split sequence (*): $0 \to E \subset M \to H \to 0$ of Banach Lie algebras. In this section we shall give conditions which imply the faithful representability of M.

PROPOSITION. *Let $\sigma_H \in \Sigma \in H^2(H, E)$ be a cocycle corresponding to (*). Denote by $\mathscr{H}$ the simply connected Lie group with Lie algebra H and let σ be the right invariant 2-form on $\mathscr{H}$ such that $\sigma(X, Y) = \sigma_H(X, Y)$ in the tangent space at the identity (that is, for $X, Y \in H$). Assume that there is a Banach space T of smooth E-valued 1-forms on $\mathscr{H}$ such that*

(a) *$\sigma = d\tau_0$ for some $\tau_0 \in T$,*

(b) *denoting by $\theta_H(X)\tau$ the Lie derivative of $\tau \in T$ with respect to the left invariant vector field on $\mathscr{H}$ defined by $X \in H$, we have $\theta_H(X)\tau \in T$ for all $\tau \in T$, $X \in H$, also $\theta_H(X) \in gl(T)$ for each $X \in H$ and moreover $\theta_H: H \to gl(T)$ is a representation of H in $gl(T)$,*

(c) *the evaluation map $T \times H \to H$ given by $(\tau, X) \mapsto \tau(X)$ is bounded (that is, $\| \tau(X) \| < K \| \tau \| \; \| X \|$ for some constant K; $\tau(X)$ being the evaluation of τ at the neutral element for the vector X*

Then there exists a faithful representation of M.

This proposition will be later applied to the extension $0 \to \Lambda^1 E \subset \Lambda^1 M \to \Lambda^1 H \to 0$. Denoting by $\hat{\mathscr{H}}$ the simply connected Lie group with Lie algebra $\Lambda^1 H$, we shall have to show that the corresponding 2-form $\hat{\sigma}$ on $\hat{\mathscr{H}}$ is exact. For this it will be necessary that the periods of $\hat{\sigma}$ on $\hat{\mathscr{H}}$ vanish, but this will be the case since $\hat{\mathscr{H}}$, being a path group, is contractible. Further, we shall need a sufficiently ample Banach space T of 1-forms on $\hat{\mathscr{H}}$ such that a solution $\hat{\tau}_0$ for $\hat{\sigma} = d\hat{\tau}_0$ exists in T. In the finite dimensional case this was obtained by representing $\mathscr{H}$ locally faithfully in some $gl(V)$ in a manner that σ became the restriction of a polynomial form on $gl(V)$, and then, via the Hopf theorem on the cohomology of such forms, a form τ_0 of finite span with $\sigma = d\tau_0$ was shown to exist. In the infinite dimensional case we shall have a locally faithful representation of $\hat{\mathscr{H}}$ in some Banach algebra $\tilde{A}$ such that $\hat{\sigma}$ becomes the restriction of some polynomial form on $\tilde{A}$ and moreover the contraction in $\hat{\mathscr{H}}$ derives from a deformation retraction

*) If one $E = L$ (as a vector space) and one takes $\omega_L([X_1, X_2]) = [X_1, X_2]$ and observes that this bracket depends only on pX_1, pX_2, one obtains Lie algebra structure on $M = L + H$ wich turns in into a topologically split extension of H, $X \mapsto (X, pX)$ is an embedding $L \to M$.

in $\tilde{A}$ preserving polynomial forms. This will permit us to construct a $\hat{\tau}_0$ satisfying $\hat{\sigma} = d\hat{\tau}_0$, and a norm on the space of right translates of $\hat{\tau}_0$.

We turn now to the proof of the Proposition. We shall construct a representation of M which is faithful on E; combining such a representation with the composite $M \underset{\varphi}{\to} H \underset{\rho}{\to} B$, one obtains a faithful representation of M. The construction of the representation of M which is faithful on E will be in close analogy with the proof of Ado's theorem for a finite dimensional Lie algebra outlined above. We shall produce a space F of E-valued functions on a local Lie group $\mathscr{M}$ with Lie algebra M such F is closed under taking directional derivatives with respect to left invariant vector fields on $\mathscr{M}$. The action of M on F by means of these derivatives will give the required representation.

4. Proof of the Proposition

Suppose H is a Lie algebra, V an abstract vector space and to every $X \in H$ there corresponds a linear map $\rho(X): V \to V$ so that $\rho([X, Y]) = \rho(X) \circ \rho(Y) - \rho(Y) \circ \rho(X)$. Then V will be called an abstract H-module. If V is a normed space, each $\rho(X): V \to V$ is bounded and the map $\rho: H \to gl(V)$ is continuous (hence ρ is a representation, when V is a Banach space), then V will be called an H-module.

