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RESEARCH ARTICLE

Parametrised moduli spaces of surfaces as infinite loop spaces

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Abstract
We study the E_2-algebra $\Lambda\mathfrak{M}_{*,1} := \coprod_{g\geqslant 0}\Lambda\mathfrak{M}_{g,1}$ consisting of free loop spaces of moduli spaces of Riemann surfaces with one parametrised boundary component, and compute the homotopy type of the group completion $\Omega B\Lambda\mathfrak{M}_{*,1}$: it is the product of $\Omega^\infty\mathbf{MTSO}(2)$ with a certain free Ω^∞-space depending on the family of all boundary-irreducible mapping classes in all mapping class groups $\Gamma_{g,n}$ with $g \geqslant 0$ and $n \geqslant 1$.

Contents

1. Introduction

The Madsen–Weiss theorem [14] can be formulated as follows: let $\mathfrak{M}_{g,1}$ denote the moduli space of Riemann surfaces of genus $g \geqslant 0$ with one parametrised boundary curve. By [19] and [4], the collection

$$\mathfrak{M}_{*,1} := \coprod_{g \geqslant 0} \mathfrak{M}_{g,1}$$

admits the structure of an E_2-algebra, more precisely an algebra over the little 2-discs operad $\mathscr{D}_2$. Madsen and Weiss identify the group completion $\Omega B\mathfrak{M}_{*,1}$ with the infinite loop space $\Omega^\infty \mathbf{MTSO}(2)$, where $\mathbf{MTSO}(2)$ is the two-dimensional oriented tangential Thom spectrum [8].

One can consider the analogous problem with $\mathfrak{M}_{*,1}$ replaced by the mapping space $\mathrm{map}(X, \mathfrak{M}_{*,1})$. This space is again a $\mathscr{D}_2$-algebra by pointwise composition, and it is our goal to understand its group completion $\Omega B\,\mathrm{map}(X, \mathfrak{M}_{*,1})$. Note that the $\mathscr{D}_2$-algebra structure extends to an algebra structure over Tillmann's surface operad built out of $\mathfrak{M}_{*,1}$, so the main theorem of [26] implies the homotopy type we wish to understand is an infinite loop space.

In this article, we will focus on the simplest non-trivial case $X = S^1$: i.e., we consider the free loop space $\Lambda\mathfrak{M}_{*,1} := \mathrm{map}(S^1, \mathfrak{M}_{*,1})$; we will briefly discuss in the appendix the general case, which is very similar.

For any discrete group Γ, one can identify $\Lambda B\Gamma \simeq \coprod_{[\gamma] \in \mathrm{Conj}(\Gamma)} BZ(\gamma, \Gamma)$, where $\mathrm{Conj}(\Gamma)$ denotes the set of conjugacy classes of Γ, and $Z(\gamma, \Gamma)$ is the centraliser of $\gamma \in \Gamma$. Note also that the isomorphism type of the group $Z(\gamma, \Gamma)$ only depends on the conjugacy class of $\gamma \in \Gamma$.

The problem we address in this paper is strongly related to analysing the structure of centralisers of elements of mapping class groups: indeed, recall that $\mathfrak{M}_{g,1}$ is a classifying space for the mapping class group $\Gamma_{g,1}$ of a smooth oriented surface of genus g with one parametrised boundary curve; we then have a homotopy equivalence

$$\Lambda\mathfrak{M}_{*,1} \simeq \coprod_{g \geqslant 0} \Lambda B\Gamma_{g,1} \simeq \coprod_{g \geqslant 0} \coprod_{[\varphi] \in \mathrm{Conj}(\Gamma_{g,1})} BZ(\varphi, \Gamma_{g,1}).$$

Results

The free loop space of the moduli space of surfaces of genus g with n parametrised boundary circles, $\Lambda\mathfrak{M}_{g,n}$, admits an action by the isometry group of the disjoint union of n oriented circles: that is, by $T^n \rtimes \mathfrak{S}_n = (S^1)^n \rtimes \mathfrak{S}_n$.

We introduce an irreducibility criterion for mapping classes that is invariant under conjugation. We then consider, for any $n \geqslant 1$ and $g \geqslant 0$, the subspace $\mathfrak{C}_{g,n} \subseteq \Lambda\mathfrak{M}_{g,n}$ of free loops whose corresponding conjugacy classes of elements in $\pi_0(\mathfrak{M}_{g,n}) \cong \Gamma_{g,n}$ are irreducible. The pointwise action of $T^n \rtimes \mathfrak{S}_n$ on $\Lambda\mathfrak{M}_{g,n}$ restricts to an action on $\mathfrak{C}_{g,n}$, and the main result of this work is the following identification, where $/\!\!/$ stands for homotopy quotient.

Theorem 1.1. *There is a weak homotopy equivalence*

$$\Omega B\Lambda\mathfrak{M}_{*,1} \simeq \Omega^\infty \mathbf{MTSO}(2) \times \Omega^\infty \Sigma^\infty_+ \coprod_{n\geqslant 1}\coprod_{g\geqslant 0} \mathfrak{C}_{g,n} /\!\!/ (T^n \rtimes \mathfrak{S}_n).$$

The key to proving this theorem is a good understanding of mapping class groups of surfaces (also with more than one boundary component) as well as an extension of classical operadic techniques to a coloured setting. As a first step, we prove a structure result for centralisers of mapping classes in $\Gamma_{g,n}$, which might be of independent interest: see Proposition 3.8.

Second, we develop a machinery for N-coloured operads with homological stability $\mathscr{O}$ containing a suboperad $\mathscr{P}$, such as a family of topological groups: the group completion of the derived relatively free algebra $\tilde{F}^{\mathscr{O}}_{\mathscr{P}}(\boldsymbol{X})$ over a $\mathscr{P}$-algebra $\boldsymbol{X}$ is computed colourwise as an infinite loop space, under suitable assumptions on $\mathscr{O}$ and $\mathscr{P}$; see Theorem 5.9. This part of the work is a generalisation of [26] and [1] to the coloured and relative setting.

The two ingredients are put together by proving that $\Lambda\mathfrak{M}_{*,1}$ is the colour-1 part of a relatively free algebra over a coloured version $\mathscr{M}$ of Tillmann's surface operad, relative to a sub-operad built out of $T^n \rtimes \mathfrak{S}_n$; the 'relative generators' are precisely the spaces $\mathfrak{C}_{g,n}$ mentioned above: see Theorem 6.5.

Related work

One approach to studying classifying spaces of diffeomorphism groups pertains to the notion of cobordism categories. It was pioneered with the breakthrough theorem by Madsen and Weiss and refined by Galatius, Madsen, Tillmann and Weiss [8].

Recall that, in the orientable setting, the cobordism category Cob_d is a topological category, with object space given by the union of all moduli spaces of closed, oriented $(d-1)$-manifolds, and morphism space given by the union of all moduli spaces of compact, oriented d-manifolds with incoming and outgoing boundary. It is natural to study two related generalisations of Cob_d, in an *equivariant* and a *parametrised* direction:

1. for a (topological) group G, we can consider the G-equivariant cobordism category Cob_d^G: objects and morphisms are, respectively, $(d-1)$- and d-manifolds endowed with an (continuous) action of G by orientation-preserving diffeomorphisms;
2. for a topological space Y, we can consider the Y-parametrised cobordism category $\mathrm{Cob}_d(Y)$: objects and morphisms are, respectively, orientable $(d-1)$- and d-manifold bundles over Y.

In the case $G = \mathbb{Z}$ and $Y = S^1$, there is a continuous functor $\mathrm{Cob}_d^{\mathbb{Z}} \to \mathrm{Cob}_d(S^1)$, given by taking mapping tori: using that every smooth bundle over S^1 is induced from a diffeomorphism, this functor is in fact a levelwise equivalence. For more general groups G and $Y = BG$, the analogous argument pertaining to G-actions and bundles over BG can fail: see [22] for a discussion of this phenomenon and counterexamples in case $G = \mathrm{SU}(2)$.

Our work can be seen as a contribution toward understanding the homotopy type of $\mathrm{Cob}_d^{\mathbb{Z}} \simeq \mathrm{Cob}_d(S^1)$: gluing a pair of pants gives a map of monoids $\Lambda\mathfrak{M}_{*,1} \to \mathrm{Cob}_2(S^1)|_{S^1}$, where $\mathrm{Cob}_2(S^1)|_{S^1}$ denotes the full subcategory of $\mathrm{Cob}_2(S^1)$ on a single object represented by a trivial S^1-bundle over S^1.

In the non-parametrised setting, the composition $\mathfrak{M}_{*,1} \to \mathrm{Cob}_2|_{S^1} \to \mathrm{Cob}_2$ is known to induce an equivalence after taking classifying spaces; we hope that in a similar way, the understanding of $B\Lambda\mathfrak{M}_{*,1}$ can shed some light on the homotopy type of $B\mathrm{Cob}_2^{\mathbb{Z}} \simeq B\mathrm{Cob}_2(S^1)$ in future work.

In the case of a finite group G, the homotopy type of Cob_d^G was recently determined by Szűcs and Galatius [9]. In work by Raptis and Steimle [20], parametrised cobordism categories $\mathrm{Cob}_d(Y)$ featured as a tool to prove index theorems; however, it was not necessary for the scopes of that work to describe the homotopy type of the classifying spaces of these categories. The much older work of Kreck on bordisms of diffeomorphisms [13] can be seen as a description of $\pi_0\big(\mathrm{Cob}_d(S^1)\big)$.

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Outline

In Section 2, we recall Alexander's method concerning arc systems. We apply these concepts to associate with a mapping class $\varphi \in \Gamma_{g,n}$ a canonical decomposition of $\Sigma_{g,n}$ along a system of simple closed curves, called the *cut locus* of φ. The goal of Section 3 is a detailed understanding of centralisers of mapping classes: this uses the canonical decomposition described in the previous section in a crucial way.

In Section 4, we recall some basic definitions and constructions related to coloured operads and introduce the coloured surface operad $\mathcal{M}$. In Section 5, we introduce the notion of a coloured operad with homological stability and prove a levelwise splitting result in the spirit of [1], which applies in particular to $\mathcal{M}$. Finally, in Section 6, we show that $\Lambda\mathfrak{M}_{*,1}$ is the colour-1 part of a relatively free $\mathcal{M}$-algebra, which in combination with the splitting result concludes the proof of Theorem 1.1.

We briefly discuss in Appendix A the analogue of Theorem 1.1 for a general parametrising space X (see Theorem A.2) as well as a weak form of naturality in X of the equivalence (see Theorem A.7); in Appendix B, we address two similar problems concerning group completion of free loop spaces, related to braid groups and symmetric groups, respectively.

2. Arc systems and the cut locus

The aim of this and the next section is to study centralisers in mapping class groups of surfaces. This interest is motivated by the following observation: for $g \geqslant 1$, the space $\Lambda\mathfrak{M}_{g,1} \simeq \Lambda B\Gamma_{g,1}$ has one connected component for each conjugacy class $[\varphi] \in \mathrm{Conj}(\Gamma_{g,1})$; this component is homotopy equivalent to $BZ(\varphi,\Gamma_{g,1})$, where we denote by $Z(\varphi,\Gamma_{g,1}) \subset \Gamma_{g,1}$ the centraliser of φ in $\Gamma_{g,1}$: that is, the subgroup of all mapping classes $\psi \in \Gamma_{g,1}$ commuting with φ.

In this section, we will first introduce some notation for surfaces and mapping class groups and then define the cut locus of a mapping class.

2.1. Surfaces and mapping class groups

We work in the entire article with smooth, oriented surfaces and orientation-preserving diffeomorphisms of surfaces.

Notation 2.1. We usually denote by $\mathcal{S}$ a smooth, compact, oriented, possibly disconnected surface, such that each component of $\mathcal{S}$ has non-empty boundary; we denote the boundary of $\mathcal{S}$ by $\partial\mathcal{S} \subset \mathcal{S}$.

The boundary $\partial\mathcal{S}$ is equipped with a decomposition $\partial\mathcal{S} = \partial^{\mathrm{in}}\mathcal{S} \sqcup \partial^{\mathrm{out}}\mathcal{S}$, into unions of connected components: the *incoming* boundary $\partial^{\mathrm{in}}\mathcal{S}$ is allowed to be empty, whereas each component of $\mathcal{S}$ is required to intersect the *outgoing* boundary in at least one curve; see Figure 1 for an example.

Both parts of the boundary are equipped with an *ordering* and a *parametrisation*: that is, there are preferred diffeomorphisms $\vartheta^{\mathrm{in}}\colon \{1,\dots,n\} \times S^1 \to \partial^{\mathrm{in}}\mathcal{S}$ and $\vartheta^{\mathrm{out}}\colon \{1,\dots,n'\} \times S^1 \to \partial^{\mathrm{out}}\mathcal{S}$, where $n = \#\pi_0(\partial^{\mathrm{in}}\mathcal{S})$ and $n' = \#\pi_0(\partial^{\mathrm{out}}\mathcal{S})$.

Note that each boundary component $c \subset \partial\mathcal{S}$ is endowed with two natural orientations: the first is induced from the orientation of $\mathcal{S}$, that is, it is the unique orientation of c that, concatenated with a

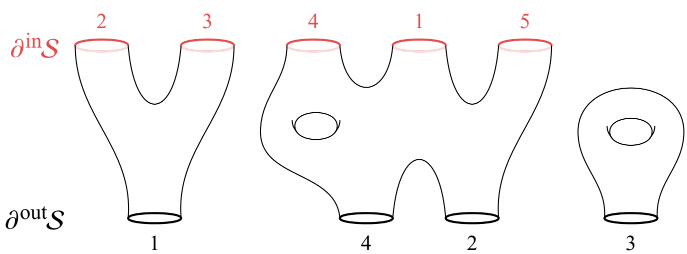

Figure 1. *A surface $\mathcal{S}$ with 5 incoming and 4 outgoing boundary curves.*

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vector field along c pointing out of $\mathcal{S}$, returns the orientation of $\mathcal{S}$; the second orientation comes from the parametrisation of c. For an *incoming* boundary component $c \subset \partial^{\mathrm{in}}\mathcal{S}$, we require that these two orientations coincide, whereas for an *outgoing* boundary component $c \subset \partial^{\mathrm{out}}\mathcal{S}$, we require that these two orientations differ.

We usually denote a surface by $(\mathcal{S}, \vartheta)$, or shortly by $\mathcal{S}$ when it is not necessary to mention the parametrisation of the boundary; here ϑ is the map $\{1, \dots, n + n'\} \times S^1 \to \partial\mathcal{S}$ obtained by concatenation of ϑ^{in} and ϑ^{out}.

Definition 2.2. Let $\Phi\colon (\mathcal{S}, \vartheta) \to (\mathcal{S}', \vartheta')$ be an orientation-preserving diffeomorphism of surfaces. We say that Φ *preserves the boundary parametrisation* if the following conditions hold:

- Φ restricts to diffeomorphisms $\Phi\colon \partial^{\mathrm{in}}\mathcal{S} \xrightarrow{\cong} \partial^{\mathrm{in}}\mathcal{S}'$ and $\Phi\colon \partial^{\mathrm{out}}\mathcal{S} \xrightarrow{\cong} \partial^{\mathrm{out}}\mathcal{S}'$.
- If $n := \#\pi_0(\partial^{\mathrm{in}}\mathcal{S}) = \#\pi_0(\partial^{\mathrm{in}}\mathcal{S}')$ and $n' := \#\pi_0(\partial^{\mathrm{out}}\mathcal{S}) = \#\pi_0(\partial^{\mathrm{out}}\mathcal{S}')$, then there exist permutations $\sigma^{\mathrm{in}} \in \mathfrak{S}_n$ and $\sigma^{\mathrm{out}} \in \mathfrak{S}_{n'}$ such that
  - $(\Phi \circ \vartheta^{\mathrm{in}})(j, \zeta) = (\vartheta')^{\mathrm{in}}(\sigma^{\mathrm{in}}(j), \zeta)$ for each $1 \leqslant j \leqslant n$ and $\zeta \in S^1$;
  - $(\Phi \circ \vartheta^{\mathrm{out}})(j, \zeta) = (\vartheta')^{\mathrm{out}}(\sigma^{\mathrm{out}}(j), \zeta)$ for each $1 \leqslant j \leqslant n'$ and $\zeta \in S^1$.

