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On the use of $\Sigma(n)$ and $\Sigma(n^2)$ in relation to integration.—

1. *The base* AB *of a right-angled triangle* ABC *is divided into* n *equal parts. On these parts as bases* n *outer rectangles* (*Fig. 1*) *and* $(n-1)$ *inner rectangles* (*Fig. 2*) *are drawn. To find* S_n *the sum of the areas of the outer set and* S_n' *the sum of the areas of the inner set.*

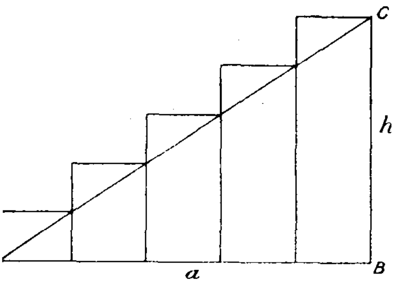

FIG. 1.

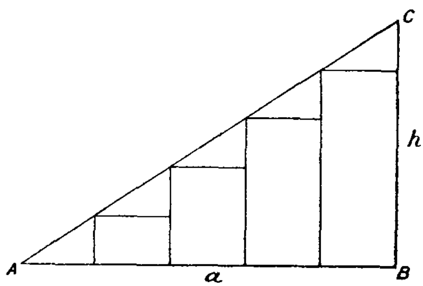

FIG. 2.

The common base is $\frac{a}{n}$, and the heights of the outer set are $\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}, \ldots, h$. S_n comprises triangle ABC plus n small triangles above AC, each of area $\frac{1}{2}\frac{a}{n} \cdot \frac{h}{n}$.

$$\therefore \quad S_n = \tfrac{1}{2}ah + \tfrac{1}{2}\frac{ah}{n}$$

and

$$S_n' = \tfrac{1}{2}ah - \tfrac{1}{2}\frac{ah}{n}.$$

As this simple method is inapplicable when area ABC is unknown, we proceed to consider a general method of summation. Multiplying each height by the base $\frac{a}{n}$ and adding, we get

$$\frac{a}{n}\cdot\frac{h}{n}+\frac{a}{n}\cdot\frac{2h}{n}+\frac{a}{n}\cdot\frac{3h}{n}+\ldots+\frac{a}{n}h$$

$$\therefore\quad S_n=\frac{ah}{n^2}(1+2+3+\ldots+n).$$

$$=\frac{ah}{n^2}\cdot\frac{n(n+1)}{2}=\tfrac{1}{2}ah+\tfrac{1}{2}\frac{ah}{n}.$$

If the rectangles in Fig. 2 be moved along $\frac{a}{n}$ to the left, they will coincide with the first $(n-1)$ rectangles of Fig. 1. Hence S_n' is got from S_n by omitting the last term

$$\therefore\quad S_n'=\frac{ah}{n^2}(1+2+3+\ldots+(n-1)\,)$$

$$=\frac{ah}{n^2}\,\frac{(n-1)n}{2}=\tfrac{1}{2}ah-\tfrac{1}{2}\frac{ah}{n}.$$

If n were a million the second terms in S and S′ would become extremely small, and the two sets of areas could not be distinguished by eye. Hence these sums show that when n is large the term $\frac{1}{2}ah$

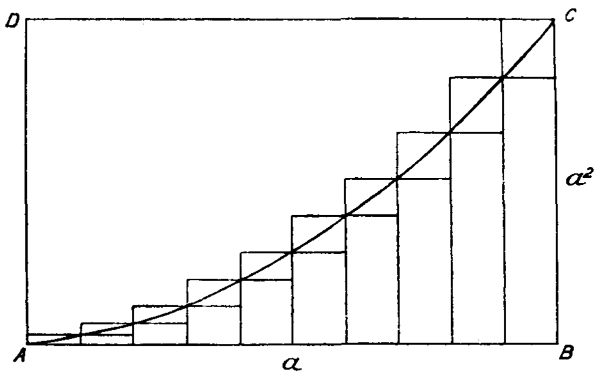

FIG. 3.

is an approximate value of area ABC. In this simple case we know that the "approximation" is really exact.