Every vector space V of 1-forms on $\mathscr{H}$ which is stable under taking Lie derivatives $\theta_H(X)$, where $X \in H$, (see (b), Proposition) is clearly made by θ_H into an abstract H-module. In particular we have assumed in (b) that the Banach space T is in this way an H-module. Let $\mathscr{M}$ be an open ball in M with centre 0 made into a local group via the Campbell–Hausdorff formula. Then $\mathscr{M}$ is a local Lie group with Lie algebra M. By the definition of σ_H, there exists a projection $\omega_M: M \to E$ such that $\sigma_H(\phi X, \phi Y) = \omega_M([X, Y])$ for all $X, Y \in M$. Let ω be the right invariant 1-form on $\mathscr{M}$ such that $\omega(X) = \omega_M(X)$ for all $X \in M$. Denote the morphism $\mathscr{H} \to \mathscr{K}$ induced by $\phi: M \to H$ also by ϕ. Then we have on $\mathscr{M}$ also the space of 1-forms $T\phi$. Let Ω be the abstract space of 1-forms spanned by ω and $T\phi$. Since $\omega(X) \neq 0$ and $\tau\phi(X) = 0$ for $0 \neq X \in E$, it follows that Ω is a direct sum of the subspaces $\mathbf{R}\omega$ and $T\phi$. Clearly Ω is a Banach space under the norm $\| \lambda\omega + \tau\phi \| = |\lambda| + \| \tau \|_T$. Our first aim is to show that Ω is an M-module with respect to the action by Lie derivatives $\theta_M(X)$; $X \in M$.

Since ω is invariant, we have $\theta_M(X)\omega = 0$ for all X. Moreover, since $\phi: \mathscr{M} \to \mathscr{H}$ maps left invariant vector fields on $\mathscr{M}$ into left invariant vector fields on $\mathscr{H}$, it follows that $T\phi$ is also stable under each $\theta_M(X)$. Thus Ω is stable, therefore it is an abstract M-module. If T is viewed as an M-module via the map $\phi: M \to H$, then it is clear that the map $T \to \Omega$ given by $\tau \mapsto \tau\phi$ is a morphism of abstract M-modules. We conclude from this data that Ω is an M-module, by the following lemma.

LEMMA. *Let V, W be Banach spaces and suppose that V is an M-module and W is an abstract M-module. Suppose further that* $a: V \to W$ *is a morphism of abstract M-modules such that W is the direct sum of aV and a trivial submodule* W_0 *and*

$$\| w + a(v) \|_W = \| w \|_W + \| v \|_V \text{ for all } w \in W_0,\ v \in V.$$

Then W is an M-module.

Indeed, by assumption, $\| \rho_V(X)v \| \leqq B \| X \| \ \| v \|$ for each $X \in M$, $v \in V$ and a constant B. Hence for each $w \in W_0$, $v \in V$

$$\| \rho_W(X)(w + a(v)) \| = \| \rho_W(X)a(v) \| = \| \rho_V(X)v \| \leqq B \| X \| \ \| v \|$$
$$\leqq B \| X \| (\| w \| + \| v \|) = B \| X \| \ \| w + a(v) \|.$$

It is well known that the Lie derivative of a closed form is closed (actually exact). Thus let Φ be the M-module of all the closed forms in Ω. Denote by F the space of E-valued functions on $\mathscr{M}$ such that $df \in \Phi$. Let M operate on F via the Lie derivatives θ_M, that is, let $\theta_M(X)f$ denote for $X \in M, f \in F$ the derivative of f in the direction of the left invariant vector field determined by X. It is easily seen that $d: F \to \Phi$ is a morphism of abstract M-modules. This is an epimorphism ($\mathscr{M}$ being diffeomorphic to an open ball in M, every closed form is exact), and its kernel is the subspace F_0 of constant functions. Moreover $d: F \to \Phi$ admits the linear section $j: \Phi \to F$ which maps Φ onto the subspace of functions vanishing at $1 \in \mathscr{M}$ (i.e. $0 \in M$). Thus F is the direct sum of F_0 and $j\Phi$ and we may define a norm in F by $\|f\|_F = \|f(1)\|_E + \| df \|_\Phi$. We wish to show that with respect to this norm F is an M-module. This will follow from the next lemma provided we know that

$$\| \theta_M(X)f - j(\theta_M(X)df) \|_F \leqq B \| df \|_\Phi \| X \|$$

for every $f \in F$ such that $f(1) = 0$ (that is, $f \in j\Phi$), $X \in M$ and some constant B. Since $\theta_M(X)f - j(\theta_M(X)df) = g$ is a constant, it is sufficient to show $\| g(1) \| \leqq B \| df \|_\Phi \| X \|$. However $j(\theta_M(X)df)$ vanishes at 1, whence $g(1) = \theta_M(X)f(1) = df(X)$. Supposing $df = \lambda\omega + \tau\phi$; $\lambda \in R$, $\tau \in T$, we find

$$\| g(1) \| = \| \lambda\omega(X) + \tau(\phi X) \| \leqq |\lambda| \ \| \omega_M \| \ \| X \| + K \| \tau \| \ \| \phi \| \ \| X \|,$$

where K is the constant in the assumption (c). The required inequality follows now by letting $B = \max(\| \omega_M \|, \| \phi \| \cdot K)$.

The following result will now imply that F is an M-module.