Note that in the previous definition, we do not require that Φ also preserves the orderings of the incoming and outgoing components of $\partial\mathcal{S}$ and $\partial\mathcal{S}'$: that is, the permutations $\sigma^{\mathrm{in}} \in \mathfrak{S}_n$ and $\sigma^{\mathrm{out}} \in \mathfrak{S}_{n'}$ may be non-trivial. To emphasise this, we distinguish between the words ‘ordering’ and ‘parametrisation’. In Section 4, when introducing the coloured operad $\mathscr{M}$, we will also consider surfaces equipped with a parametrisation of *collar neighbourhoods* of the incoming and the outgoing boundary.

Notation 2.3. For all $g \geqslant 0$ and $n \geqslant 1$, we fix a model surface $\Sigma_{g,n}$: it is a connected surface of genus g with n outgoing and no incoming boundary components. We say that $\mathcal{S}$ is of type $\Sigma_{g,n}$ if there exists a diffeomorphism $\mathcal{S} \to \Sigma_{g,n}$ preserving the boundary parametrisation.

Definition 2.4. The *mapping class group* $\Gamma(\mathcal{S}, \partial\mathcal{S})$ is the group of isotopy classes of diffeomorphisms $\Phi\colon \mathcal{S} \to \mathcal{S}$ that fix the boundary pointwise: that is, $\Phi \circ \vartheta = \vartheta$. Such a Φ is called a *diffeomorphism of* $(\mathcal{S}, \partial\mathcal{S})$. For $\mathcal{S} = \Sigma_{g,n}$, we also write $\Gamma_{g,n}$ for $\Gamma(\mathcal{S}, \partial\mathcal{S})$. We usually denote isotopy classes by small Greek letters φ and use capital Greek letters for diffeomorphisms.

Remark 2.5. Note that any diffeomorphism $\Xi\colon \mathcal{S} \to \mathcal{S}'$ induces an identification of the groups $\Gamma(\mathcal{S}, \partial\mathcal{S}) \cong \Gamma(\mathcal{S}', \partial\mathcal{S}')$ by conjugation with Ξ: the mapping class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$, represented by the diffeomorphism Φ, corresponds to the mapping class $\varphi^{\Xi} \in \Gamma(\mathcal{S}', \partial\mathcal{S}')$, represented by $\Xi \circ \Phi \circ \Xi^{-1}$.

Definition 2.6. Let $\mathfrak{H} \subset \mathfrak{S}_{\pi_0(\partial^{\mathrm{out}}\mathcal{S})} \times \mathfrak{S}_{\pi_0(\partial^{\mathrm{in}}\mathcal{S})}$ be a subgroup, where ‘$\mathfrak{S}$’ denotes the symmetric group on the finite set given as index. We define the *extended mapping class group* $\Gamma^{\mathfrak{H}}(\mathcal{S})$ as the group of isotopy classes of diffeomorphisms $\Phi\colon \mathcal{S} \to \mathcal{S}$ that preserve the orientation of $\mathcal{S}$ and the boundary parametrisation, and permute the boundary components of $\partial^{\mathrm{out}}\mathcal{S}$ and $\partial^{\mathrm{in}}\mathcal{S}$ according to a pair of permutations in $\mathfrak{H}$.

If we take $\mathfrak{H} = \mathfrak{S}_{\pi_0(\partial^{\mathrm{out}}\mathcal{S})} \times \mathfrak{S}_{\pi_0(\partial^{\mathrm{in}}\mathcal{S})}$, we also write $\Gamma(\mathcal{S})$ for the extended mapping class group. If $\mathcal{S} = \Sigma_{g,n}$, we also write $\Gamma^{\mathfrak{H}}_{g,n} = \Gamma^{\mathfrak{H}}(\mathcal{S})$ and $\Gamma_{g,(n)} = \Gamma(\mathcal{S})$ for the extended mapping class groups.

Note that we have an extension $1 \to \Gamma(\mathcal{S}, \partial\mathcal{S}) \to \Gamma^{\mathfrak{H}}(\mathcal{S}) \to \mathfrak{H} \to 1$.

Definition 2.7. If G is a group, we denote by $\mathrm{Conj}(G)$ the set of conjugacy classes of G. For a group element $\gamma \in G$, we denote by $Z(\gamma, G) \subseteq G$ the *centraliser* of γ: that is, the subgroup of all elements $\gamma' \in G$ that commute with γ.

Notation 2.8. We fix, once and for all, for all conjugacy classes in $\mathrm{Conj}(\Gamma_{g,n})$, a representative of the class. We denote by $\mathfrak{g}\colon \Gamma_{g,n} \to \Gamma_{g,n}$ the function of sets assigning to each element of $\Gamma_{g,n}$ the representative of its class.

Definition 2.9. Let $\mathcal{S}$ be a surface. We denote by $\mathfrak{M}(\mathcal{S})$ the moduli space of Riemann structures on $\mathcal{S}$; two Riemann structures on $\mathcal{S}$ are considered equivalent if there is a diffeomorphism $\Psi\colon \mathcal{S} \to \mathcal{S}$ fixing

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$\partial\mathcal{S}$ pointwise and pulling back one Riemann structure to the other. If $\mathcal{S} = \Sigma_{g,n}$, we also write $\mathfrak{M}_{g,n}$ for the moduli space $\mathfrak{M}(\Sigma_{g,n})$.

The hypothesis that every connected component of $\mathcal{S}$ has non-empty boundary implies that $\mathfrak{M}(\mathcal{S})$ is a classifying space for the group $\Gamma(\mathcal{S}, \partial\mathcal{S})$.

2.2. Arcs and the Alexander method

For the rest of this section, we fix a connected surface $\mathcal{S}$ of type $\Sigma_{g,n}$, with $g \geqslant 0$ and $n \geqslant 1$, and focus on the mapping class group $\Gamma(\mathcal{S}, \partial\mathcal{S})$. Given a mapping class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$, we construct a system of simple closed curves on $\mathcal{S}$ cutting $\mathcal{S}$ into two subsurfaces W and Y: the subsurface $W \subset \mathcal{S}$ is the *white* subsurface and is, up to isotopy, the maximal subsurface of $\mathcal{S}$ satisfying the following conditions:

- all connected components of W touch $\partial\mathcal{S}$;
- φ can be represented by a diffeomorphism of $\mathcal{S}$ fixing W pointwise.

We start by recalling some standard facts about embedded arcs in surfaces. The material of this subsection is taken, up to minor changes, from [7]. For the following definition see [7, §1.2.7].

Definition 2.10. An *arc* in $\mathcal{S}$ is a smooth embedding $\alpha\colon [0;1] \hookrightarrow \mathcal{S}$ such that $\alpha^{-1}(\partial\mathcal{S}) = \{0,1\}$ and α is transverse to $\partial\mathcal{S}$. Two arcs are *disjoint* if their images are disjoint (also at the endpoints). An arc is *essential* if it does not cut $\mathcal{S}$ in two parts, one of which is a disc.

Two arcs α and α' are *directly isotopic* if $\alpha(0) = \alpha'(0)$, $\alpha(1) = \alpha'(1)$, and there is an isotopy of embeddings $[0;1] \longrightarrow \mathcal{S}$ that is stationary on $\{0,1\}$ and connects α to α'. Two arcs are *inversely isotopic* if the previous holds after reparametrising one of the two arcs in the opposite direction. Two arcs are *isotopic* if they are directly or inversely isotopic; we write $\alpha \sim \alpha'$ if α and α' are isotopic.

Two arcs α and β are in *minimal position* if they are disjoint at their endpoints, they intersect transversely, and the number of intersection points in $\alpha \cup \beta$ is minimal among all choices of $\alpha' \sim \alpha$ and $\beta' \sim \beta$ with α' and β' transverse.

Note that we only consider isotopy classes of arcs relative to their endpoints; two arcs sharing one endpoint are never considered in minimal position (and, by convention, cannot be isotoped to be in minimal position). In particular, unless $\mathcal{S}$ is a disc, there are more than countably many isotopy classes of essential arcs in $\mathcal{S}$.

If $\chi(\mathcal{S}) = 2 - 2g - n \leqslant 0$, then according to [7, §1.2.7], the following statement holds: given a collection of essential arcs $\alpha_1, \ldots, \alpha_k$ in $\mathcal{S}$, which have all distinct endpoints and are pairwise non-isotopic, one can replace each α_i with an arc $\alpha'_i \sim \alpha_i$ so that $\alpha'_1, \ldots, \alpha'_k$ are pairwise in minimal position. In fact, it suffices to choose a Riemannian metric of constant curvature on $\mathcal{S}$ such that $\partial\mathcal{S}$ is geodesic, and replace each α_i with its geodesic representative relative to the endpoints: the hypothesis on $\chi(\mathcal{S})$ ensures that we get a non-positively curved metric, so that geodesic representatives are unique; moreover, geodesic representatives are automatically pairwise in minimal position.

Among all connected surfaces with non-empty boundary, the only one with positive Euler characteristic is $\Sigma_{0,1}$, that is, the disc: note that the statement holds vacuously also for the disc, which contains no essential arc. The following is a special case of the Alexander method [7, Prop. 2.8].

Proposition 2.11. *Let $\alpha_0, \ldots, \alpha_k$ and β be a collection of essential arcs in $\mathcal{S}$, and assume the following:*

- *all arcs are pairwise in minimal position;*
- *the arcs $\alpha_0, \ldots, \alpha_k$ are pairwise disjoint.*

Let Φ be a diffeomorphism of $(\mathcal{S}, \partial\mathcal{S})$, and suppose that Φ fixes each of $\alpha_0, \ldots, \alpha_k$ and β up to isotopy relative to the endpoints. Then Φ can be isotoped to a diffeomorphism Φ' of $\mathcal{S}$ that fixes $\alpha_0 \cup \cdots \cup \alpha_k \cup \beta$ pointwise.

In Proposition 2.11, one can enhance the requirement on Φ' to be the following: the map Φ' fixes pointwise a small neighbourhood $U \subset \mathcal{S}$ of the union $\alpha_0 \cup \cdots \cup \alpha_k \cup \beta \cup \partial\mathcal{S}$. Here and in the following,

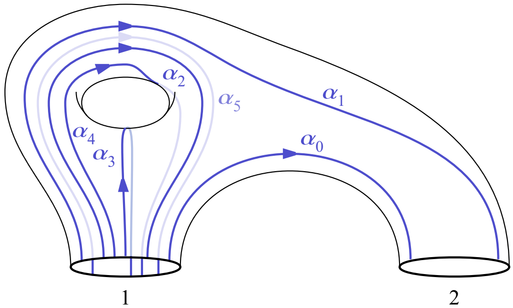

Figure 2. *A maximal collection of six pairwise non-parallel arcs $\alpha_0, \ldots, \alpha_5$ on a surface of type $\Sigma_{1,2}$.*

a *small neighbourhood* is required to deformation retract onto $\alpha_0 \cup \cdots \cup \alpha_k \cup \beta \cup \partial\mathcal{S}$ by restriction of an ambient homotopy, defined on $\mathcal{S}$ and stationary on $\alpha_0 \cup \cdots \cup \alpha_k \cup \beta \cup \partial\mathcal{S}$.

Remark 2.12. The Alexander method, as stated in [7], only applies under the additional hypothesis that the arcs $\alpha_0, \ldots, \alpha_k$ and β are pairwise non-isotopic. We remark, however, that this hypothesis is not essential.

To see this, suppose that $\alpha_0, \ldots, \alpha_k$ and β is a collection of arcs as in Proposition 2.11: up to reordering the arcs α_i, we can assume that there is $0 \leqslant k' \leqslant k$ such that the arcs $\alpha_0, \ldots, \alpha_{k'}$ and β are pairwise non-isotopic, and, moreover, each α_i with $i \geqslant k' + 1$ is isotopic to some α_j with $j \leqslant k'$.

We can then apply the Alexander method to the collection $\alpha_0, \ldots, \alpha_{k'}$ and β, obtaining a diffeomorphism Φ' that fixes a small neighbourhood U of $\alpha_0 \cup \cdots \cup \alpha_{k'} \cup \beta$ pointwise. We then argue as follows: for each index $i \geqslant k' + 1$, there is an index $j \leqslant k'$ such that α_i and α_j are isotopic and in minimal position: this implies that they cobound (together with two segments in $\partial\mathcal{S}$) a rectangle in $\mathcal{S}$, and, up to shrinking, we can assume that this rectangle lies already inside U: that is, we can assume that α_i is fixed pointwise by Φ' as well.

2.3. The cut locus of a mapping class

In this subsection, we fix a class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$, represented by a diffeomorphism Φ, and study the isotopy classes of arcs and curves that it fixes.

Definition 2.13. Two arcs α and α' in $\mathcal{S}$ are *directly parallel* if they are disjoint and there is an *embedding* $[0;1] \times [0;1] \hookrightarrow \mathcal{S}$ restricting to α on $[0;1] \times \{0\}$ and to α' on $[0;1] \times \{1\}$, and restricting to an embedding $\{0,1\} \times [0;1] \hookrightarrow \partial\mathcal{S}$.

Two arcs α and α' are *inversely parallel* if the previous holds after reparametrising one of the two arcs in the opposite direction. Two arcs α and α' are *parallel* if they are directly or inversely parallel.

Note that in the previous definition, we do not insist that the embedding $[0;1] \times [0;1] \hookrightarrow \mathcal{S}$ is orientation-preserving; see Figure 2.

Definition 2.14. The *fixed-arc complex of* φ is an abstract simplicial complex whose vertices are all isotopy classes of essential arcs α in $\mathcal{S}$ fixed by φ. A collection of isotopy classes of arcs $\alpha_0, \ldots, \alpha_k$ spans a k-simplex if the arcs $\alpha_0, \ldots, \alpha_k$ can be isotoped to disjoint, pairwise non-parallel arcs $\alpha'_0, \ldots, \alpha'_k$.

The mapping class φ is called ∂*-irreducible* if its fixed-arc complex is empty and if $\mathcal{S}$ is not of type $\Sigma_{0,1}$.

Example 2.15. Every isotopy class of essential arcs in $\mathcal{S}$ is fixed (up to isotopy) by the identity $\mathbb{1} \in \Gamma(\mathcal{S}, \partial\mathcal{S})$. Therefore $\mathbb{1}$ is not ∂-irreducible, provided that $\mathcal{S}$ admits some essential arc; if $\mathcal{S}$ does not admit essential arcs, then $\mathcal{S}$ is a disc $\Sigma_{0,1}$ and we have prescribed, also in this case, that $\mathbb{1} \in \Gamma_{0,1}$ is *not* ∂-irreducible. The reason to regard $\mathbb{1} \in \Gamma_{0,1}$ as not being ∂-irreducible will become clear later; for the moment we make a simple comparison and say that, in a similar way, $1 \in \mathbb{Z}$ is not considered a prime number.

Example 2.16. The fixed-arc complex of $\varphi \in \Gamma_{0,2} \cong \mathbb{Z}$ is empty if $\varphi \neq \mathbb{1}$ and consists of uncountably many vertices, joined by no higher simplex if $\varphi = \mathbb{1}$. Therefore every non-trivial element in $\Gamma_{0,2}$ is ∂-irreducible.

Example 2.17. For $g \geqslant 1$ the boundary Dehn twist $T_\partial \in \Gamma_{g,1}$ is ∂-irreducible, though there are plenty of isotopy classes of *simple closed curves* in $\Sigma_{g,1}$ that are fixed by T_∂: in fact, *all* simple closed curves are fixed, up to isotopy, by T_∂. Nevertheless, *no isotopy class* of essential arcs is fixed by φ; here it is crucial to consider isotopy classes of arcs relative to the endpoints.