2. *To find an approximation to the area* ABC *where* AC *is given by* $y=x^2$.

Divide AB into ten equal parts, and draw outer and inner rectangles in one figure. To get the height at any point we must square the value of x at that point. The values of x at the points of division are $\frac{a}{10}, \frac{2a}{10}, \frac{3a}{10}$, etc, so that the heights (outer) are $\left(\frac{a}{10}\right)^2, \left(\frac{2a}{10}\right)^2, \left(\frac{3a}{10}\right)^2$, etc. Multiply by the common base $\frac{a}{10}$ and add.

$$\therefore \quad S_{10}=\frac{a}{10}\cdot\left(\frac{a}{10}\right)^2+\frac{a}{10}\left(\frac{2a}{10}\right)^2+\frac{a}{10}\left(\frac{3a}{10}\right)^2+\dots+\frac{a}{10}(a)^2$$

$$=\frac{a^3}{10^3}(1^2+2^2+3^2+\dots+10^2)$$

$$=\frac{a^3}{10^3}\cdot\frac{10\,.\,11\,.\,21}{6}=\frac{77}{200}a^3.$$

For the inner nine rectangles omit the last term.

$$\therefore \quad S'_{10}=\frac{a^3}{10^3}(1^2+2^2+3^2+\dots+9^2)$$

$$=\frac{a^3}{10^3}\cdot\frac{9\,.\,10\,.\,19}{6}=\frac{57}{200}a^3.$$

The area ABC lies in value between S and S′, not exactly midway, but in this case slightly less since AC is curving upward. Hence $\frac{67}{200}a^3$ is the approximate area of ABC, somewhat on the large side.

We can improve this value by taking n divisions and making n large. The abscissae of the points of division become $\frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}$, etc., and hence the heights are $\left(\frac{a}{n}\right)^2, \left(\frac{2a}{n}\right)^2, \left(\frac{3a}{n}\right)$, etc.

Multiply by $\frac{a}{n}$ and add for the whole area.

$$\therefore \quad S_n=\frac{a}{n}\left(\frac{a}{n}\right)^2+\frac{a}{n}\left(\frac{2a}{n}\right)^2+\frac{a}{n}\left(\frac{3a}{n}\right)^2+\dots+\frac{a}{n}(a)^2$$

$$=\frac{a^3}{n^3}(1^2+2^2+3^2+\dots+n^2)$$

$$=\frac{a^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}=\tfrac{1}{3}a^3+\tfrac{1}{3}a^3\left(\frac{3}{2n}+\frac{1}{2n^2}\right),$$

and $$S_n' = \frac{a^3}{n^3}(1^2+2^2+3^2+\ldots+(n-1)^2)$$

$$= \frac{a^3}{n^3}\cdot\frac{(n-1)n(2n-1)}{6} = \tfrac{1}{3}a^3 - \tfrac{1}{3}a^3\left(\frac{3}{2n}-\frac{1}{2n^2}\right).$$

It is of special importance that these expressions should be put in the second form, *i.e.* a fixed part plus or minus a variable part with n only in the denominator. The mean of S_n and S_n', namely $\frac{1}{3}a^3+\frac{1}{6}\frac{a^3}{n^2}$, will give a better approximation than S_n or S_n', and it will be too large. Hence if we take away the small term $\frac{a^3}{6n^2}$ the remainder $\frac{1}{3}a^3$ must be a good approximation to area ABC when n is large. Adopting this value then we have area ABC $= \frac{1}{3}$ area ABCD.

3. *To find an approximation to the area* ABC *when* AC *is given by* $y=\sqrt{x}$.