LEMMA. *Let W, V be normed vector spaces which are also abstract M-modules. Suppose that V is an M-module and* $d: W \to V$ *is a morphism of abstract M-modules such that* Ker d *is a trivial submodule (that is, with trivial action of M) and there is a linear section* $j: V \to W$ *of d satisfying*

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(i) $\| w + j(v) \|_W = \| w \|_W + \| v \|_V$ *for all* $w \in \operatorname{Ker} d$, $v \in V$;

(ii) $\| \rho_W(X) j(v) - j(\rho_V(X) v) \| \leqq B \| v \| \, \| X \|$ *for all* $v \in V$, $X \in M$.

Then W is an M-module.

Indeed, for $w \in \operatorname{Ker} d$, $v \in V$ and $X \in M$,

$$\| \rho_W(X)(w + j(v)) \| = \| \rho_W(X) j(v) \| \leqq \| \rho_W(X) j(v) - j(\rho_V(X) v) \|$$
$$+ \| j(\rho_V(X) v) \| \leqq B \| v \| \, \| X \| + \| \rho_V(X) v \| \leqq (B + K_0) \| v \| \, \| X \|$$
$$\leqq (B + K_0)(\| w \| + \| v \|) \| X \| = (B + K_0) \| w + j(v) \| \, \| X \|,$$

where K_0 is the norm of $\rho_V \colon M \to gl(V)$.

Thus F is an M-module. To complete the proof of the Proposition note that

$$d\omega(X_1, X_2) = \omega_M([X_1, X_2]) = \sigma_H(\phi X_1, \phi X_2) = \sigma(\phi X_1, \phi X_2),$$

which implies by the invariance of ω and σ that $d\omega = \sigma\phi$. This and $\sigma = d\tau_0$ implies $\omega - \tau_0 \phi \in \Phi$. Let $f \in F$ be such that $df = \omega - \tau_0 \phi$. Then for each $X \in E$,

$$(\theta_M(X) f)(1) = (Xf)(1) = \omega(X) - \tau_0(\phi X) = \omega_M(X) = X,$$

so that $\Theta_M(X) f \neq 0$ whenever $X \neq 0$. Thus we have a representation of M in $gl(F)$ which is faithful on E. (If F is not a Banach space, the M-module structure on F can be extended to the completion $\bar{F}$).

As we remarked in concluding §3, this implies that M is faithfully representable.

5. Polynomial differential forms

In this section we gather auxiliary results which will be needed in the proof of the Theorem.

5.1. Generalities. Let E, V be Banach spaces. An E-valued *n-linear functional* on V is an n-linear function w on V with values in E which satisfies a boundedness condition $\| w(x_1, \cdots, x_n) \| \leqq B \| x_1 \| \cdots \| x_n \|$ for some $B < \infty$ and all $x_1, \cdots, x_n \in V$. The least such B is denoted by $\| w \|$.

The restriction of an n-linear functional w to the diagonal in V^n induces a function on V which is called a *homogeneous polynomial* of degree n (the degree is determined by the polynomial). Given w, there is exactly one symmetric n-linear functional $\bar{w}$ such that w and $\bar{w}$ induce the same polynomial and $\| \bar{w} \| \leqq \| w \|$ (symmetrization does not increase the norm). With a homogeneous polynomial is therefore associated a unique symmetric n-linear functional, they will be denoted by the same letter w; the value of the polynomial at x will be denoted by $w(x)$ or $w(x, \cdots, x)$; the norm of the polynomial is understood to be the norm of the associated symmetric functional. Since we have to consider only homogeneous polynomials, we take "polynomial" always in that sense. With the above norm, the space of polynomials of a fixed degree is a Banach space.

We shall identify in the usual way the tangent space to V at $x \in V$ with V itself. Then each $X \in V$ determines a tangent field on V, termed the *constant field*, and also denoted by X. The derivative of a homogeneous polynomial w of degree n in the direction of such constant tangent field X is the polynomial of degree $n-1$ defined by

$$(Xw)(x) = \frac{d}{dt} w(x + tX) \Big|_{t=0} = nw(X, x, \cdots, x) = nw(x, \cdots . x, X).$$

It follows easily that $\| Xw \| \leqq n \| w \| \; \| X \|$.

Suppose ω is an E-valued differential form of degree q on V. The value taken by ω on the constant fields $X_1, \cdots, X_q$ on V at $x \in V$ will be denoted by $\omega(x; X_1, \cdots, X_q)$. Considering the latter as a function of x, we have that ω is a map of V into the Banach space of skew-symmetric q-linear E-valued functionals on V. If ω is a polynomial on V of degree k, then ω will be called a *polynomial differential form* of type (k, q). We have in such case a unique $(k+q)$-linear functional w on V, symmetric in the first k variables and skew-symmetric in the last q variables such that

$$\omega(x; X_1, \cdots, X_q) = w(x, \cdots, x, X_1, \cdots, X_q).$$

According to the above notation, $X\omega(x; X_1, \cdots, X_q)$ is the derivative of the function $x \mapsto \omega(x; X_1, \cdots, X_q)$ in the direction of the constant vector field X. Thus, by the usual definition, the exterior derivative $d\omega$ is the polynomial form of type $(k-1, q+1)$ given by

$$\begin{aligned} d\omega(x; X_1, \cdots, X_{q+1}) & \\ &= \Sigma(-1)^{j-1} X_j \omega(x; X_1, \cdots, \hat{X}_j, \cdots, X_{q+1}) \\ &= k\, \Sigma(-1)^{j-1} w(x, \cdots, x, X_j, X_1, \cdots, \hat{X}_j, \cdots, X_{q+1}). \end{aligned}$$