Note that a simplex in the fixed-arc complex of a class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ has dimension at most $-3\chi(\mathcal{S})-1$ if $\chi(\mathcal{S}) < 0$, and at most 0 if $\mathcal{S}$ is of type $\Sigma_{0,2}$. In the second case, just note that each two disjoint essential arcs in $\Sigma_{0,2}$ are parallel. In the first case, let $\alpha_0, \dots, \alpha_k$ be disjoint, pairwise non-parallel and essential arcs in $\mathcal{S}$; then cutting $\mathcal{S}$ along the arcs α_i yields a surface whose connected components are either hexagons or connected surfaces of negative Euler characteristic. Up to adjoining more arcs (and thus increase k), we can assume to have only hexagons. Each hexagon has 3 sides coming from $\partial\mathcal{S}$ and 3 sides coming from the cuts. If $\ell \geqslant 1$ denotes the number of hexagons, we have $3\ell = 2(k+1)$, as each arc contributes to 2 hexagons; moreover $\chi(\mathcal{S}) = \ell - (k+1)$, implying $3\chi(\mathcal{S}) = 3\ell - 3(k+1) = -k-1$.

Construction 2.18. *Let $\mathcal{S}$ be not of type $\Sigma_{0,1}$, let $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$, and let $\alpha_0, \dots, \alpha_k$ be disjoint, essential, pairwise non-parallel arcs in $\mathcal{S}$, representing a maximal simplex in the fixed-arc complex of φ. Let U be a closed, small neighbourhood of the union $\alpha_0 \cup \cdots \cup \alpha_k \cup \partial\mathcal{S}$. The complement $\mathcal{S} \setminus U$ consists of many regions, some of which may be discs: let $W \subset \mathcal{S}$ denote the union of U and all discs in $\mathcal{S} \setminus U$. Then W is a closed, possibly disconnected subsurface of $\mathcal{S}$; we denote by Y the closure of $\mathcal{S} \setminus W$. If ∂W denotes the union of all boundary components of W, and ∂Y denotes the union of all boundary components of Y, then ∂W takes the form $\partial\mathcal{S} \cup c_1 \cup \cdots \cup c_h$, for some $h \geqslant 0$ and some curves $c_1, \dots, c_h \subset \mathcal{S}$; similarly $\partial Y = c_1 \cup \cdots \cup c_h$. The curves $c_1, \dots, c_h$ inherit a canonical boundary orientation from Y, which is oriented as subsurface of $\mathcal{S}$.*

Definition 2.19. For φ and $\alpha_0, \dots, \alpha_k$ as above, we define the *cut locus* of φ, relative to the simplex $\alpha_0, \dots, \alpha_k$, as the isotopy class of the multicurve $c_1, \dots, c_h$, denoted $[c_1, \dots, c_h]$. Here and in the following, a *multicurve* is an unordered collection of disjoint and *oriented* simple closed curves, and two multicurves are considered isotopic if there is an ambient isotopy bringing the first to the second.

The two regions W and Y are called the associated *white* and *yellow* regions or subsurfaces of $\mathcal{S}$, and they depend, as subsets of $\mathcal{S}$, on a choice of a multicurve representing the cut locus. If $\varphi = \mathbb{1} \in \Gamma_{0,1}$, we declare the cut locus to be empty and W to be the entire surface $\Sigma_{0,1}$.

See Figure 3 for an example of the cut locus of a mapping class obtained as a simple product of Dehn twists. Note that it is possible that two curves c_i and c_j cobound a cylinder in Y: in this case, the two curves are isotopic as non-oriented simple closed curves, but the isotopy bringing c_i to c_j, spanned by the cylinder of Y, ends with an orientation-reversing diffeomorphism $c_i \cong c_j$, as the two curves inherit their orientation from Y while being on *opposite* sides of the cylinder contained in Y. Hence the isotopy classes of the curves $c_1, \dots, c_h$, considered as oriented curves, are all distinct.

Note that the cut locus of the identity $\mathbb{1} \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ is *empty*, and the white and yellow decomposition consists of a white region $W = \mathcal{S}$ and an empty yellow region. Vice versa, the cut locus of a non-trivial mapping class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ is always non-empty.

Definition 2.19 depends a priori on a choice of a maximal simplex in the fixed-arc complex of φ; there is, moreover, a subtle detail that we should check to guarantee that Definition 2.19 is well-posed: suppose that the disjoint arcs $\alpha'_0, \dots, \alpha'_k$ are isotopic to the disjoint arcs $\alpha_0, \dots, \alpha_k$ (that is, $\alpha'_i \sim \alpha_i$ for all $0 \leqslant i \leqslant k$), so that the two collections of arcs represent the same maximal simplex in the fixed-arc complex of φ; then we need to check that the two collections of arcs give rise to the same collection of isotopy classes of oriented, disjoint simple closed curves $c_1, \dots, c_h$.

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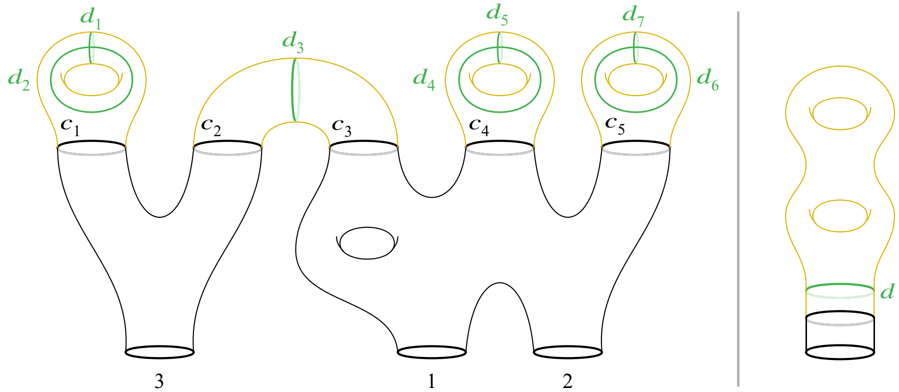

Figure 3. *Two examples of a decomposition into the 'yellow' and the 'white' region according to a fixed mapping class φ. In the first case, φ is given by the product of the Dehn twists along the curves $d_1, \dots, d_7$, and in the second case, it is just the Dehn twist along the single green curve d. In the second case, the mapping class φ is ∂-irreducible, the cut locus consists of the only isotopy class of d, oriented as a boundary of the yellow region, and the white region is just a collar neighbourhood of $\partial\mathcal{S}$.*

We will prove directly that the cut locus only depends on φ, and not on the chosen *maximal* simplex (and its representative) in the fixed-arc complex of φ.

Lemma 2.20. *Let φ and $\alpha_0, \dots, \alpha_k$ be as in Definition 2.19, and let β be an arc whose endpoints are disjoint from the endpoints of $\alpha_0, \dots, \alpha_k$. Suppose that the isotopy class of β is fixed by φ; then β can be isotoped, relative to its endpoints, to an arc $\hat{\beta}$ lying in a small neighbourhood U of $\alpha_0 \cup \dots \cup \alpha_k \cup \partial\mathcal{S}$.*

Proof. Up to isotoping β to another arc, we can assume that β is in minimal position with respect to the arcs $\alpha_0, \dots, \alpha_k$. By Proposition 2.11, we can represent φ by a diffeomorphism Φ that fixes a closed neighbourhood U' of the union $\alpha_0 \cup \dots \cup \alpha_k \cup \beta \cup \partial\mathcal{S}$. Let $U \subset U'$ be a small neighbourhood of $\alpha_0 \cup \dots \cup \alpha_k \cup \partial\mathcal{S}$.

If β is disjoint from the arcs α_i, by maximality of the simplex $\alpha_0, \dots, \alpha_k$ in the fixed-arc complex of φ, we obtain that either β is not essential (and can then be isotoped inside a small neighbourhood of $\partial\mathcal{S}$, hence inside U) or β is parallel to one of the arcs α_i (and can then be isotoped to a small neighbourhood of $\partial\mathcal{S} \cup \alpha_i$, hence inside U).

If β is not disjoint from the arcs α_i, let $\ell \geqslant 1$ be the number of transverse intersections of β with $\alpha_0 \cup \dots \cup \alpha_k$: by induction, let us suppose that the statement of the lemma holds whenever β is replaced by an arc that can be isotoped so as to have at most $\ell - 1$ intersection points with the arcs α_i. Suppose that p is one intersection point of β with one arc α_i: suppose further that the segment $[\alpha_i(0); p] \subset \alpha_i$ contains no other point of $\alpha_i \cap \beta$ in its interior (this means that p is an *outermost* point of $\alpha_i \cap \beta$ along α_i).

We can operate a surgery on β and produce two arcs β' and β'' also contained in U' and transverse to $\alpha_0 \cup \dots \cup \alpha_k$; see Figure 4: the arc β' is obtained by smoothing the concatenation of the segments $[\beta(0); p] \subset \beta$ and $[p; \alpha_i(0)] \subset \alpha_i$, whereas the arc β'' is obtained by smoothing the concatenation of the segments $[\alpha_i(0); p] \subset \alpha_i$ and $[p; \beta(1)] \subset \beta$. We assume that β' and β'' are disjoint and have an endpoint on a small interval of $\partial\mathcal{S}$ centred at $\alpha_i(0)$, on opposite sides with respect to $\alpha_i(0)$.

Both β' and β'' are contained in U', hence they are fixed pointwise by Φ: in particular the isotopy classes of β' and β'' are fixed by Φ. Moreover, each of β' and β'' has strictly less than ℓ intersections with $\alpha_0 \cup \dots \cup \alpha_k$, and hence, by inductive hypothesis, each of β' and β'' can be isotoped to lie in U, relative to its endpoints.

Let $\hat{\beta}'$ and $\hat{\beta}''$ be the two arcs obtained in this way, and assume that $\hat{\beta}'$ and $\hat{\beta}''$ are transverse. If $\hat{\beta}'$ and $\hat{\beta}''$ are not in minimal position, they must form some bigon in $\mathcal{S}$; the possibility of half-bigons in the sense of [7, §1.2.7] is irrelevant, since we consider arcs up to isotopy relative to the

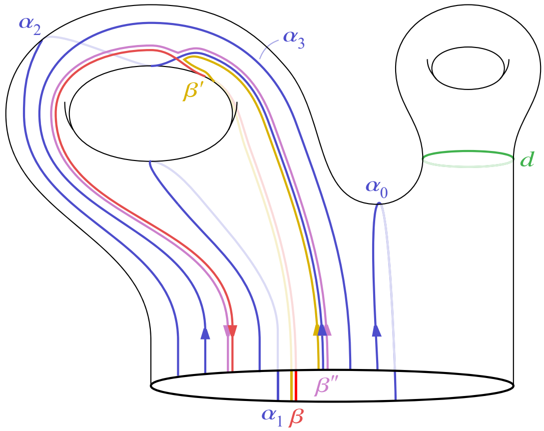

Figure 4. *If φ is the Dehn twist along the curve d, then the blue arcs $\alpha_0, \ldots, \alpha_3$ constitute a maximal simplex in the fixed-arc complex of φ; the subset U is a small neighbourhood of the union of the blue arcs and the black boundary curve. The red arc β intersects α_2 transversally, and the surgery produces the yellow and violet arcs β' and β''.*

endpoints. Clearly, we can simplify all bigons formed by $\hat{\beta}'$ and $\hat{\beta}''$ without losing that these two arcs are contained in U.

Suppose therefore that $\hat{\beta}'$ and $\hat{\beta}''$ are in minimal position: then they are disjoint because β' and β'' were disjoint (in particular, it is automatic that $\hat{\beta}'$ and $\hat{\beta}''$ do not form half-bigons). The arc β is homotopic, relative to its endpoints, to the concatenation of the arcs $\hat{\beta}'$ and $\hat{\beta}''$, which can be connected using a small segment near $\alpha_i(0)$. Note that this concatenation gives an embedded arc $\hat{\beta}$ with the same endpoints as β; since homotopic arcs relative to their endpoints are also isotopic relative to their endpoints, we have that β is isotopic to $\hat{\beta}$; moreover, $\hat{\beta}$ lies in U. □

Proposition 2.21. *Let $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$, and let $\alpha_0, \ldots, \alpha_k$ and $\beta_0, \ldots, \beta_{k'}$ be two sequences of arcs representing two different maximal simplices in the fixed-arc complex of φ; we assume that all arcs $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_{k'}$ are in minimal position. Then the associated cut loci constitute the same isotopy class of an oriented multicurve in $\mathcal{S}$.*

Proof. Let U_α be a closed small neighbourhood of $\alpha_0 \cup \cdots \cup \alpha_k \cup \partial\mathcal{S}$ and U_β be a closed small neighbourhood of $\beta_0 \cup \cdots \cup \beta_{k'} \cup \partial\mathcal{S}$; let W_α and W_β be obtained from U_α and U_β by adjoining the disc components of $\mathcal{S} \setminus U_\alpha$ and $\mathcal{S} \setminus U_\beta$, respectively (see Definition 2.19).

By Lemma 2.20, we can find an isotopy of the identity of $(\mathcal{S}, \partial\mathcal{S})$ bringing U_α in the interior of U_β, and hence in the interior of W_β: without loss of generality, in the following assume $U_\alpha \subseteq U_\beta \subseteq W_\beta$. If D is a disc component of $\mathcal{S} \setminus U_\alpha$, then $D \setminus U_\beta$ is a union of discs contained in $\mathcal{S} \setminus U_\beta$, and therefore $D \subset W_\beta$. It follows that $W_\alpha \subseteq W_\beta$. Since every component of W_β touches $\partial\mathcal{S}$, the map $\pi_0(\partial\mathcal{S}) \to \pi_0(W_\beta)$ is surjective. This map factors through the canonical map $\pi_0(W_\alpha) \to \pi_0(W_\beta)$, as $\partial\mathcal{S} \subset W_\alpha$. We conclude that $\pi_0(W_\alpha) \to \pi_0(W_\beta)$ is surjective.

By the same argument we can find an isotopy of the identity of $\mathcal{S}$ bringing U_β in the interior of U_α, and hence W_β in the interior of W_α: as a consequence we can obtain a surjection $\pi_0(W_\beta) \to \pi_0(W_\alpha)$, showing that $\#\pi_0(W_\beta) = \#\pi_0(W_\alpha)$ and that the map $\pi_0(W_\alpha) \to \pi_0(W_\beta)$ induced by the inclusion is in fact a bijection.

We next prove that each component V of $W_\beta \setminus W_\alpha$ is a cylinder with one boundary curve equal to some c_i and one boundary curve equal to some c'_j. Fix a component $\bar{W}_\beta \subset W_\beta$, and let $\bar{W}_\alpha \subset W_\alpha$ be the unique component of W_α contained in $\bar{W}_\beta$. We know that, conversely, $\bar{W}_\beta$ can be embedded in $\bar{W}_\alpha$: since the genus is weakly increasing along embeddings of orientable surfaces with boundary, the surfaces

$\bar{W}_\alpha$ and $\bar{W}_\beta$ must have the same genus; this implies in particular every component V of $\bar{W}_\beta \setminus \bar{W}_\alpha$ has genus 0 and touches at most one curve $c_i \in \partial\bar{W}_\alpha$.

Since $\bar{W}_\alpha$ and $\bar{W}_\beta$ are connected, we have that a component V of $\bar{W}_\beta \setminus \bar{W}_\alpha$ touches exactly one curve $c_i \subset \partial\bar{W}_\alpha$, and since V cannot be a disc, we obtain that V is a surface of genus 0 with at least two boundary components; more precisely, one boundary component of V is a curve $c_i \subset \partial\bar{W}_\alpha$, and all other boundary components are curves $c'_j \subset \partial\bar{W}_\beta$.

This proves in particular that the number of boundary components of $\bar{W}_\beta$ is greater or equal to the number of boundary components of $\bar{W}_\alpha$; we can again reverse the rôles of α and β and conclude that $\bar{W}_\alpha$ and $\bar{W}_\beta$ have the same number of boundary component; this in turn implies that every connected component V of $\bar{W}_\beta \setminus \bar{W}_\alpha$ is a cylinder with one boundary curve equal to some c_i and one boundary curve equal to some c'_j.