Interchange the axes in Fig. 3, then ACD is the area required. Let AD be given equal to c. Then AB $=\sqrt{c}$. But by last result area ACD $=\frac{2}{3}$ area ABCD $=\frac{2}{3}c\sqrt{c}$.

4. *Examples on approximate summation.*

(i) Let the rectangular areas in Figs. 1, 2 make a complete revolution about AB, thus forming sets of circular discs. Find the sums of the volumes in each case when there are ten divisions, and deduce an approximation to the volume of a cone of height a and base radius h.

Find a better approximation by using n divisions and making n large.

(ii) Figs. 1 and 2. Let the "moment" of a rectangle about its base be defined to be the product of its area and the distance of its centre from the base. Find the total moment about AB for both figures when the number of divisions is twelve, and deduce an approximation to the moment of triangle ABC about AB.

(iii) Draw a quadrant of the circle $x^2+y^2=a^2$, and divide the base into six equal parts. Draw the sets of rectangles and let them generate discs by making a complete revolution about the base.

Calculate the sums of these volumes, and find an approximate volume of the hemisphere.

Improve the result by summing for n discs.

(iv) Find the approximate moment of a quadrant of a circle about its base.

(v) In a pile of nine square stone slabs each of thickness 6 inches laid on one another, the side of the top slab measures 4 feet and that of the bottom 10 feet. The sides of the whole nine are in arithmetic progression. Calculate the volume of the pile, and show that it exceeds nine times the volume of the middle slab.

5. *Exact values.*

We saw in § 1 that the "approximation" obtained by taking the fixed term of the series was really exact. In § 2 the figure shows that the trapezoidal area exceeds area ABC just as $\frac{1}{3}a^3 + \frac{1}{6}\frac{a^3}{n^2}$ exceeds $\frac{1}{3}a^3$, the excess in each case being very small when n is very large. Again, denoting by $\triangle$, $\triangle'$ the respective sums of the trianglar-shaped pieces above and below AC, Fig. 3 shows that $\triangle > \triangle'$, just as $\frac{1}{3}a^3\left(\frac{3}{2n} + \frac{1}{2n^2}\right) > \frac{1}{3}a^3\left(\frac{3}{2n} - \frac{1}{2n^2}\right)$, and the *areas* diminish just as these *expressions* diminish with increase of n. Hence the relation between the several *areas* is strictly parallel to that between the several *terms* if $\frac{1}{3}a^3$ is the exact value of area ABC.

THEOREM.—*If* S_n *consists of a fixed term along with a variable term which decreases continually when n increases, the fixed term measures the exact area of* ABC.

In § 2 write

$$S_n = \tfrac{1}{3}a^3 + \tfrac{1}{3}a^3\left(\frac{3}{2n} + \frac{1}{2n^2}\right)$$

$$S'_n = \tfrac{1}{3}a^3 - \tfrac{1}{3}a^3\left(\frac{3}{2n} - \frac{1}{2n^2}\right).$$

Then if area ABC is not equal to $\frac{1}{3}a^3$, let it equal $\frac{1}{3}a^3 + e$ where e is fixed. Since $S_n =$ area ABC $+ \triangle$ we get

$$\triangle = \tfrac{1}{3}a^3\left(\frac{3}{2n} + \frac{1}{2n^2}\right) - e$$

$$\triangle' = \tfrac{1}{3}a^3\left(\frac{3}{2n} - \frac{1}{2n^2}\right) + e.$$

If e is positive we can choose n so large that $\triangle$ becomes negative; if e is negative we can similarly make $\triangle'$ become negative. This is true in every case where the variable term decreases *continually*. In either case the result is absurd geometrically unless $e=0$. Therefore $\mathrm{ABC}=\frac{1}{3}a^3$, the fixed term in S_n or S'_n.

Hence the rule for finding the area enclosed by a given curve, two ordinates and the x-axis; divide the base into n equal parts, write down the areas of one set of rectangles, sum the series, and arrange in the form $a+\left(\frac{b}{n}+\frac{c}{n^2}+\dots\right)$; then a is the required area.