5.2. A Homotopy operator Let V, W be Banach spaces. A family of bounded linear maps $P(s): V \to W$; $0 \leqq s \leqq 1$ will be called *strongly continuous* if all curves $s \mapsto P(s)x$; $x \in V$ are continuous. It is well known ([2], Chap, V, Th. 5) that then $\{\| P(s) \| \mid 0 \leqq s \leqq 1\}$ is a bounded set. $P(s)$ will be called a *strong C^1-family* if $P'(s)x$ exists for each s and each x and both families $P(s)$, $P'(s)$ are strongly continuous. For a strong C^1-family $P(s)$ and for $0 \leqq s \leqq 1$ we define a homotopy operator D_s which assigns to a polynomial form ω on W of type (k, q) the polynomial form $D_s\omega$ on V of type $(k+1, q-1)$ according to the formula

$$D_s\omega(x; X_1, \cdots, X_{q-1}) = \int_0^s \omega(P(t)x; P'(t)x, P(t)X_1, \cdots, P(t)X_{q-1})dt.$$

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Clearly $\| D_s\omega \| \leqq B^{k+q}\| \omega \|$ if B is a joint bound for $P(s)$ and $P'(s)$.

LEMMA. $dD_s\omega + D_s d\omega = \omega P(s) - \omega P(0)$ *for* $0 \leqq s \leqq 1$.

The proof is a standard computation. We shall need a corollary concerning path-valued forms.

For $P(s)$ and ω as in the lemma, let $\tilde{\omega}$ denote the $\Gamma^1 E$-valued form on V such that $\tilde{\omega}(x; X_1, \cdots, X_q)$ is the C^1-path in E given by

$$s \mapsto (\omega P(s))\,(x; X_1, \cdots, X_q) = \omega(P(s)x; P(s)X_1, \cdots, P(s)X_q).$$

Then $\tilde{\omega}$ is again a polynomial form of the same type as ω. Let $\tilde{D}\omega$ be the $\Gamma^1 E$-valued form on V such that $\tilde{D}\omega(x; X_1, \cdots, X_{q-1})$ is the path

$$s \mapsto D_s\omega(x; X_1, \cdots, X_{q-1}).$$

$\tilde{D}\omega$ is a polynomial form of the same type as $D\omega$.

COROLLARY. *Regarding the E-valued form* $\omega P(0)$ *as being* $\Gamma^1 E$*-valued, due to the inclusion* $E \subset \Gamma^1 E$ *which identifies a constant path with its one-point image, we have for every strong* C^1*-family* $P(s)\colon V \to W;\ 0 \leqq s \leqq 1$, *that*

$$\tilde{\omega} - \omega P(0) = d\tilde{D}\omega + \tilde{D}d\omega.$$

PROOF. For fixed $x, X_1, \cdots, X_q \in V$, each side of this equation is a path in E. Evaluating at s, $\tilde{\omega}(x; X_1, \cdots, X_q)(s) = (\omega P(s))(x; X_1, \cdots, X$, and

$$(\tilde{D}d\omega)(x; X_1, \cdots, X_q)(s) = D_s d\omega(x; X_1, \cdots, X_q).$$

All we need to show is then

$$d\tilde{D}\omega(x; X_1, \cdots, X_q)(s) = dD_s\omega(x; X_1, \cdots, X_q).$$

Now if w_s denotes the linear functional for which $D_s\omega(x; X_1, \cdots, X_{q-1}) = w_s(x, \cdots, x, X_1, \cdots X_{q-1})$, then the linear functional w defining $\tilde{D}\omega$ is given by

$$w(x_1, \cdots, x_{k+q})(s) = w_s(x_1, \cdots, x_{k+q}),$$

so that for each s

$$\begin{aligned} d\tilde{D}\omega(x; X_1, \cdots, X\ )(s) \\ &= (k\,\Sigma(-1)^{j-1}w(x, \cdots, x, X_j, X_1, \cdots, \hat{X}_j, \cdots, X_q))(s) \\ &= k\,\Sigma(-1)^{j-1}w_s(x, \cdots, x, X_j, X_1, \cdots, \hat{X}_j, \cdots, X_q) \\ &= dD_s\omega(x; X_1, \cdots, X_q). \end{aligned}$$

The above will be used for the strong C^1-family $P(s)\colon \Gamma^1 W \to W$ of evaluation maps $P(s)\gamma = \gamma(s)$. In this case we have $\| P(s) \|$, $\| P'(s) \| \leqq 1$.

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5.3. Invariant forms on Banach algebras The proof of the Theorem involves the right invariant form σ on $\mathscr{H}$ which was defined in the Proposition. We shall relate it to a polynomial form by the following observation.