Thus, we obtain a bijection between the curves $c_1, \ldots, c_h$ and the curves $c'_1, \ldots, c'_{h'}$, showing in particular that $h = h'$; note also that the bijection associates with each curve c_i a curve c'_j in the same isotopy class of oriented curves. □

When referring to the *cut locus* of a mapping class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$, we will henceforth mean the cut locus of φ with respect to any maximal simplex in the fixed-arc complex of φ.

The cut locus of a mapping class behaves well under conjugation of the mapping class, as explained in the following Lemma.

Lemma 2.22. *Let $\varphi, \psi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ be mapping classes, let Ψ be a diffeomorphism representing ψ, and let $[c_1, \ldots, c_h]$ be the cut locus of φ; then $[\Psi(c_1), \ldots, \Psi(c_h)]$ is the cut locus of $\psi\varphi\psi^{-1}$. In particular, if φ and ψ commute, then ψ preserves the cut locus of φ as an unordered collection of isotopy classes of oriented simple closed curves.*

Proof. Let $\alpha_0, \ldots, \alpha_k$ be disjoint arcs representing a maximal simplex in the fixed-arc complex of φ, and represent φ by a diffeomorphism Φ fixing pointwise a small neighbourhood U of $\alpha_0 \cup \cdots \cup \alpha_k \cup \partial\mathcal{S}$; finally, represent the cut locus of φ by curves $c_1, \ldots, c_h$ contained in U. Then Ψ induces an isomorphism from the fixed-arc complex of φ to the fixed arc complex of $\psi\varphi\psi^{-1}$; in particular $\Psi(\alpha_0), \ldots, \Psi(\alpha_k)$ are disjoint arcs representing a maximal simplex in the fixed-arc complex of $\psi\varphi\psi^{-1}$. Moreover, $\Psi(U)$ is a small neighbourhood of $\Psi(\alpha_0) \cup \cdots \cup \Psi(\alpha_k) \cup \partial\mathcal{S}$, which is fixed pointwise by the representative $\Psi\Phi\Psi^{-1}$ of $\psi\varphi\psi^{-1}$. We can represent the cut locus of φ by curves $c_1, \ldots, c_h \subset \partial U$; then

$$\Psi(c_1), \ldots, \Psi(c_h) \subset \partial\Psi(U)$$

are automatically curves representing the cut locus of $\psi\varphi\psi^{-1}$, according to Construction 2.18. □

3. Centralisers of mapping classes

We fix a surface $\mathcal{S} \cong \Sigma_{g,n}$ and a mapping class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S}) \cong \Gamma_{g,n}$ as in the previous section. In this section, we prove a structural result for the centraliser $Z(\varphi, \Gamma(\mathcal{S}, \partial\mathcal{S}))$ of φ in $\Gamma(\mathcal{S}, \partial\mathcal{S})$; see Proposition 3.8.

3.1. Yellow components, similarity and irreducibility

We fix oriented simple closed curves $c_1, \ldots, c_h \subset \mathcal{S}$ representing the cut locus of φ, and we let $W \cup Y$ be the associated decomposition of $\mathcal{S}$ into its white and yellow regions.

Recall from Construction 2.18 that the curves $c_1, \ldots, c_h$ inherit an orientation from Y: we fix an orientation-compatible parametrisation of each curve c_i: that is, an identification with S^1. Note that in this way $c_1, \ldots, c_h$ are *incoming* curves for W and *outgoing* for Y. In fact, we have $\partial Y = \partial^{\mathrm{out}} Y = c_1 \cup \cdots \cup c_h = \partial^{\mathrm{in}} W$, whereas $\partial^{\mathrm{out}} W = \partial^{\mathrm{out}}\mathcal{S} = \partial\mathcal{S}$.

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We fix a representative $\Phi\colon \mathcal{S} \to \mathcal{S}$ of φ, which fixes the white region W pointwise: to see that this is possible, choose arcs $\alpha_0, \dots, \alpha_k$ in a maximal simplex of the fixed-arc complex of φ, choose a first representative Φ' of φ fixing pointwise a small neighbourhood U of $\alpha_0 \cup \dots \cup \alpha_k \cup \partial\mathcal{S}$, construct W and Y starting from U, and use that the group $\mathrm{Diff}(D^2, \partial D^2)$ of diffeomorphisms of a disc relative to its boundary is contractible, and in particular connected, in order to isotope Φ' relative to U to a diffeomorphism Φ fixing W pointwise.

We note that this representative Φ is unique up to an isotopy that is stationary on W: to see this, first note that there is a fibration

$$\mathrm{Diff}(Y, \partial Y) \hookrightarrow \mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}) \xrightarrow{\mathfrak{p}} \mathrm{Emb}_{\partial^{\mathrm{out}}}(W, \mathcal{S}),$$

where $\mathrm{Emb}_{\partial^{\mathrm{out}}}(W, \mathcal{S})$ denotes the space of embeddings of W into $\mathcal{S}$ restricting to the identity on the boundary $\partial^{\mathrm{out}} W = \partial\mathcal{S}$. A result of Earle–Schatz [5] ensures that $\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S})$ has contractible components for every compact orientable surface $\mathcal{S}$ such that every connected component of $\mathcal{S}$ is connected; in particular, $\mathrm{Diff}(Y, \partial Y)$ also has contractible components. A result of Gramain [10, Thm. 5] ensures that for disjoint, properly embedded arcs $\alpha_0, \dots, \alpha_k \subset \mathcal{S}$, the space $\mathrm{Emb}_{\coprod \partial\alpha_i}(\coprod \alpha_i, \mathcal{S})$ has also contractible components; this, together with contractibility of $\mathrm{Diff}(D^2, \partial D^2)$, implies that also $\mathrm{Emb}_{\partial^{\mathrm{out}}}(W, \mathcal{S})$ has contractible connected components. Thus, all spaces involved in the above fibration have contractible connected components; in particular the component $\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S})_\varphi$ intersects the fibre $\mathfrak{p}^{-1}(W \subset \mathcal{S}) \cong \mathrm{Diff}(Y, \partial Y)$ in a connected or empty subspace, and the representative Φ of φ witnesses that this intersection is non-empty, hence contractible, in particular connected.

For each path component $P \subseteq Y$, the diffeomorphism Φ restricts to $\Phi|_P\colon P \to P$, giving an element $\varphi_P \in \Gamma(P, \partial P)$.

Definition 3.1. Two path components P and P' of Y are *similar* if there is a diffeomorphism $\Xi\colon P \to P'$ preserving the boundary parametrisation and such that $\varphi_P = (\varphi_{P'})^{\Xi}$. Note that the path components of ∂P are not equipped with a preferred order, as well as the path components of $\partial P'$; yet Definition 2.2 is meaningful here. See also Remark 2.5.

Notation 3.2. We write $Y = \coprod_{i=1}^{r} \coprod_{j=1}^{s_i} Y_{i,j}$, where $Y_{1,1}, \dots, Y_{r,s_r} \subseteq Y$ are the connected components of Y and $Y_{i,j}$ is similar to $Y_{i',j'}$ if and only if $i = i'$. We also let $Y_i := \coprod_{j=1}^{s_i} Y_{i,j}$. Here $r \geqslant 0$ is the number of similarity classes of components of Y (it can be 0 if Y is empty, i.e., if φ is the identity mapping class), whereas $s_i \geqslant 1$ is the number of components of Y belonging to the i^{th} similarity class.

For each $1 \leqslant i \leqslant r$, there are unique $g_i \geqslant 0$ and $n_i \geqslant 1$ such that $Y_{i,j}$ is of type Σ_{g_i,n_i}. We denote by $\varphi_{i,j} \in \Gamma(Y_{i,j}, \partial Y_{i,j})$ the class represented by the restriction $\Phi|_{Y_{i,j}}$, Moreover, we fix diffeomorphisms $\Xi_{i,j}\colon Y_{i,j} \to \Sigma_{g_i,n_i}$ and assume that $\Xi_{i,j}$ preserves the boundary parametrisation.

The conjugation by $\Xi_{i,j}$ induces an identification $\Gamma(Y_{i,j}, \partial Y_{i,j}) \to \Gamma_{g_i,n_i}$, under which $\varphi_{i,j}$ corresponds to some element $\bar{\varphi}_{i,j} := (\varphi_{i,j})^{\Xi_{i,j}} \in \Gamma_{g_i,n_i}$. Up to replacing $\Xi_{i,j}$ by another diffeomorphism $Y_{i,j} \to \Sigma_{g_i,n_i}$, we can assume that $\bar{\varphi}_{i,j} \in \Gamma_{g,n}$ *coincides* with $\mathfrak{g}(\bar{\varphi}_{i,j})$: that is, it is the representative of its own conjugacy class (see Notation 2.8). Note that the diffeomorphism replacing $\Xi_{i,j}$ is not required to induce the same bijection of sets of boundary components as $\Xi_{i,j}$. It can be helpful to remark that $\pi_0(\partial Y_{i,j})$ is not equipped a priori with a preferred order, and only after choosing $\Xi_{i,j}$, we obtain an order on $\pi_0(\partial Y_{i,j})$ by pulling back the canonical order on $\pi_0(\partial\Sigma_{g_i,n_i})$. Under the assumption that the diffeomorphisms $\Xi_{i,j}$ are well-chosen, we also have $\bar{\varphi}_{i,j} = \bar{\varphi}_{i,j'}$ for all $1 \leqslant i \leqslant r$ and $1 \leqslant j, j' \leqslant s_i$. We therefore write $\bar{\varphi}_i := \bar{\varphi}_{i,j}$.

Note that, counting the components of ∂Y, we obtain $h = \sum_{i=1}^{r} n_i \cdot s_i$.

Lemma 3.3. *In the situation above, for each* $1 \leqslant i \leqslant r$ *and* $1 \leqslant j \leqslant s_i$*, we have that* $\varphi_{i,j} \in \Gamma(Y_{i,j}, \partial Y_{i,j})$ *is* ∂*-irreducible.*

Proof. Suppose that there is an essential arc $\beta \subset Y_{i,j}$ (that is, the endpoints of β are on $\partial Y_{i,j}$) that is fixed up to isotopy by $\varphi_{i,j}$. Then we can isotope Φ relative to $\mathcal{S} \setminus \mathring{Y}_{i,j}$ so that Φ fixes β pointwise; without loss of generality, we assume that Φ already fixes β pointwise.

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We can extend β to an arc $\alpha \subset \mathcal{S}$ with endpoints on $\partial\mathcal{S}$ by joining the endpoints of β inside W with $\partial\mathcal{S}$. Then Φ fixes α pointwise. By Lemma 2.20, we can isotope α into the region W. This implies that α is not in minimal position with respect to $\partial Y_{i,j}$, and therefore α must form a bigon with the multicurve $\partial Y_{i,j}$. Since α intersects $\partial Y_{i,j}$ in exactly two points, namely the endpoints of β, there must be a bigon with one side equal to β and the other contained in $\partial Y_{i,j}$. This bigon is contained in $Y_{i,j}$, contradicting the assumption that β is essential in $Y_{i,j}$. $\square$

3.2. *The group* $\tilde{Z}(\varphi)$ *and its relation to* $Z(\varphi, \Gamma(\mathcal{S}, \partial\mathcal{S}))$

In this subsection, we introduce a certain group $\tilde{Z}(\varphi)$ built out of small mapping class groups and symmetric groups. The peculiarity of $\tilde{Z}(\varphi)$ is that it admits a natural map $\varepsilon\colon \tilde{Z}(\varphi) \to Z(\varphi, \Gamma(\mathcal{S}, \partial\mathcal{S})) \subset \Gamma(\mathcal{S}, \partial\mathcal{S})$. In the next subsection, we will identify the kernel of ε, and we will prove in the final subsection that ε is surjective.

Recall that $\varphi_Y \in \Gamma(Y, \partial Y)$ denotes the mapping class represented by $\Phi|_Y$. We consider the centraliser $Z(\varphi_Y) \subset \Gamma(Y)$ of φ_Y in the extended mapping class group $\Gamma(Y)$ using the natural inclusion $\Gamma(Y, \partial Y) \subset \Gamma(Y)$. Note that $\Gamma(Y)$ admits a natural map to $\mathfrak{S}_h \cong \mathfrak{S}_{\pi_0(\partial Y)}$ given by the action of mapping classes on boundary components. This map restricts to a map $Z(\varphi_Y) \to \mathfrak{S}_h$.

Similarly, we can consider the extended mapping class group $\Gamma^{\mathfrak{S}_h}(W)$ that contains mapping classes of W that fix $\partial^{\mathrm{out}} W = \partial\mathcal{S}$ pointwise but may permute the h incoming boundary components of W: here we identify $\mathfrak{S}_h \cong \mathfrak{S}_{\pi_0(\partial^{\mathrm{in}} W)} \subset \mathfrak{S}_{\pi_0(\partial^{\mathrm{out}} W)} \times \mathfrak{S}_{\pi_0(\partial^{\mathrm{in}} W)}$.

Definition 3.4. We define $\tilde{Z}(\varphi)$ as the fibre product

$$\tilde{Z}(\varphi) := \Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} Z(\varphi_Y).$$

Gluing Y and W along $\partial Y = \partial^{\mathrm{in}} W$ yields a map of groups

$$\hat{\varepsilon}\colon \Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} \Gamma(Y) \to \Gamma(\mathcal{S}, \partial\mathcal{S}).$$

Explicitly, for a couple of mapping classes (ψ_W, ψ_Y), we choose representatives $\Psi_W : W \to W$ and $\Psi_Y : Y \to Y$. The fact that ψ_W and ψ_Y project to the same permutation of $\pi_0(\partial^{\mathrm{in}} W) = \pi_0(\partial Y) \cong \{1, \dots, h\}$, together with the fact that both Ψ_W and Ψ_Y preserve the boundary parametrisation, implies that $\Psi_Y|_{\partial Y} = \Psi_W|_{\partial^{\mathrm{in}} W}$, and hence we can glue the two diffeomorphisms to a diffeomorphism of $\mathcal{S}$ (we skip all details about smoothing the output homeomorphism near the gluing curves).

Lemma 3.5. *The restriction* $\varepsilon = \hat{\varepsilon}|_{\tilde{Z}(\varphi)}$ *has image inside* $Z(\varphi, \Gamma(\mathcal{S}, \partial\mathcal{S})) \subset \Gamma(\mathcal{S}, \partial\mathcal{S})$.

Proof. Note that $(\mathbb{1}_W, \varphi_Y)$ is a central element of $\tilde{Z}(\varphi)$: indeed, given a pair $(\psi_W, \psi_Y) \in \tilde{Z}(\varphi)$, we have that $\psi_Y \in Z(\varphi_Y)$, so ψ_Y commutes with φ_Y, and clearly $\mathbb{1}_W$ commutes with ψ_W in $\Gamma^{\mathfrak{S}_h}(W)$. Applying ε, we obtain that $\varepsilon(\psi_W, \psi_Y)$ commutes with $\varepsilon(\mathbb{1}_W, \varphi_Y) = \varphi$. $\square$

In the following lemma, we decompose $Z(\varphi_Y)$, which is the second factor appearing in the formula for $\tilde{Z}(\varphi)$.