6. *Meaning of certain series.*

(i) The series $1\,.\,1+2\,.\,3+3\,.\,5+\dots+n(2n-1)$. In Fig. 1 let $\mathrm{AB}=\mathrm{BC}=5$ inches so that the rectangles have areas 1, 2, 3, 4, 5 square inches respectively. The distances of their centres from AD are $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$ inches; hence their moments about AD are $1\times\frac{1}{2}$, $2\times\frac{3}{2}$, etc., and the sum

$$=\tfrac{1}{2}(1\,.\,1+2\,.\,3+3\,.\,5+4\,.\,7+5\,.\,9).$$

Thus the series means twice the moment about AD of a system of n rectangles arranged as in Fig. 1. If n be large the whole area is nearly triangular; hence the series may also be interpreted as an approximation to twice the moment of a triangle about a parallel to its base through the vertex.

Cor. Show that the series $1\,.\,2+2\,.\,3+3\,.\,4+\dots+n(n+1)$, and $1^2+2^2+3^2+\dots+n^2$ are *upper* and *lower* approximations to the same area system.

(ii) The series

$$\frac{1}{1(1+h)}+\frac{1}{(1+h)(1+2h)}+\frac{1}{(1+2h)(1+3h)}+\dots+\frac{1}{(1+(n-1)h)(1+nh)}$$

Plot the graph of $\frac{1}{x}$; then the ordinates when $x=1$, $1+h$, $1+2h,\dots$ are $\frac{1}{1}, \frac{1}{1+h}, \frac{1}{1+2h}, \dots$. Draw the set of outer rectangles beginning at $x=1$. Their areas are $\frac{h}{1}, \frac{h}{1+h}, \frac{h}{1+2h}, \dots$. Their centres are distant from OX $\frac{1}{2}\,.\,1$, $\frac{1}{2}\cdot\frac{1}{1+h}$, $\frac{1}{2}\cdot\frac{1}{1+2h}$, ... Hence the total moment about OX is

$$\frac{h}{2}+\frac{h}{2(1+h)^2}+\frac{h}{2(1+2h)^2}+\ldots+\frac{h}{2(1+\overline{n-1}h)^2}.$$

This *exceeds* the moment of the area under the curve, so that if we take the distances from OX too small, namely, $\frac{1}{2}\cdot\frac{1}{1+h}$, $\frac{1}{2}\cdot\frac{1}{1+2h}$, etc., the result will be a better approximation for the curve. In this case the sum is

$$\frac{h}{2(1+h)}+\frac{h}{2(1+h)(1+2h)}+\frac{h}{2(1+2h)(1+3h)}+\ldots$$

or $\frac{h}{2}$ times the proposed series. Hence the series stated means an approximate value of $\frac{2}{h}$ times the moment about OX of the area under the graph of $\frac{1}{x}$ and between the ordinates at $x=1$ and $x=1+nh$.

G. D. C. STOKES.

The equation of the polar of a point with respect to a circle.—The equation of the polar of a point (x', y') with respect to a circle $x^2+y^2=r^2$ can be found very easily by the use of

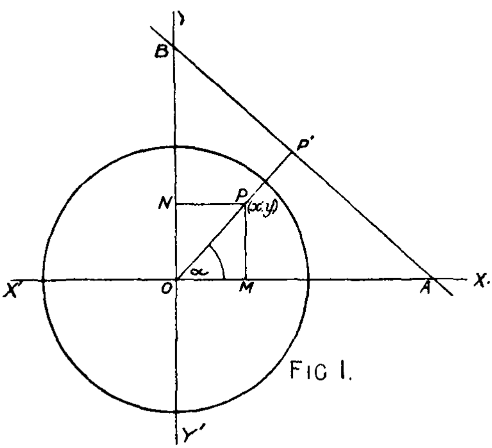

FIG 1.

either the perpendicular or the intercept form of the equation of a straight line.

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