Lemma. *Let B be an associative Banach algebra with unity and let B_1 be the Lie group of invertible elements of B. Then there is a morphism β of B_1 into the Lie group A_1 of invertible elements of some associative Banach algebra A such that for each right invariant q-form σ_B on B_1 there is a polynomial form σ_A on A of type (q,q) satisfying $\sigma_A\beta = \sigma_B$.*

The proof will employ the following remark: If h is a skew-symmetric q-linear functional on a Banach space V and $Z: V \times V \to V$ is a bounded bilinear composition (that is, $\| Z(x,X) \| \leqq K\| x \| \, \| X \|$ for a constant K), then h_Z defined by

$$h_Z(x; X_1, \cdots, X_q) = h(Z(x,X_1), Z(x,X_2), \cdots, Z(x,X_q))$$

is a polynomial differential form of type (q,q). Indeed, $w(x_1, \cdots, x_q, X_1, \cdots, X_q) = h(Z(x_1,X_1), \cdots, Z(x_q,X_q))$ is a $2q$-linear functional with $\| w \| \leqq K^q \| h \|$. Symmetrizing with respect to $x_1, \cdots, x_q$ yields a functional $\bar{w}$ which is symmetric with respect to $x_1, \cdots, x_q$, skew-symmetric with respect to $X_1, \cdots, X_q$ and $h_Z(x; X_1, \cdots, X_q) = \bar{w}(x, \cdots, x, X_1, \cdots X_q)$.

Now, to prove the Lemma, let B^* be the opposite algebra of B (that is, with the same elements and multiplication $a * b = ba$). On the direct sum $A = B \oplus B^*$ define the skew-symmetric q-linear functional $h((a_1,b_1), \cdots, (a_q,b_q)) = \sigma_B(1; a_1, \cdots, a_q)$ and the bilinear composition $Z((x_1,y_1), (X_1,Y_1)) = (X_1 y_1, 0)$. Put $\sigma_A = h_Z$, which is a polynomial form of type (q,q) on A. Explicitly,

$$\begin{aligned}\sigma_A((x,y); (X_1,Y_1), \cdots, (X_q,Y_q)) &= h((X_1 y, 0), \cdots, (X_q y, 0)) \\ &= \sigma_B(1; X_1 y, \cdots, X_q y).\end{aligned}$$

Define $\beta: B_1 \to A$ by $\beta(x) = (x, x^{-1})$. Then

$$\begin{aligned}\sigma_A\beta(x; X_1 \cdots, , X_q) &= \sigma_A((x,x^{-1}); (X_1, -x^{-1}X_1x^{-1}), \cdots, (X_1, -x^{-1}X_qx^{-1})) \\ &= \sigma_B(1; X_1x^{-1}, \cdots, X_qx^{-1}) = \sigma_B(x; X_1, \cdots, X_q)\end{aligned}$$

by the right invariance of σ_B.

5.4. The Lie algebra module of polynomial forms. A differential form v on the Lie group A_1 of invertible elements of a Banach algebra A with unity will be called a polynomial form if v is the restriction of a polynomial form on A. Since the restriction determines the form uniquely (A_1 being open in A), we denote both by the same letter. Let V be the Banach space of polynomial forms of a fixed type (k,q) on A_1. As usual, identify the Lie algebra of A_1 with A — the tangent space at identity.

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LEMMA. *For each $X \in A$ and $v \in V$ let $\theta(X)v$ be the Lie derivative of v in the direction of the left invariant vector field on A_1 determined by X. Then V is an A-module under the action θ.*

PROOF. Let $w(x_1, \cdots, x_{k+q})$ be a $(k+q)$-linear functional on A, symmetric in the first k variables, skew-symmetric in the remaining q variables such that

$$v(x; X_1, \cdots, X_q) = w(x, \cdots, x, \; X_1, \cdots, X_r).$$

Define

$$(\theta(X)w)(x_1, \cdots, x_{k+q}) = \sum_{i=1}^{k+q} w(x_1, \cdots, x_i X, \cdots, x_{k+q}).$$

Then $\theta(X)w$ is a functional which is symmetric in the first k variables and anti-symmetric in the remaining q variables. Moreover $(\theta(X)v)(x; X_1, \cdots, X_q) = (\theta(X)w)(x, \cdots, x, X_1, \cdots, X_q)$, whence

$$\| \theta(X)v \| = \| \theta(X)w \| \leqq (k+q) \| X \| \; \| w \| = (k+q) \; \| X \| \; \| v \|.$$

COROLLARY. *Let A, V be as above and let $\pi: \mathscr{H} \to A_1$ be a morphism of Lie groups. Put $T = V\pi$ and define the norm in T by $\| \tau \| = \inf\{\| v \|; v\pi = \tau, v \in V\}$. For $X \in H$, $\tau \in T$ let $\theta_H(X)\tau$ be the Lie derivative in the direction of the corresponding left invariant field on $\mathscr{H}$. Then the action θ_H turns T into an H-module, moreover the evaluation map $T \times H \times \cdots \times H \to E$ given by $(\tau, X_1, \cdots, X_q) \mapsto \tau(X_1, \cdots, X_q)$ is bounded.*

PROOF. Consider the epimorphism $V \to T$ given by $v \to v\pi$. Its kernel V_0 is composed of those forms v such that $v(y; Y_1, \cdots, Y_q) = 0$ whenever $y = \pi(x)$ for some $x \in \mathscr{H}$ and $Y_1, \cdots, Y_q$ are images of vectors tangent to $\mathscr{H}$ at x. Hence V_0 is a closed subspace of V. V is an A-module by the Lemma. If $\pi_H: H \to A$ denotes the tangent map to π at the identity $1 \in \mathscr{H}$, then π_H turns V into an H-module. V_0 is invariant under the action of H, hence V/V_0 is an H-module. By definition, T and V/V_0 are isomorphic both as abstract H-modules and as Banach spaces, so T is an H-module.