Lemma 3.6. *There is an isomorphism of groups*

$$Z(\varphi_Y) \cong \prod_{i=1}^{r} Z(\bar{\varphi}_i) \wr \mathfrak{S}_{s_i},$$

where $Z(\bar{\varphi}_i) \subset \Gamma_{g_i,(n_i)}$ *is the centraliser in the extended mapping class group, and where* $Z(\bar{\varphi}_i) \wr \mathfrak{S}_{s_i} = \left(Z(\bar{\varphi}_i)\right)^{s_i} \rtimes \mathfrak{S}_{s_i}$ *denotes the wreath product.*

Proof. Let $\psi_Y \in Z(\varphi_Y)$ be a centralising mapping class, and represent ψ_Y by a diffeomorphism Ψ_Y preserving the boundary parametrisation. Furthermore, let $P, P' \subset Y$ be two connected components with $\Psi_Y(P) = P'$; then restricting the commutativity of ψ_Y and φ_Y to these two components, we obtain

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the equality $\varphi_Y|_P^{\Psi_Y} = \varphi_Y|_{P'}$ in $\Gamma(P', \partial P')$. This implies that P and P' are similar, and using Notation 3.2, we have that each Y_i is Ψ_Y-invariant and therefore

$$Z(\varphi_Y) = \prod_{i=1}^{r} Z(\varphi|_{Y_i}),$$

where $\varphi|_{Y_i}$ is defined using that Y_i is a φ-invariant union of connected components of Y. Now fix $1 \leqslant i \leqslant r$; using the diffeomorphisms $\Xi_{i,j}$ for varying $1 \leqslant j \leqslant s_i$, we can identify the surface Y_i with $\coprod_{1\leqslant j\leqslant s_i} \Sigma_{g_i,n_i}$ and thus identify $\Gamma(Y_i)$ with $\Gamma_{g_i,(n_i)} \wr \mathfrak{S}_{s_i}$. Thus, $\varphi|_{Y_i}$ corresponds to the element

$$(\bar{\varphi}_i, \ldots, \bar{\varphi}_i) \in (\Gamma_{g_i,(n_i)})^{s_i} \subset \Gamma_{g_i,(n_i)} \wr \mathfrak{S}_{s_i}.$$

It follows that $Z(\varphi|_{Y_i})$ is isomorphic to $Z(\bar{\varphi}_i) \wr \mathfrak{S}_{s_i}$. □

We conclude the subsection by analysing the actual subgroup of $\mathfrak{S}_h$ over which the fibre product $\tilde{Z}(\varphi)$ lives. Now we will focus on the case in which W is connected because the exposition is a bit easier: indeed, the natural map $\Gamma^{\mathfrak{S}_h}(W) \to \mathfrak{S}_h$ is surjective under this hypothesis on W, so we just have to describe the image of the map $Z(\varphi_Y) \to \mathfrak{S}_h$.

Notation 3.7. We denote by $\mathfrak{H}_i \subset \mathfrak{S}_{n_i}$ the image of $Z(\bar{\varphi}_i)$ under the natural map $\Gamma_{g_i,(n_i)} \to \mathfrak{S}_{n_i}$.

The proof of Lemma 3.6 shows that the image of $Z(\varphi_Y) \to \mathfrak{S}_h \cong \mathfrak{S}_{\pi_0(\partial Y)}$ is the subgroup $\prod_i \mathfrak{H}_i \wr \mathfrak{S}_{s_i}$ consisting of those permutations of the set $\pi_0(\partial Y)$ that preserve each subset $\pi_0(\partial Y_i)$ for each $1 \leqslant i \leqslant r$, and send each subset $\pi_0(\partial Y_{i,j})$ to some subset $\pi_0(\partial Y_{i,j'})$ in a way that, under the identifications $\pi_0(\partial Y_{i,j}) \cong \{1, \ldots, n_i\} \cong \pi_0(\partial Y_{i,j'})$, gives a permutation in $\mathfrak{S}_{n_i}$, which can also be attained by projecting an element $\bar{\psi}_i \in Z(\bar{\varphi}_i)$.

3.3. The kernel of ε

Recall the gluing homomorphism $\hat{\varepsilon}\colon \Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} \Gamma(Y) \to \Gamma(\mathcal{S}, \partial\mathcal{S})$ from the previous subsection. We proceed by identifying the kernel of $\hat{\varepsilon}$. Note that $\hat{\varepsilon}$ has its image in the subgroup $\Gamma(\mathcal{S}, \partial\mathcal{S})_{[c_1,\ldots,c_h]}$ of $\Gamma(\mathcal{S}, \partial\mathcal{S})$ containing all mapping classes ψ that preserve the cut locus $[c_1, \ldots, c_h]$ of φ: that is, send each oriented homotopy class of a curve c_i to the oriented homotopy class of some (possibly different) curve c_j. If $(\psi_W, \psi_Y) \in \Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} \Gamma(Y)$ belongs to the kernel of $\hat{\varepsilon}$, then in particular $\hat{\varepsilon}(\psi_W, \psi_Y)$ acts trivially on the components of the cut locus. It follows that both ψ_W and ψ_Y project to the identity element in $\mathfrak{S}_h$: that is, (ψ_W, ψ_Y) is in fact contained in the subgroup $\Gamma(W, \partial W) \times \Gamma(Y, \partial Y)$ of $\Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} \Gamma(Y)$. Hence the kernel of $\hat{\varepsilon}$ coincides with the kernel of the restriction of $\hat{\varepsilon}$ to $\Gamma(W, \partial W) \times \Gamma(Y, \partial Y)$.

We can now use [7, Thm. 3.18], in the version for disconnected surfaces: since no component of W or Y is a disc, the kernel of

$$\hat{\varepsilon}\colon \Gamma(W, \partial W) \times \Gamma(Y, \partial Y) \to \Gamma(\mathcal{S}, \partial\mathcal{S})$$

is generated by the couples $(D_{c_i}, D_{c_i}^{-1})$, where D_{c_i} denotes the Dehn twist about the curve c_i.

Since each component of W has at least one outgoing boundary, whereas the curves c_i are incoming for W, we can apply [7, Lem. 3.17] to the first coordinates of the elements $(D_{c_i}, D_{c_i}^{-1})$ and conclude that they generate a subgroup of $\Gamma(W, \partial W) \times \Gamma(Y, \partial Y)$ isomorphic to $\mathbb{Z}^h$. Finally, we note that the elements $(D_{c_i}, D_{c_i}^{-1})$ belong to the subgroup $\tilde{Z}(\varphi)$, as $D_{c_i}^{-1} \in \Gamma(Y, \partial Y)$, being the inverse of a boundary twist, is a central element and in particular it commutes with φ_Y. It follows that the kernel of ε is the free abelian group of rank h generated by the elements $(D_{c_i}, D_{c_i}^{-1})$.

3.4. Surjectivity of ε

We now prove that the map $\varepsilon\colon \tilde{Z}(\varphi) \to Z(\varphi, \Gamma(\mathcal{S}, \partial\mathcal{S}))$ is surjective. In order to do so, let $\psi \in Z(\varphi, \Gamma(\mathcal{S}, \partial\mathcal{S})) \subset \Gamma(\mathcal{S}, \partial\mathcal{S})$ be a centralising mapping class (see Definition 2.7). Then, by Lemma 2.22, ψ preserves the cut locus of φ.

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We can fix a representative $\Psi\colon \mathcal{S} \to \mathcal{S}$ of ψ that permutes the curves $c_1, \dots, c_h$ preserving their parametrisation. In particular, Ψ restricts to a diffeomorphism of W and of Y, respectively. Moreover, Ψ fixes pointwise $\partial\mathcal{S} = \partial^{\mathrm{out}}W$, and both $\Psi|_W$ and $\Psi|_Y$ are diffeomorphisms preserving the boundary parametrisation of W and Y, respectively. Consider now the mapping class $\varphi_Y \in \Gamma(Y, \partial Y)$ represented by $\Phi|_Y$, and note that also $(\Psi|_Y) \circ (\Phi|_Y) \circ (\Psi^{-1}|_Y)$ represents a mapping class in $\Gamma(Y, \partial Y)$, which we denote by $\varphi_Y^{\Psi|_Y}$.

We claim that $\varphi_Y = \varphi_Y^{\Psi|_Y}$ holds in $\Gamma(Y, \partial Y)$. To see this, note that gluing with the identity in $\Gamma(W, \partial W)$ gives a map $\lambda_Y^{\mathcal{S}}\colon \Gamma(Y, \partial Y) \to \Gamma(\mathcal{S}, \partial\mathcal{S})$, which is injective by [7, Thm. 3.18]; the claim follows from the observation that $\lambda_Y^{\mathcal{S}}(\varphi_Y^{\Psi|_Y}) = \varphi^{\Psi}$, which by the hypothesis on Ψ is equal to $\varphi = \lambda_Y^{\mathcal{S}}(\varphi_Y)$.

It follows that $\Psi|_Y$ represents a mapping class $\psi_Y \in \Gamma(Y)$ that belongs to $Z(\varphi_Y)$. Similarly, $\Psi|_W$ represents a class in $\Gamma^{\mathfrak{S}_h}(W)$, and it is clear that ψ_W and ψ_Y project to the same element of $\mathfrak{S}_h$: that is, the couple (ψ_W, ψ_Y) gives an element of $\tilde{Z}(\varphi)$. It is also evident that $\varepsilon(\psi_W, \psi_Y) = \psi$. This implies that ε is surjective. Putting together the discussion of this and the previous subsections, we conclude the following.

Proposition 3.8. *Let $g \geqslant 0, n \geqslant 1$, let $\mathcal{S}$ be a surface of type $\Sigma_{g,n}$, let $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$, let $c_1, \dots, c_h$ be a system of oriented curves representing the cut locus of φ, and let W and Y be the corresponding white and yellow regions of $\mathcal{S}$, using Notation 3.2. Then there is an isomorphism of groups induced by ε*

$$\tilde{Z}(\varphi)/\mathbb{Z}^h = \left(\Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} \textstyle\prod_i Z(\bar{\varphi}_i) \wr \mathfrak{S}_{s_i}\right)\Big/\mathbb{Z}^h \xrightarrow{\cong} Z(\varphi, \Gamma(\mathcal{S}, \partial\mathcal{S})),$$

where $\mathbb{Z}^h$ is the free abelian group generated by the elements $(D_{c_i}, D_{c_i}^{-1})$. If, moreover, W is connected, we can use Notation 3.7 and rewrite the isomorphism as

$$\left(\Gamma^{\prod_i \mathfrak{H}_i \wr \mathfrak{S}_{s_i}}(W) \times^{\prod_i \mathfrak{H}_i \wr \mathfrak{S}_{s_i}} \textstyle\prod_i Z(\bar{\varphi}_i) \wr \mathfrak{S}_{s_i}\right)\Big/\mathbb{Z}^h \xrightarrow{\cong} Z(\varphi, \Gamma(\mathcal{S}, \partial\mathcal{S})).$$

4. Generalities on coloured operads

In this section, we establish the operadic framework that we will use in the rest of the article. The reader who is well-acquainted with the language of coloured operads may skip this interlude and go directly to Section 5.

4.1. Notation and diagram categories

By 'space', we mean a topological space that is compactly generated and has the weak Hausdorff property. Let **Top** be the category of spaces; it is Cartesian closed, complete, and cocomplete.

For a topologically enriched category $\mathbf{I}$ and two objects $k, n \in \mathrm{ob}(\mathbf{I})$, we denote by $\mathbf{I}\binom{k}{n}$ the space of morphisms from k to n, and we denote the identity of n by $\mathbb{1}_n \in \mathbf{I}\binom{n}{n}$.

Notation 4.1. Let **Inj** be the small category with objects $\underline{r} = \{1, \dots, r\}$ for all non-negative integers $r \in \{0, 1, 2, \dots\}$, and with morphisms $\underline{r} \to \underline{r}'$ being all injective maps of finite sets. Moreover, let $\mathbf{\Sigma} \subseteq \mathbf{Inj}$ be the subcategory of all bijective maps.

The category **Inj** is spanned by two sorts of maps: on the one hand, *permutations* $\tau \in \mathfrak{S}_r$, which constitute the category $\mathbf{\Sigma}$, and on the other hand, the *top cofaces* $d^r : \underline{r-1} \to \underline{r}$, where for each $1 \leqslant i \leqslant r$, we denote by d^i the unique strictly monotone function whose image does not contain the element $i \in \underline{r}$. Whenever we apply a contravariant functor to **Inj**, we write $d_i := (d^i)^*$.

Notation 4.2. Let N be a fixed set, $r \geqslant 0$, and let $K = (k_1, \dots, k_r)$ be a tuple of elements of N. We write $\#K := r$ for the *length* of K. If $u\colon \underline{s} \hookrightarrow \underline{r}$ is a map in **Inj**, we write $u^*K := \big(k_{u(1)}, \dots, k_{u(s)}\big)$. If $\boldsymbol{X} := (X_n)_{n \in N}$ is a family of objects in a category with finite products, we write $\boldsymbol{X}(K) := X_{k_1} \times \dots \times X_{k_r}$;

For tuples $K = (k_1, \dots, k_r)$ and $K' = (k'_1, \dots, k'_{r'})$ of elements of N, we denote by $\mathbf{Inj}\binom{K}{K'} \subset \mathbf{Inj}\binom{\underline{r}}{\underline{r}'}$ the subset of all $u\colon \underline{r} \hookrightarrow \underline{r}'$ with $u^*K' = K$, and we write $\mathbf{\Sigma}\binom{K}{K'} := \mathbf{Inj}\binom{K}{K'} \cap \mathbf{\Sigma}\binom{\underline{r}}{\underline{r}'}$.

Definition 4.3. Let N be an index set and let $\boldsymbol{G} := (G_n)_{n\in N}$ be a family of topological groups. We define the *wreath product* $\boldsymbol{G} \wr \mathbf{Inj}$ as the following topologically enriched category:

1. the objects of $\boldsymbol{G} \wr \mathbf{Inj}$ are all tuples $K = (k_1, \dots, k_r)$ with $r \geqslant 0$ and $k_i \in N$;
2. for two tuples K and K', we define $(\boldsymbol{G} \wr \mathbf{Inj})\binom{K}{K'} = \boldsymbol{G}(K) \times \mathbf{Inj}\binom{K}{K'}$: that is, a morphism $K \to K'$ in $\boldsymbol{G} \wr \mathbf{Inj}$ is a pair $(\boldsymbol{\gamma}, u)$ consisting of a tuple $\boldsymbol{\gamma} \in \boldsymbol{G}(K)$ and an injection $u \colon \underline{r} \hookrightarrow \underline{r}'$ satisfying $K = u^*K'$;
3. we let $(\boldsymbol{\gamma}', u') \circ (\boldsymbol{\gamma}, u) := (u^*\boldsymbol{\gamma}' \cdot \boldsymbol{\gamma}, u' \circ u)$, where $u^*\boldsymbol{\gamma}' = (\gamma'_{u(1)}, \dots, \gamma'_{u(r)})$, and '$\cdot$' denotes component-wise multiplication.

For each tuple K, we define $\boldsymbol{G}[K] \subseteq \boldsymbol{G} \wr \mathbf{Inj}$ as the full subcategory spanned by objects of the form τ^*K for $\tau \in \mathfrak{S}_r$. Moreover, we let $\boldsymbol{G} \wr \boldsymbol{\Sigma}$ be the subgroupoid with morphism spaces given by $\boldsymbol{G}(K) \times \boldsymbol{\Sigma}\binom{K}{K'}$.

If $(G_n)_{n\in N}$ is the trivial sequence $G_n = 1$ of groups, then we write $N \wr \mathbf{Inj}$ for the wreath product, and we also write $N \wr \boldsymbol{\Sigma}$, respectively $N[K]$ for the corresponding subgroupoids.

Example 4.4. If $\boldsymbol{X} = (X_n)_{n\in N}$ is a sequence of spaces, then we obtain a functor

$$\boldsymbol{X} \colon N \wr \boldsymbol{\Sigma} \to \mathbf{Top}, \quad K \mapsto \boldsymbol{X}(K) = X_{k_1} \times \cdots \times X_{k_r}.$$

Construction 4.5. *Let $\boldsymbol{X} := (X_n)_{n\in N}$ be a family of based spaces, together with based left actions of G_n on X_n for each $n \in N$. Then the functor from Example 4.4 can be extended to a functor $\boldsymbol{G} \wr \mathbf{Inj} \to \mathbf{Top}$ as follows. For each injective map $u \colon \underline{r} \hookrightarrow \underline{r}'$, each fibre has at most one element; therefore we obtain an extension $N \wr \mathbf{Inj} \to \mathbf{Top}$ by*

$$u_*(x_1, \dots, x_r) := (x_{u^{-1}(1)}, \dots, x_{u^{-1}(r')}),$$

where we define $x_\varnothing$ to be the basepoint.

Moreover, $\boldsymbol{G}(K)$ acts on $\boldsymbol{X}(K)$ component-wise, so for a morphism $(\boldsymbol{\gamma}, u)$ in $\boldsymbol{G} \wr \mathbf{Inj}$, we can define $(\boldsymbol{\gamma}, u)_*(x) := u_*(\boldsymbol{\gamma} \cdot x)$.