If $\tau \in T; X_1, \cdots, X_q \in H$, then

$$\| \tau(X_1, \cdots, X_q) \| = \| v(1; \pi_H X_1, \cdots, \pi_H X_q) \| \leqq \| v \| \; \| 1 \| \; \| \pi_H \| \; \| X_1 \| \cdots \| X_q \|$$

for each $v \in V$ such that $\tau = v\pi$. Thus $\|v\|$ can be replaced here by $\| \tau \|$, showing that the evaluation map $T \times H \times \cdots \times H \to E$ is bounded.

REMARK. Let σ be the 2-form on $\mathscr{H}$ introduced in the Proposition. Suppose there is a Banach algebra A, a morphism of Lie groups $\pi: \mathscr{H} \to A_1$ and a polynomial form τ_A on A such that $\sigma = d\tau_A \pi$. Then M is faithfully representable. Indeed, if V is the space of forms on A of the same type as τ_A and $\tau_0 = \tau_A \pi$, $T = V\pi$, then the assumptions (a), (b), (c) of the Proposition are satisfied.

6. Proof of the Theorem

Let ^ denote the image under the functor Λ^1. Thus we have the topologically split exact sequence

$$(\hat{*}) \qquad 0 \to \hat{E} \subset \hat{M} \underset{\hat{\phi}}{\to} \hat{H} \to 0.$$

By assumption, there exists a cocycle σ_H describing (*) which harmonizes with a faithful representation $\rho: H \to B$ into a Banach algebra B. Evidently $(\hat{*})$ is described by the cocycle $\hat{\sigma}_H: \hat{H} \times \hat{H} \to \hat{E}$.

We shall apply the Proposition to the exact sequence $(\hat{*})$ and to the cocycle $\hat{\sigma}_H$. Thus let $\hat{\mathscr{H}}$ be the simply connected Lie group with Lie algebra $\hat{H}$ and let $\hat{\sigma}$ be the right invariant 2-form on $\hat{\mathscr{H}}$ such that $\hat{\sigma}(\gamma_1, \gamma_2) = \hat{\sigma}_H(\gamma_1, \gamma_2)$ for any γ_1, γ_2 tangent to $\hat{\mathscr{H}}$ at the identity (that is, $\gamma_1, \gamma_2 \in \hat{H}$). By the remark concluding §5 it will be sufficient to find a Banach algebra $\tilde{A}$, a polynomial form $\hat{\tau}_A$ on $\tilde{A}$ and a morphism of Lie groups $\bar{\pi}: \hat{\mathscr{H}} \to \tilde{A}_1$ such that $\hat{\sigma} = d\tilde{\tau}_A\bar{\pi}$.

Let $\rho: \mathscr{H} \to B_1$ denote the morphism of Lie groups corresponding to the given representation $\rho: H \to B$. If σ is the 2-form on $\mathscr{H}$ defined in the Proposition, then there is evidently a right invariant 2-form on B_1 whose composite with ρ is σ. Applying to this situation the Lemma in §5.3, we conclude that there is a Banach algebra A, a polynomial form σ_A on A of type (2, 2) and a morphism $\pi: \mathscr{H} \to A_1$ such that $\sigma = \sigma_A \pi$.

Let $\pi_H: H \to A$ be the tangent map to π at the identity. Then the evaluation of $\sigma = \sigma_A \pi$ at the identity yields

$$(6.1) \qquad \sigma_H(Y_1, Y_2) = \sigma_A(1; \pi_H(Y_1), \pi_H(Y_2)); \; Y_1, Y_2 \in H.$$

Denote by ~ the application of the functor Γ^1. Clearly $\tilde{A}$ is a Banach algebra with identity and $\hat{\sigma}_A$ is an $\tilde{E}$-valued polynomial form on the Lie group $\tilde{A}_1$. From (6.1) follows that the bilinear functional $\tilde{\sigma}_H: \tilde{H} \times \tilde{H} \to \tilde{E}$ is obtained from $\tilde{\sigma}_A$ by the morphism $\tilde{\pi}_H: \tilde{H} \to \tilde{A}$, so that

$$(6.2) \qquad \tilde{\sigma}_H(\gamma_1, \gamma_2) = \tilde{\sigma}_A(1; \tilde{\pi}_H(\gamma_1), \tilde{\pi}_H(\gamma_2)) \text{ for every } \gamma_1, \gamma_2 \in \tilde{H}.$$

Let $\tilde{\mathscr{H}}$ be the simply connected Lie group with Lie algebra $\tilde{H}$ and let $\tilde{\pi}: \tilde{\mathscr{H}} \to \tilde{A}_1$ be the morphism of Lie groups induced by $\tilde{\pi}_H$. Further, let $\tilde{\sigma}$ be the right invariant form on $\tilde{\mathscr{H}}$ corresponding to $\tilde{\sigma}_H$. We wish to show that the equality $\sigma = \sigma_A \pi$ on $\mathscr{H}$ implies the equality $\tilde{\sigma} = \tilde{\sigma}_A \tilde{\pi}$ on $\tilde{\mathscr{H}}$. The identity (6.2) shows that this is indeed so when the forms are evaluated on vectors tangent to $\tilde{\mathscr{H}}$ at 1, i.e. on $\gamma_1, \gamma_2 \in \tilde{H}$. Hence it will suffice to show that $\tilde{\sigma}_A \tilde{\pi}$ is right invariant on $\mathscr{K}$. Now the right invariance of $\sigma_A \pi$ on $\tilde{\mathscr{H}}$ can be expressed by writing that

$$\sigma_A(y; \pi_H(X_1)y, \pi_H(X_2)y) = \sigma_A(1; \pi_H(X_1), \pi_H(X_2))$$

whenever $y \in \mathscr{H}$, $X_1, X_2 \in H$. Thus $\tilde{\sigma}_A \tilde{\pi}$ is right invariant.

Let $j:\hat{H} \subset \tilde{H}$ be the inclusion and denote the induced morphism of Lie groups also by j. It is clear that $\tilde{\sigma}_H = \hat{\sigma}_H j$ and thus $\hat{\sigma} = \tilde{\sigma} j$. The morphism of $\mathscr{H}$ into the group of invertible elements of a Banach algebra which we are seeking is now defined by $\bar{\pi} = \tilde{\pi} j : \hat{\mathscr{H}} \to \tilde{\mathscr{H}} \to \tilde{A}_1$. Clearly $\tilde{\sigma} = \tilde{\sigma}_A \, \tilde{\pi}$ implies, by composition with j, that $\hat{\sigma} = \tilde{\sigma}_A \bar{\pi}$.

It remains to find a polynomial form $\tilde{\tau}_A$ on $\tilde{A}$ such that $(\tilde{\sigma}_A - d\tilde{\tau}_A)\bar{\pi} = 0$. For this purpose we use the strong C^1-family $P(s): \tilde{A} \to A$; $0 \leqq s \leqq 1$, given by $P(s)\gamma = \gamma(s)$ for each path $\gamma \in \tilde{A}$. We note that

$$\tilde{\sigma}_A(\gamma_0;\gamma_1,\gamma_2)(s) = \sigma_A(P(s)\gamma_0;\ P(s)\gamma_1, P(s)\gamma_2) = (\sigma_A P(s))(\gamma_0;\gamma_1,\gamma_2)$$

for $\gamma_0, \gamma_1, \gamma_2 \in \tilde{A}$. There is a family of Lie algebra morphisms $P_H(s): \hat{H} \to H$ given by $P_H(s)\mu = \mu(s)$ for each $\mu \in \hat{H}$. Let $P_{\mathscr{H}}(s): \hat{\mathscr{H}} \to \mathscr{H}$ be the corresponding family of morphisms of Lie groups. We note that, for each $x \in \hat{\mathscr{H}}$, the curve $s \mapsto P_{\mathscr{H}}(s)x$ in $\mathscr{H}$ is of class C^1. Moreover $P_{\mathscr{H}}(0)$ is the trivial map. *These observations show that the assumptions of the following lemma are satisfied.* Hence $\tilde{\tau}_A$ may be taken, as given by the lemma.

LEMMA. *Let $P_{\mathscr{H}}(s): \tilde{\mathscr{H}} \to \mathscr{H}$; $0 \leqq s \leqq 1$ be a family of morphisms between two Lie groups such that for each $x \in \tilde{\mathscr{H}}$ the curve $s \mapsto P_{\mathscr{H}}(s)x$ is of class C^1 and moreover $P_{\mathscr{H}}(0)$ is the trivial morphism. Let further $\tilde{A}$, A be Banach spaces and $P(s): \tilde{A} \to A$ a strong C^1-family of linear maps. Assume also that there are smooth maps $\pi: \mathscr{H} \to A$* and *$\bar{\pi}: \tilde{\mathscr{H}} \to \tilde{A}$ such that for each s the diagram*

$$\begin{array}{ccc} \tilde{\mathscr{H}} & \xrightarrow{P_{\mathscr{H}}(s)} & \mathscr{H} \\ \bar{\pi}\downarrow & & \downarrow \pi \\ \tilde{A} & \xrightarrow[P(s)]{} & A \end{array}$$

commutes. Suppose σ_A is a polynomial differential n-form on A with values in a Banach space E. Define the polynomial differential form $\tilde{\sigma}_A$ on $\tilde{A}$ with values in $\tilde{E} = \Gamma^1 E$ by letting $\tilde{\sigma}_A(\gamma_0;\gamma_1,\cdots,\gamma_n)$ be the path in E given by $\tilde{\sigma}_A(\gamma_0;\gamma_1,\cdots,\gamma_n)(s) = (\sigma_A P(s))(\gamma_0;\gamma_1,\cdots,\gamma_n)$ for every $\gamma_0, \gamma_1, \cdots, \gamma_n \in \tilde{A}$. Assume finally that the form $\sigma = \sigma_A \pi$ on $\mathscr{H}$ is closed.