Definition 4.6. Let $\mathbf{I}$ be a small and topologically enriched category and let $H \colon \mathbf{I}^{\mathrm{op}} \times \mathbf{I} \to \mathbf{Top}$ be a functor. Then we define the *coend* to be the coequaliser

$$\int^{c\in\mathbf{I}} H(c,c) := \operatorname{coeq}\left(\coprod_{c,c'} \mathbf{I}\binom{c}{c'} \times H(c', c) \underset{(f,x)\mapsto H(\mathbb{1}_{c'}, f)(x)}{\overset{(f,x)\mapsto H(f,\mathbb{1}_c)(x)}{\rightrightarrows}} \coprod_{c} H(c,c) \right).$$

4.2. Coloured operads

We assume that the reader is familiar with the classical notion of an operad, as it is for example presented in [15]; in particular, the visualisation of operations by trees is taken for granted.

We will give a brief introduction to the notion of a *coloured* operad mostly for the purpose of fixing the notation we will use later. For a detailed introduction to coloured operads, we refer the reader to [28].

Definition 4.7. Let N be a fixed set. An *N-coloured operad* is a family of functors $\mathcal{O}\binom{-}{n} \colon (N \wr \boldsymbol{\Sigma})^{\mathrm{op}} \to \mathbf{Top}$ for each $n \in N$, together with:

1. choices of identities $\mathbb{1}_n \in \mathcal{O}\binom{n}{n}$;
2. composition maps for each $n, k_i, l_{ij} \in N$, which are of the form

$$\mathcal{O}\binom{k_1, \dots, k_r}{n} \times \prod_{i=1}^{r} \mathcal{O}\binom{l_{i1}, \dots, l_{is_i}}{k_i} \to \mathcal{O}\binom{l_{11}, \dots, l_{rs_r}}{n}, \quad (\mu; \mu'_1, \dots, \mu'_r) \mapsto \mu \circ (\mu'_1, \dots, \mu'_r);$$

such that the usual coherence axioms from [28, §11.2] are satisfied. For $\mu \in \mathcal{O}\binom{k_1, \dots, k_r}{n}$, we will call n the *output*, $(k_1, \dots, k_r)$ the *input profile*, and $\#\mu := r$ the *arity* of μ. For the first few values of r, we call μ *nullary*, *unary*, respectively *binary* if μ has arity 0, 1, respectively 2. For the empty tuple, we will

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write $\mathcal{O}\binom{}{n}$ for the space of nullaries. We say that $\mathcal{O}$ is $\mathfrak{S}$*-free* if for each $K = (k_1, \ldots, k_r)$, the subgroup $\mathfrak{S}_K \subseteq \mathfrak{S}_r$, which fixes the tuple K acts freely on $\mathcal{O}\binom{K}{n}$.

We call $\mathcal{O}$ *monochromatic* if $N = *$ is just a singleton. In this case, we also write $\mathcal{O}(r) := \mathcal{O}\binom{*,\ldots,*}{*}$ for the space of r-ary operations. For an N-coloured operad $\mathcal{O}$ and a fixed colour $n \in N$, we also consider the monochromatic operad $\mathcal{O}|_n$ with operation spaces $(\mathcal{O}|_n)(r) := \mathcal{O}\binom{n,\ldots,n}{n}$.

Additionally, we will use the short notation for 'partial' composition: for $\mu \in \mathcal{O}\binom{k_1,\ldots,k_r}{n}$ and $\mu' \in \mathcal{O}\binom{l_1,\ldots,l_s}{k_i}$, we will write

$$\mu \circ_i \mu' := \mu \circ (\mathbb{1}_{k_1}, \ldots, \mathbb{1}_{k_{i-1}}, \mu', \mathbb{1}_{k_{i+1}}, \ldots, \mathbb{1}_{k_r}) \in \mathcal{O}\binom{k_1,\ldots,k_{i-1},l_1,\ldots,l_s,k_{i+1},\ldots,k_r}{n}.$$

Example 4.8. 1. The little d-discs operads $\mathcal{D}_d$ for $1 \leqslant d \leqslant \infty$ are examples of monochromatic operads. In our setting, we put $\mathcal{D}_d(0) = \{\mathfrak{v}\}$, the single nullary operation given by an empty configuration of discs (thus, $\mathfrak{v}$ stands for 'void').

2. Each small topologically enriched category $\mathbf{I}$ is a coloured operad with colour set $\mathrm{ob}(\mathbf{I})$ and only unaries.

Definition 4.9. Let $\mathcal{O}$ be an N-coloured operad. An $\mathcal{O}$*-algebra* is an N-indexed family $\boldsymbol{X} := (X_n)_{n\in N}$ of spaces, together with maps

$$\mathcal{O}\binom{K}{n} \times \boldsymbol{X}(K) \to X_n, \quad (\mu; x_1, \ldots, x_r) \mapsto \mu(x_1, \ldots, x_r)$$

such that the usual coherence axioms from [28, §13] are satisfied.

A morphism $f\colon \boldsymbol{X} \to \boldsymbol{X}'$ of $\mathcal{O}$-algebras is a family $(f_n\colon X_n \to X'_n)_{n\in N}$ of maps satisfying $f_n(\mu(x_1, \ldots, x_r)) = \mu(f_{k_1}(x_1), \ldots, f_{k_r}(x_r))$ for all operations μ and all elements x_i. This gives rise to the category $\mathcal{O}$-$\mathbf{Alg}$.

Example 4.10. For each N-coloured operad $\mathcal{O}$, we have an $\mathcal{O}$-algebra by the family $(\mathcal{O}\binom{}{n})_{n\in N}$. This algebra is initial in $\mathcal{O}$-$\mathbf{Alg}$ by construction, so we call it the *initial* $\mathcal{O}$*-algebra*. For $1 \leqslant d \leqslant \infty$, the initial $\mathcal{D}_d$-algebra is just a single point.

Definition 4.11. For a fixed colour set N, a morphism $\rho\colon \mathcal{P} \to \mathcal{O}$ of N-coloured operads is a family $\rho_n^-\colon \mathcal{P}\binom{-}{n} \Rightarrow \mathcal{O}\binom{-}{n}$ of transformations such that we have, abbreviating $\rho := \rho_n^K$ for all K and n,

1. $\rho(\mathbb{1}_n^{\mathcal{P}}) = \mathbb{1}_n^{\mathcal{O}}$ for each $n \in N$;
2. $\rho(\mu) \circ (\rho(\mu'_1), \ldots, \rho(\mu'_r)) = \rho(\mu \circ (\mu'_1, \ldots, \mu'_r))$.

Each operad morphism $\rho\colon \mathcal{P} \to \mathcal{O}$ gives rise to a *base-change adjunction*

$$\rho_!\colon \mathcal{O}\text{-}\mathbf{Alg} \rightleftarrows \mathcal{P}\text{-}\mathbf{Alg} : \rho^*$$

as follows: each $\mathcal{O}$-algebra is a $\mathcal{P}$-algebra by restriction. For the converse, we consider the *absolute* adjunction $F^{\mathcal{O}}\colon \mathbf{Top}^N \rightleftarrows \mathcal{O}\text{-}\mathbf{Alg} : U^{\mathcal{O}}$, where $U^{\mathcal{O}}$ just forgets the action, and where for each N-indexed family $\boldsymbol{X}$ of spaces, we define $F^{\mathcal{O}}(\boldsymbol{X})_n := \int^{K\in N\wr\Sigma} \mathcal{O}\binom{K}{n} \times \boldsymbol{X}(K)$. Then each $\mathcal{P}$-algebra $\boldsymbol{X}$ can be presented as the reflexive coequaliser of $F^{\mathcal{P}}U^{\mathcal{P}}F^{\mathcal{P}}U^{\mathcal{P}}\boldsymbol{X} \rightrightarrows F^{\mathcal{P}}U^{\mathcal{P}}\boldsymbol{X}$, whence the induced $\mathcal{O}$-algebra $\rho_!\boldsymbol{X}$ is the reflexive coequaliser of $F^{\mathcal{O}}U^{\mathcal{P}}F^{\mathcal{P}}U^{\mathcal{P}}\boldsymbol{X} \rightrightarrows F^{\mathcal{O}}U^{\mathcal{P}}\boldsymbol{X}$, compare [2, §4].

Intuitively, $\rho_!\boldsymbol{X}$ is a quotient of the free $\mathcal{O}$-algebra over $\boldsymbol{X}$ by the existing $\mathcal{P}$-action on $\boldsymbol{X}$. This adjunction clearly respects compositions: if $\rho\colon \mathcal{Q} \to \mathcal{P}$ and $\rho'\colon \mathcal{P} \to \mathcal{O}$ are two morphisms of N-coloured operads, then we clearly have $(\rho' \circ \rho)^* = \rho^* \circ \rho'^*$, so by the uniqueness of left adjoints, we also have $(\rho' \circ \rho)_! \cong \rho'_! \circ \rho_!$. When ρ is clear from the context, we also write $F_{\mathcal{P}}^{\mathcal{O}}\colon \mathcal{P}\text{-}\mathbf{Alg} \rightleftarrows \mathcal{O}\text{-}\mathbf{Alg} : U_{\mathcal{P}}^{\mathcal{O}}$ for the base-change adjunction.

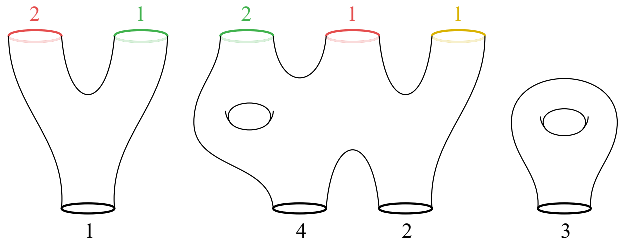

Figure 5. *An element in* $\mathscr{M}\binom{2,1,2}{4}$*. Note that the colours green, yellow and red only indicate which inputs belong together, while the actual 'colours' of the inputs are* 2*,* 1 *and* 2*, respectively.*

4.3. The coloured surface operad

We define an $\mathbb{N}_{\geqslant 1}$-coloured operad $\mathscr{M}$, which is a 'coloured' version of Tillmann's surface operad [26]; see Figure 5.

We first recall Segal's cobordism category $\mathbf{M}$ [25], which is a topologically enriched category: objects of $\mathbf{M}$ are non-negative integers $n \geqslant 0$; a morphism from n to n' is represented by a (possibly disconnected) Riemann surface W with n incoming and n' outgoing boundary components; the surface W is equipped with a choice of collar neighbourhoods $U^{\mathrm{in}}_{\partial W}$ and $U^{\mathrm{in}}_{\partial W}$ of $\partial^{\mathrm{in}}W$ and $\partial^{\mathrm{out}}W$, respectively; these neighbourhoods are equipped with:

1. a *holomorphic* parametrisation $\tilde{\vartheta}^{\mathrm{in}} \colon \{1, \ldots, n\} \times S^1 \times [0;1) \to U^{\mathrm{in}}_{\partial W}$; and
2. an *antiholomorphic* parametrisation $\tilde{\vartheta}^{\mathrm{out}} \colon \{1, \ldots, n'\} \times S^1 \times [0;1) \to U^{\mathrm{out}}_{\partial W}$.

Note that restricting $\tilde{\vartheta}^{\mathrm{in}}$ and $\tilde{\vartheta}^{\mathrm{out}}$ to $\{1, \ldots, n\} \times S^1 \times 0$ and $\{1, \ldots, n'\} \times S^1 \times 0$, respectively, we obtain parametrisations of $\partial^{\mathrm{in}}W$ and $\partial^{\mathrm{out}}W$, respectively, as in Notation 2.1.

The space of morphisms $\mathbf{M}\binom{n}{n'}$ is the moduli space of conformal classes of such Riemann surfaces W, considered up to biholomorphism compatible with the choice of parametrised collar neighbourhoods of the incoming and outgoing boundary. We usually denote by $(W, \tilde{\vartheta})$ a morphism, or shortly by W when it is not necessary to mention the parametrisation of the collar neighbourhood of the boundary; here $\tilde{\vartheta} \colon \{1, \ldots, n + n'\} \times S^1 \times [0;1) \to W$ is obtained by concatenation of $\tilde{\vartheta}^{\mathrm{in}}$ and $\tilde{\vartheta}^{\mathrm{out}}$.

The composition of two morphisms $(W, \tilde{\vartheta}) \colon n \to n'$ and $(W', \tilde{\vartheta}') \colon n' \to n''$ is given by gluing the Riemann surfaces $W \setminus \partial^{\mathrm{out}}W$ and $W' \setminus \partial^{\mathrm{in}}W'$, using the identification $U^{\mathrm{out}}_{\partial W} \setminus \partial^{\mathrm{out}}W \cong U^{\mathrm{in}}_{\partial W'} \setminus \partial^{\mathrm{in}}W'$ given by

$$\tilde{\vartheta}^{\mathrm{out}}(j, \zeta, t) \equiv (\tilde{\vartheta}')^{\mathrm{in}}(j, \zeta, 1 - t),$$

for all $1 \leqslant j \leqslant n'$, $\zeta \in S^1$ and $0 < t < 1$. The resulting surface W'' is also endowed with collar neighbourhoods of the incoming and outgoing boundaries whose parametrisations are given by $\tilde{\vartheta}^{\mathrm{in}}$ and $(\tilde{\vartheta}')^{\mathrm{out}}$, respectively. The identity of $n \in \mathbf{M}$ is described in Construction 4.13.

Definition 4.12. For each $n \geqslant 1$, the Lie group $T^n \rtimes \mathfrak{S}_n = (S^1)^n \rtimes \mathfrak{S}_n$ will be denoted by R_n.

We can regard R_n as a kind of *twisted torus*; it is the isometry group of $\coprod_n S^1$.

Construction 4.13. *We can embed R_n in the endomorphism space $\mathbf{M}\binom{n}{n}$; see Figure 6: given an element $(z_1, \ldots, z_n, \sigma) \in R_n$, we consider the morphism $(W, \tilde{\vartheta}) \colon n \to n$ given by the following:*

1. *as a Riemann surface, W is $\{1, \ldots, n\} \times S^1 \times [0;1]$, with the canonical Riemann structure;*
2. *we let $U^{in}_{\partial W} = W \setminus \partial^{out}W$, and $\tilde{\vartheta}^{in} \colon \{1, \ldots, n\} \times S^1 \times [0;1) \hookrightarrow W$ is the canonical inclusion;*
3. *we let $U^{out}_{\partial W} = W \setminus \partial^{in}W$, and let $\tilde{\vartheta}^{out} \colon \{1, \ldots, n\} \times S^1 \times [0;1) \hookrightarrow W$ be*

$$\tilde{\vartheta}^{out}(j, \zeta, t) = \left(\sigma^{-1}(j), z_j \cdot \zeta, 1 - t\right).$$

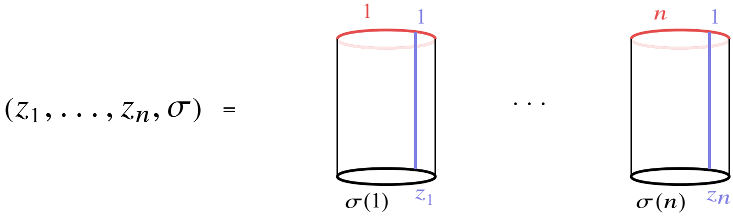

Figure 6. *An instance of $R_n \hookrightarrow \mathbf{M}\binom{n}{n}$.*

This assignment embeds in fact R_n as a group into the automorphisms of $n \in \mathbf{M}$; in particular, the identity of n can be described as the image of the unit of R_n along the embedding; see Figure 6.

Consider now the subcategory $\mathbf{M}_\partial \subseteq \mathbf{M}$ containing all objects and those cobordisms W whose components have non-empty outgoing boundary. In detail, for fixed $k, n \geqslant 0$, the morphism space $\mathbf{M}_\partial(k,n)$ looks as follows: $\pi_0(\mathbf{M}_\partial(k,n))$ is indexed by the number $1 \leqslant l \leqslant n$ of path components, an unordered partition $\{1,\dots,n\} = \boldsymbol{n}_1 \sqcup \dots \sqcup \boldsymbol{n}_l$ into non-empty subsets with $\min(\boldsymbol{n}_j) < \min(\boldsymbol{n}_{j+1})$, an ordered partition $\{1,\dots,k\} = \boldsymbol{k}_1 \sqcup \dots \sqcup \boldsymbol{k}_l$ into possibly empty sets, and genera $g_1,\dots,g_l \geqslant 0$. If we let $n_j := \#\boldsymbol{n}_j$ and $k_j := \#\boldsymbol{k}_j$, then the corresponding path component is homotopy equivalent to $\mathfrak{M}_{g_1,k_1+n_1} \times \dots \times \mathfrak{M}_{g_l,k_l+n_l}$, by restricting collar parametrisations to boundary parametrisations.

Let $n, n' \in \mathbf{M}_\partial$, and note that the embedding $R_n \subset \mathbf{M}\binom{n}{n}$ has image inside $\mathbf{M}_\partial\binom{n}{n}$. In the following lemma, we consider the right action of R_n on $\mathbf{M}_\partial\binom{n}{n'}$ by precomposition.

Lemma 4.14. *The group R_n acts freely on the space $\mathbf{M}_\partial\binom{n}{n'}$.*

Proof. Let $(W,\tilde{\vartheta})\colon n \to n'$ be a morphism in $\mathbf{M}\binom{n}{n'}$, let $(z_1,\dots,z_n,\sigma) \in R_n$, and suppose that $(W,\tilde{\vartheta}) = (W,\tilde{\vartheta}) \circ (z_1,\dots,z_n,\sigma)$. Note that the morphism $(W,\tilde{\vartheta}) \cdot (z_1,\dots,z_n,\sigma)$ is represented by the pair $(W',\tilde{\vartheta}')$, where:

1. $W' = W$ and $(\tilde{\vartheta}')^{\mathrm{out}} = \tilde{\vartheta}^{\mathrm{out}}$;
2. $(\tilde{\vartheta}')^{\mathrm{in}}$ is the postcomposition of $\tilde{\vartheta}^{\mathrm{in}}\colon \{1,\dots,n\} \times S^1 \times [0;1) \to W$ with the automorphism of $\{1,\dots,n\} \times S^1 \times [0;1)$ given by $(j,\zeta,t) \mapsto (\sigma^{-1}(j), z_j \cdot \zeta, t)$.

If $\psi\colon W \to W'$ is a diffeomorphism exhibiting the equivalence of $(W,\tilde{\vartheta})$ and $(W',\tilde{\vartheta}')$ in $\mathbf{M}_\partial\binom{n}{n'}$, then the first two conditions imply that ψ restricts to the identity of $U^{\mathrm{out}}_{\partial W}$: that is, on the image of $\tilde{\vartheta}^{\mathrm{out}}$. Since ψ is a holomorphic map, it must be the identity on a closed and open subset of W; since each connected component of W has non-empty outgoing boundary, and thus intersects $U^{\mathrm{out}}_{\partial W}$, we conclude that ψ must be the identity of W.

It then follows that the automorphism of $\{1,\dots,n\}\times S^1\times[0;1)$ given by $(j,\zeta,t) \mapsto (\sigma^{-1}(j), z_j\cdot\zeta, t)$ is in fact the identity of $\{1,\dots,n\} \times S^1 \times [0;1)$, and this implies that $(z_1,\dots,z_n,\sigma)$ is the identity of R_n, as desired. □

Definition 4.15. We consider $\mathbf{M}_\partial$ as a symmetric monoidal category with monoidal sum being the disjoint union; since the monoidal sum behaves as the usual sum of natural numbers on objects, we have an associated coloured operad $\mathscr{M}$ with colours $\mathbb{N}_{\geqslant 1} = \{1,2,\dots\}$ and

$$\mathscr{M}\binom{k_1,\dots,k_r}{n} := \mathbf{M}_\partial\binom{k_1+\dots+k_r}{n}.$$

Note that the restriction $\mathscr{M}|_1$ to the colour 1 is exactly Tillmann's surface operad [26]. For each $\mathscr{M}$-algebra $\boldsymbol{X} = (X_n)_{n\geqslant 1}$, the space X_1 is an algebra over the classical surface operad $\mathscr{M}|_1$.

Example 4.16. In contrast to the little disc operads, the initial $\mathscr{M}$-algebra is non-trivial: for instance, its colour-1 part $\mathscr{M}\binom{}{1}$ homotopy equivalent to the familiar collection of moduli spaces

$$\mathscr{M}\binom{}{1} = \mathbf{M}_\partial\binom{0}{1} \simeq \coprod_{g\geqslant 0}\mathfrak{M}_{g,1}.$$

4.4. Tensor products and based operads

Definition 4.17. In [3, §II.3], Boardman and Vogt constructed a tensor product for operads. We are only interested in the following special case: let $\mathscr{A}$ be a monochromatic operad and $\mathbf{I}$ be a small, topologically enriched category with object set N. Then $\mathscr{A}\otimes\mathbf{I}$ is an N-coloured operad with operation spaces

$$(\mathscr{A}\otimes\mathbf{I})\binom{k_1,\dots,k_r}{n} = \mathscr{A}(r)\times\prod_{i=1}^{r}\mathbf{I}\binom{k_i}{n},$$

together with the following structure, where we denote operations in $\mathscr{A}\otimes\mathbf{I}$ by $\mu\otimes(\nu_1,\dots,\nu_r)$ with $\mu\in\mathscr{A}(r)$ and $\nu_i\in\mathbf{I}\binom{k_i}{n}$:

1. symmetric actions $\tau^*(\mu\otimes(\nu_1,\dots,\nu_r)) = (\tau^*\mu)\otimes(\nu_{\tau(1)},\dots,\nu_{\tau(r)})$;
2. identities $(\mathbb{1}^{\mathscr{A}},\mathbb{1}_n^{\mathbf{I}})$;
3. compositions

$$\begin{aligned}&(\mu\otimes(\nu_1,\dots,\nu_r))\circ\big(\mu_1\otimes(\nu_{1,1},\dots,\nu_{1,s_1}),\dots,\mu_r\otimes(\nu_{r,1},\dots,\nu_{r,s_r})\big)\\ &:= (\mu\circ(\mu_1,\dots,\mu_r))\otimes\big(\nu_1\circ\nu_{1,1},\dots,\nu_1\circ\nu_{1,s_1},\dots,\nu_r\circ\nu_{r,1},\dots,\nu_r\circ\nu_{r,s_r}\big).\end{aligned}$$

For $n\in N$, we also abbreviate $\mu\otimes n := \mu\otimes(\mathbb{1}_n,\dots,\mathbb{1}_n)\in(\mathscr{A}\otimes\mathbf{I})\binom{n,\dots,n}{n}$. Note that $(\mathscr{A}\otimes\mathbf{I})$-algebras are the same as enriched functors $\mathbf{I}\to\mathscr{A}$-**Alg**.

This construction is bifunctorial: if $\rho_1\colon\mathscr{A}\to\mathscr{A}'$ is a morphism of operads and $\rho_2\colon\mathbf{I}\to\mathbf{I}'$ is a functor that is the identity on objects, then we get a morphism $\rho_1\otimes\rho_2\colon\mathscr{A}\otimes\mathbf{I}\to\mathscr{A}'\otimes\mathbf{I}'$.

Example 4.18. Regard N as the discrete category with objects N. Then we get

$$(\mathscr{A}\otimes N)\binom{k_1,\dots,k_r}{n} = \begin{cases}\mathscr{A}(r) & \text{for } k_1=\dots=k_r=n\\ \varnothing & \text{else,}\end{cases}$$

and $(\mathscr{A}\otimes N)$-algebras are just N-indexed families of $\mathscr{A}$-algebras. One example that will be of particular importance for us later is the operad $\mathscr{D}_1\otimes N$, which has a copy of the little 1-discs operad $\mathscr{D}_1$ in each colour $n\in N$.

Definition 4.19. Consider the monochromatic operad $\mathscr{B}$ with only two operations, namely the identity $\mathscr{B}(1)=\{\mathbb{1}\}$ and a single nullary $\mathscr{B}(0)=\{\mathfrak{v}\}$. Then $(\mathscr{B}\otimes N)$-algebras are the same as families $\boldsymbol{X}=(X_n)_{n\in N}$ of based spaces.

A *based N-coloured operad* is an N-coloured operad $\mathscr{O}$, together with an operad morphism $\mathscr{B}\otimes N\to\mathscr{O}$. A morphism of based N-coloured operads is an operad map $\rho\colon\mathscr{O}\to\mathscr{P}$ commuting with the two maps from $\mathscr{B}\otimes N$.

Remark 4.20. 1. A based N-coloured operad is the same as an N-coloured operad $\mathscr{O}$, together with a choice of nullary operation $\mathfrak{v}_n\in\mathscr{O}\binom{}{n}$ for each colour $n\in N$, and a morphism $\rho\colon\mathscr{O}\to\mathscr{P}$ of based operads has to additionally satisfy $\rho(\mathfrak{v}_n^{\mathscr{O}})=\rho(\mathfrak{v}_n^{\mathscr{P}})$.
2. The nullaries $\mathfrak{v}_n$ of a based operad can be used to 'block' inputs by precomposition with them. More precisely, for each input profile $K=(k_1,\dots,k_r)$ and $1\leqslant i\leqslant r$, we have a map

$$d_i\colon\mathscr{O}\binom{K}{n}\to\mathscr{O}\binom{d_iK}{n},\quad \mu\mapsto\mu\circ_i\mathfrak{v}_{k_i}.$$

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In this way the functors $\mathcal{O}\binom{-}{n}\colon (N\wr\boldsymbol{\Sigma})^{\mathrm{op}}\to\mathbf{Top}$ can be extended to $\mathcal{O}\binom{-}{n}\colon (N\wr\mathbf{Inj})^{\mathrm{op}}\to\mathbf{Top}$. Using these functors, one can give a concise description of the free $\mathcal{O}$-algebra over a family $\boldsymbol{X}:=(X_n)_{n\in N}$ of based spaces: for each $n\in N$, we have

$$F^{\mathcal{O}}_{\mathcal{B}\otimes N}(\boldsymbol{X})_n\cong\int^{K\in N\wr\mathbf{Inj}}\mathcal{O}\binom{K}{n}\times\boldsymbol{X}(K).$$

Example 4.21. 1. The little discs operads $\mathcal{D}_d$ have precisely one nullary operation and are thus canonically based. The same applies to $\mathcal{D}_d\otimes N$ for each colour set N.

2. For each small and topologically enriched category $\mathbf{I}$, the tensor product $\mathcal{B}\otimes\mathbf{I}$ differs from $\mathbf{I}$ only by the single nullary operation $\mathfrak{v}_n\in(\mathcal{B}\otimes\mathbf{I})\binom{}{n}$ for each colour n. Note that $(\mathcal{B}\otimes\mathbf{I})$-algebras are precisely functors $\mathbf{I}\to\mathbf{Top}_*$ to the category of based topological spaces.
3. As a particular case of the previous example, let $\boldsymbol{G}=(G_n)_{n\in N}$ be a sequence of groups, and consider $\boldsymbol{G}$ as a groupoid. Then a $(\mathcal{B}\otimes\boldsymbol{G})$-algebra is a sequence of based spaces $(X_n)_{n\in N}$ with a basepoint-preserving left action of G_n on X_n for all $n\in N$.

5. Infinite loop spaces from coloured operads with homological stability

In this section, we address the following problem: if $\mathcal{O}$ is an N-coloured operad with homological stability (which will be made precise soon), $\mathbf{I}$ is a topological category together with a map $\mathcal{B}\otimes\mathbf{I}\to\mathcal{O}$, and if $\boldsymbol{X}=(X_n)_{n\in N}$ an $(\mathcal{B}\otimes\mathbf{I})$-algebra, what can we say about the homotopy type of $F^{\mathcal{O}}_{\mathcal{G}}(\boldsymbol{X})$? By answering this question, we extend the methods from [1, §5], where the monochromatic and non-relative case was treated: that is, $\mathbf{I}=N=*$, so $\mathcal{B}\otimes\mathbf{I}=\mathcal{B}$.

We briefly summarise the strategy of [1, §5]: in a first step, a notion of (monochromatic) 'operad with homological stability' is introduced: such an operad $\mathcal{O}$ comes in particular with a morphism of operads $\iota\colon\mathcal{D}_1\to\mathcal{O}$, satisfying the weak homotopy commutativity condition, which demands that $\iota(\mathcal{D}_1(2))\subseteq\mathcal{O}(2)$ lies in a single path component[1]; hence it makes sense to consider group completions of $\mathcal{O}$-algebras.

In a second step, the authors of [1] focus on operads with homological stability $\mathcal{O}$, which come with a map $\pi\colon\mathcal{O}\to\mathcal{D}_\infty$ of operads under $\mathcal{D}_1$. Thus, we have for each based space X two maps of $\mathcal{O}$-algebras:

1. $F^{\mathcal{O}}_{\mathcal{B}}(X)\to F^{\mathcal{O}}_{\mathcal{B}}(*)=\mathcal{O}(0)$ induced by $X\to *$;
2. $F^{\mathcal{O}}_{\mathcal{B}}(X)\to\pi^*F^{\mathcal{D}_\infty}_{\mathcal{B}}(X)$, the unit of the base-change adjunction.

Intuitively, the first map forgets the space X, while the second map forgets the operad $\mathcal{O}$. In [1, Thm. 5.4], it is shown that the product map induces a weak equivalence $\Omega BF^{\mathcal{O}}_{\mathcal{B}}(X)\to\Omega B\mathcal{O}(0)\times\Omega^\infty\Sigma^\infty X$ on group completions, after identifying the group completion of $F^{\mathcal{D}^\infty}_{\mathcal{B}}(X)$ with $\Omega^\infty\Sigma^\infty X$.

Finally, an operad with homological stability $\mathcal{O}$ admits a replacement by another operad with homological stability $\mathcal{O}':=\mathcal{O}\times\mathcal{D}_\infty$, which has a comparison map $\pi\colon\mathcal{O}'\to\mathcal{D}_\infty$, and under mild extra assumptions, the free algebras $F^{\mathcal{O}}_{\mathcal{B}}(X)$ and $F^{\mathcal{O}'}_{\mathcal{B}}(X)$ are equivalent as A_∞-algebras and thus have equivalent group completions.

5.1. *Coloured operads with homological stability*

Definition 5.1. An *operad under* $\mathcal{D}_1$ is an N-coloured operad $\mathcal{O}$ together with an operad morphism $\iota\colon\mathcal{D}_1\otimes N\to\mathcal{O}$ satisfying the weak homotopy commutativity condition levelwise, meaning that $\iota((\mathcal{D}_1\otimes N)\binom{n,n}{n})\subseteq\mathcal{O}\binom{n,n}{n}$ is contained in a single path component.

1. If $\mathcal{O}$ is an operad under $\mathcal{D}_1$, then $\mathcal{O}$ is based by $\mathfrak{v}_n:=\iota(\mathfrak{v}\otimes n)\in\mathcal{O}\binom{}{n}$. This gives rise to the input blocking maps

$$\beta\colon\mathcal{O}\binom{k_1,\dots,k_r}{n}\to\mathcal{O}\binom{}{n},\quad \mu\mapsto\mu(\mathfrak{v}_{k_1},\dots,\mathfrak{v}_{k_r}).$$

[1]In principle, any A_∞-operad would suffice; we restrict to $\mathcal{D}_1$ for simplicity.

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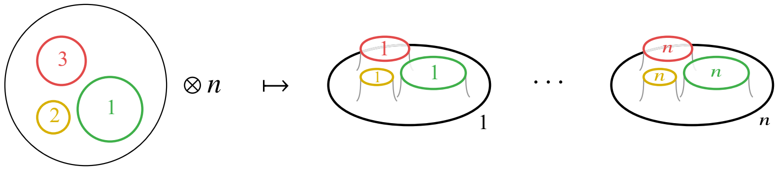

Figure 7. *An instance of* $\iota_{\mathcal{M}}\colon (\mathcal{D}_2 \otimes \mathbb{N}_{\geqslant 1})\binom{n,n,n}{n} \to \mathcal{M}\binom{n,n,n}{n}$.