Let $\tilde{D}$ be the contraction operator which maps E-valued forms on A to $\tilde{E}$-valued forms on $\tilde{A}$, as defined in §5.2. We assert that then the form $\tilde{\tau}_A = \tilde{D}\sigma_A$ satisfies $(\tilde{\sigma}_A - d\tilde{\tau}_A)\bar{\pi} = 0$.

Indeed, according to the contraction formula in §5.2, we have on $\tilde{A}$

$$\tilde{\sigma}_A - \sigma_A P(0) = d\tilde{D}\sigma_A + \tilde{D}d\sigma_A,$$

where $\sigma_A P(0)$ is regarded as $\tilde{E}$-valued, due to $E \subset \tilde{E}$. Thus it will suffice now to prove that $\sigma_A P(0)\bar{\pi} = 0 = \tilde{D}d\sigma_A\bar{\pi}$.

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Now $\sigma_A P(0)\bar{\pi} = \sigma_A \pi P_{\mathscr{H}}(0) = 0$ since $P_{\mathscr{H}}(0)$ is the trivial map. Further, for each s, and for any vectors $X_1, \cdots, X_n$ tangent to $\tilde{\mathscr{H}}$ at $x \in \tilde{\mathscr{H}}$,

$$\tilde{D}(d\sigma_A)\bar{\pi}(X_1, \cdots, X_n)(s) = \tilde{D}(d\sigma_A)(\bar{\pi}(x); \bar{\pi}(X_1), \cdots, \bar{\pi}(X_1))(s)$$

$$= D_s(d\sigma_A)(\bar{\pi}(x);\ \bar{\pi}(X_1), \cdots, \bar{\pi}(X_n))$$

$$= \int_0^s d\sigma_A(P(t)\bar{\pi}(x);\ P'(t)\bar{\pi}(x),\ P(t)\bar{\pi}(X_1), \cdots, P(t)\bar{\pi}(X_n))dt.$$

Here $P(t)\bar{\pi}(x) = \pi\ P_{\mathscr{H}}(t)x$ and since the curve $t \mapsto P_{\mathscr{H}}(t)x$ is differentiable, also $P'(t)\bar{\pi}(x) = \pi(P_{\mathscr{H}}(t)x)'$. It follows that for each t, $d\sigma_A$ in the integral is evaluated on $\pi\mathscr{H}$ on π-images of tangent vectors to $\mathscr{H}$. This evaluation gives 0 since $\sigma_A\pi$ is by assumption a closed form.

REMARK. Consider the sequence (*) $0 \to E \subset M \to H \to 0$ of §2. Identifying $x \in E$ with the set of all paths $f \in \Lambda^1 E$ for which $f(1) = x$ we obtain an inclusion $E \subset \Lambda^1 M / \Omega^1 E$, where $\Omega^1 E \subset \Lambda^1 E$ is the subspace of loops. This leads to the exact sequence

$$0 \to E \subset \Lambda^1 M / \Omega^1 E \to \Lambda^1 H \to 0.$$

The foregoing proof can be modified so that it works for this sequence in place of $0 \to \Lambda^1 E \to \Lambda^1 M \to \Lambda^1 H \to 0$ (see beginning of §6). Thus the assumptions taken in the Theorem of §2 imply the faithful representability of $\Lambda^1 M / \Omega^1 E$.

7. An example

We show that the Theorem works in the case of the Lie algebra M mentioned in §1 which does not have a faithful representation. By construction of M (see [3]), the centre E of M is 1-dimensional and $H = M/E$ is the loop algebra Ω^1so(3) of continuously differentiable loops in so(3) based at the origin. Because E is 1-dimensional, any E-valued form may be regarded as a real valued form. There is a projection $\omega_M: M \to E$ such that $\omega_M[X_1, X_2] = \sigma_H(\phi X_1, \phi X_2)$ for X_1, $X_2 \in M$ and σ_H is the skew-symmetric form on $H = \Omega^1$ so(3) defined by

$$\sigma_H(\lambda_1, \lambda_2) = \int_0^1 (\lambda_1(t), \lambda_2'(t))dt,$$

where (,) is the Killing inner product in so(3). We shall now exhibit a Banach algebra B, together with a representation $\rho: H \to B$ and a bilinear functional σ_B on B such that $\sigma_H = \sigma_B \rho$.

Put $B = \Gamma^1 gl(3, R)$. B is a Banach algebra. Further

$$H = \Omega^1 \text{ so}(3) \subset \Gamma^1 gl(3, R)$$

and this natural inclusion is a representation. Extend the Killing norm in so(3) to some norm in $gl(3, R)$ and the Killing inner product to some bilinear form on $gl(3, R)$ and continue to denote this form by (,). Then σ_B defined by

$$\sigma_B(\gamma_1, \gamma_2) = \int_0^1 (\gamma_1(t), \gamma_2'(t))dt$$

is a bilinear functional on B and the restriction of σ_B to Ω^1so(3) is just σ_H.

Hence we may conclude by the Theorem that although M is not faithfully representable, $\Lambda^1 M$ has a faithful representation.

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