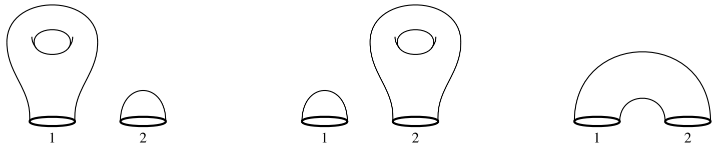

Figure 8. *The three generators* e_2^1, e_2^2 *and* $e_2^{1,2}$ *of* $\pi_0(\mathcal{M}\binom{}{2})$.

2. If $\mathcal{O}$ is an operad under $\mathcal{D}_1$, then each $\mathcal{O}$-algebra $\boldsymbol{M}$ is levelwise an H-commutative $\mathcal{D}_1$-algebra, so for each $n \in N$, the set $\pi_0(M_n)$ is an abelian monoid whose (unique) binary operation we denote by '$\curlyvee$'; there is a bar construction of M_n, and by the group completion theorem [18], we have an isomorphism $H_\bullet(\Omega B M_n) \cong H_\bullet(M_n)[\pi_0(M_n)^{-1}]$.

A morphism $\rho\colon \mathcal{O} \to \mathcal{O}'$ of operads $\iota\colon \mathcal{D}_1 \otimes N \to \mathcal{O}$ and $\iota'\colon \mathcal{D}_1 \otimes N \to \mathcal{O}'$ under $\mathcal{D}_1$ is a morphism of operads such that $\rho \circ \iota = \iota'$ holds. In that case, for each $\mathcal{O}'$-algebra $\boldsymbol{X}$, the levelwise group completions of the $\mathcal{O}'$-algebra $\boldsymbol{X}$ and of the $\mathcal{O}$-algebra $\rho^*(\boldsymbol{X})$ coincide, as they only depend on the $\mathcal{D}_1 \otimes N$-structure.

Notation 5.2. Let $\iota\colon \mathcal{D}_1 \otimes N \to \mathcal{O}$ be an operad under $\mathcal{D}_1$. We fix an operation $\mathfrak{p} \in \mathcal{D}_1(2)$, and we abbreviate $\mu \curlyvee \mu' := \iota(\mathfrak{p} \otimes n) \circ (\mu, \mu')$ for operations $\mu \in \mathcal{O}\binom{K}{n}$ and $\mu' \in \mathcal{O}\binom{K'}{n}$ If $(M_n)_{n \in N}$ is an $\mathcal{O}$-algebra and $x, x' \in M_n$, then we also write $x \curlyvee x' := \iota(\mathfrak{p} \otimes n)(x, x')$.

The notation '$\curlyvee$' is pictorially inspired by the following example:

Example 5.3. Recall Definition 4.15 and Example 4.18. We have a map of operads $\iota_{\mathcal{M}}\colon \mathcal{D}_2 \otimes \mathbb{N}_{\geqslant 1} \to \mathcal{M}$ given by applying the classical inclusion of $\mathcal{D}_2$ into Tillmann's surface operad level-wise; see Figure 7. This morphism restricts to a map $\mathcal{D}_1 \otimes \mathbb{N}_{\geqslant 1} \to \mathcal{M}$ of operads satisfying the weak homotopy commutativity condition and thus turns $\mathcal{M}$ into an operad under $\mathcal{D}_1$.

The input blocking maps $\beta\colon \mathcal{M}\binom{K}{n} \to \mathcal{M}\binom{}{n}$ are induced by capping each ingoing boundary curve with a disc, and the abelian monoid $\pi_0(\mathcal{M}\binom{}{n})$ contains all isomorphism types of surfaces $\mathcal{S}$ with n ordered outgoing boundary curves and no incoming boundary curve with $\mathcal{S}$ possibly disconnected, such that each path component of $\mathcal{S}$ has non-empty boundary. The addition on $\pi_0(\mathcal{M}\binom{}{n})$ is given by gluing n pairs of pants; the neutral element is given by an ordered collection of n discs. The abelian monoid $\pi_0(\mathcal{M}\binom{}{n})$ is finitely generated: for instance, it can be generated by the following elements e_n^i and $e_n^{i,j}$; see Figure 8:

1. For $1 \leqslant i \leqslant n$, we let e_n^i be the isomorphism type of surfaces with n path components, such that the component carrying the i ${}^{\text{th}}$ boundary curve has genus 1, whereas all others components are discs.
2. For each $1 \leqslant i < j \leqslant n$, we let $e_n^{i,j}$ be the isomorphism type of surfaces with $n-1$ path components, all of genus 0, such that one component is a cylinder carrying the i ${}^{\text{th}}$ and the j ${}^{\text{th}}$ boundary curve.

Construction 5.4 (Stable operation space). *We call an N-coloured operad $\mathcal{O}$ under $\mathcal{D}_1$ admissibly graded if for each $n \in N$, the abelian monoid $\pi_0(\mathcal{O}\binom{}{n})$ is finitely generated and, for each source profile $(k_1, \ldots, k_r)$, the degree map*

$$|\cdot|\colon \mathcal{O}\binom{k_1,\ldots,k_r}{n} \xrightarrow{\beta} \mathcal{O}\binom{}{n} \to \pi_0(\mathcal{O}\binom{}{n}), \quad \mu \mapsto |\mu| = \pi_0(\beta(\mu))$$

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is surjective.[2] *In this case we write for each $\delta \in \pi_0(\mathcal{O}\binom{}{n})$*

$$\mathcal{O}\binom{k_1,\dots,k_r}{n}^{\delta} := \left\{\mu \in \mathcal{O}\binom{k_1,\dots,k_r}{n};\ |\mu| = \delta\right\} \neq \emptyset.$$

We choose for each $n \in N$ a finite generating set $E_n \subseteq \pi_0(\mathcal{O}\binom{}{n})$ and let e_n be the sum of all elements from E_n. If we fix a nullary operation $\tilde{e}_n \in \mathcal{O}\binom{}{n}$ with $|\tilde{e}_n| = e_n$, called propagator, then we obtain for each component $\delta \in \pi_0(\mathcal{O}\binom{}{n})$ and each input profile K a stabilising map

$$\mathrm{stab}\colon \mathcal{O}\binom{K}{n}^{\delta} \to \mathcal{O}\binom{K}{n}^{\delta \curlyvee e_n}, \quad \mu \mapsto \mu \curlyvee \tilde{e}_n.$$

From this, we can form the space of stable operations from K to n,

$$\mathcal{O}\binom{K}{n}^{\infty} := \mathrm{hocolim}\left(\mathcal{O}\binom{K}{n}^{0} \xrightarrow{\mathrm{stab}} \mathcal{O}\binom{K}{n}^{e_n} \xrightarrow{\mathrm{stab}} \mathcal{O}\binom{K}{n}^{2e_n} \xrightarrow{\mathrm{stab}} \cdots\right).$$

Definition 5.5. Let $\mathcal{O}$ be an N-coloured operad under $\mathcal{D}_1$, which is admissibly graded. By the associativity of the operadic composition, input blocking and stabilisation commute: that is, for each δ, the square

$$\begin{tikzcd}
\mathcal{O}\binom{k_1,\dots,k_r}{n}^{\delta} \arrow[r, "\mathrm{stab}"] \arrow[d, "\beta"'] & \mathcal{O}\binom{k_1,\dots,k_r}{n}^{\delta\curlyvee e_n} \arrow[d, "\beta"] \\
\mathcal{O}\binom{}{n}^{\delta} \arrow[r, "\mathrm{stab}"'] & \mathcal{O}\binom{}{n}^{\delta\curlyvee e_n}.
\end{tikzcd}$$

commutes. Hence we obtain a stable input blocking $\beta_n^K : \mathcal{O}\binom{K}{n}^{\infty} \to \mathcal{O}\binom{}{n}^{\infty}$ for each input profile K, which depends, up to homotopy, only on the path component from which the propagator is chosen: that is, on the choice of generating set E_n.

We call $\mathcal{O}$ an *operad with homological stability* if there is a choice of generating sets such that all stable input blockings β_n^K induce isomorphisms in integral homology.

Example 5.6. The coloured surface operad $\mathcal{M}$ is admissibly graded, and we may use the generating sets from Example 5.3.

It is even an operad with homological stability: here we use that multiplying with the propagator automatically yields a connected cobordism and increases the genus by at least one; see Figure 9, so the stable input blocking is the capping map $\mathfrak{M}_{\infty,n+k_1+\dots+k_r} \to \mathfrak{M}_{\infty,n}$ between stable moduli spaces of Riemann surfaces: this is a homology equivalence by Harer's stability theorem [11].

5.2. *Derived base-change and a splitting result*

Recall that we want to establish an analogue of [1, Thm. 5.4] for the coloured case *and relative* case: that is, we want to consider relatively free algebras, relative to a map $\mathcal{P} \to \mathcal{O}$ of based operads, where we will soon restrict to the case $\mathcal{P} = \mathcal{B} \otimes \mathbf{I}$ for an enriched category $\mathbf{I}$.

The most convenient setting for such a discussion does not use the strict functor $F_{\mathcal{P}}^{\mathcal{O}}$, but a homotopically better behaved one, which we denote by $\tilde{F}_{\mathcal{P}}^{\mathcal{O}}$. This simplifies many point-set issues, and with regard to our original problem, it will turn out to be equivalent to the space we want to understand.

The functor $\tilde{F}_{\mathcal{P}}^{\mathcal{O}}$ can be constructed by considering the model structure on the categories of $\mathcal{O}$- and $\mathcal{P}$-algebras as in [2], but we decided to give an explicit description. Here we assume that the reader is familiar with monads and their two-sided bar constructions, as introduced in [17].

[2] In the monochromatic setting, the degree map is automatically surjective: by the weak homotopy commutativity condition, $\pi_0(\mathcal{O})$ contains the commutative operad $\mathcal{C}om$, and if we write $\mathcal{C}om(r) = \{\mu_r\}$, then $\beta(\mu_{r+1} \circ_1 \delta) = \mu_1 \circ \delta = \mathbb{1} \circ \delta = \delta$ for each $\delta \in \pi_0(\mathcal{O}(0))$. For the coloured case, though, it seems necessary to additionally assume this property.

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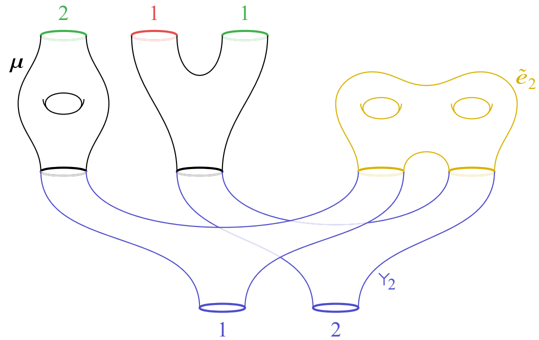

Figure 9. *A single stabilisation step on $\mathcal{M}\binom{2,1}{2}$. Note that $|\tilde{e}_2| = e_2^1 \curlyvee e_2^2 \curlyvee e_2^{1,2}$ is the isomorphism class of surfaces of type $\Sigma_{2,2}$.*

Construction 5.7. *Let $\mathcal{P} \to \mathcal{O}$ be a morphism of based N-coloured operads. Then we obtain monads $\mathbb{O} := U^{\mathcal{O}}_{\mathcal{B}\otimes N} F^{\mathcal{O}}_{\mathcal{B}\otimes N}$ and $\mathbb{P} := U^{\mathcal{P}}_{\mathcal{B}\otimes N} F^{\mathcal{P}}_{\mathcal{B}\otimes N}$ on $\mathbf{Top}_*^N$, and $\mathbb{O}$ is a left $\mathbb{P}$-functor by the transformation $\mathbb{O}\mathbb{P} \Rightarrow \mathbb{O}^2 \Rightarrow \mathbb{O}$. For each $\mathcal{P}$-algebra $\boldsymbol{X}$, we consider the two-sided bar construction $B_\bullet(\mathbb{O}, \mathbb{P}, \boldsymbol{X})$ with p-simplices $B_p(\mathbb{O}, \mathbb{P}, \boldsymbol{X}) = \mathbb{O}\mathbb{P}^p U^{\mathcal{P}}_{\mathcal{B}\otimes N}\boldsymbol{X}$, which is an N-coloured simplicial space, and define the derived free algebra $\tilde{F}^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X}) := |B_\bullet(\mathbb{O}, \mathbb{P}, \boldsymbol{X})|$ to be its levelwise geometric realisation. Then $\tilde{F}^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X})$ is itself an $\mathcal{O}$-algebra with multiplication*

$$\mathbb{O}|B_\bullet(\mathbb{O}, \mathbb{P}, \boldsymbol{X})| \cong |B_\bullet(\mathbb{O}^2, \mathbb{P}, \boldsymbol{X})| \to |B_\bullet(\mathbb{O}, \mathbb{P}, \boldsymbol{X})|,$$

where the first identification is due to [17, Lem. 9.7] and the last map is given by $|B_\bullet(\kappa, \mathbb{P}, \boldsymbol{X})|$ for the operadic composition $\kappa\colon \mathbb{O}^2 \Rightarrow \mathbb{O}$. Second, we have a morphism $\tilde{F}^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X}) \to F^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X})$ of $\mathcal{O}$-algebras, by noticing that $F^{\mathcal{O}}_{\mathcal{P}}(\boldsymbol{X})$ is the reflexive coequaliser of $B_1(\mathbb{O}, \mathbb{P}, \boldsymbol{X}) \rightrightarrows B_0(\mathbb{O}, \mathbb{P}, \boldsymbol{X})$; see Definition 4.11.

Before stating our main theorem, let us fix once and for all the point-set requirements we want to assume:

Setting 5.8. *Throughout this section, we consider the following:*

1. *Let $\mathcal{O}$ be an N-coloured $\mathfrak{S}$-free operad with homological stability such that, additionally, the inclusions $\{\mathbb{1}_n\} \hookrightarrow \mathcal{O}\binom{n}{n}$ are cofibrations.*
2. *Let $\mathbf{I}$ be a topologically enriched category with object set N such that the inclusions $\{\mathbb{1}_n\} \hookrightarrow \mathbf{I}\binom{n}{n}$ are cofibrations. We assume that there is a map $\mathcal{B} \otimes \mathbf{I} \to \mathcal{O}$ of based N-coloured operads.*
3. *Let $\boldsymbol{X} = (X_n)_{n\in N}$ be an $(\mathcal{B} \otimes \mathbf{I})$-algebra, or in other words, an enriched functor $X_\bullet\colon \mathbf{I} \to \mathbf{Top}_*$, and we assume that each X_n is well-based.*

Moreover, we assume that all involved spaces are Hausdorff.

Of course, the example we have in mind is $\mathcal{O}$ being the surface operad $\mathcal{M}$ and $\mathbf{I}$ being the family $\boldsymbol{R} = (R_n)_{n\geqslant 1}$ of twisted tori. We want to show the following theorem:

Theorem 5.9 (Splitting theorem). *In the above Setting 5.8, we have, for each $n \in N$, a weak equivalence of loop spaces*

$$\Omega B\tilde{F}^{\mathcal{O}}_{\mathcal{B}\otimes\mathbf{I}}(\boldsymbol{X})_n \simeq \Omega B\mathcal{O}\binom{}{n} \times \Omega^\infty\Sigma^\infty \mathrm{hocolim}_{\mathbf{I}}(X_\bullet).$$

The proof of Theorem 5.9 will occupy the rest of this section. Let us start by establishing a map that compares the two sides. To do so, we start by constructing an N-coloured version of the E_∞-operad $\mathcal{D}_\infty$ and show that we can, without loss of generality, assume that there is a comparison map from $\mathcal{O}$ to it:

Construction 5.10. *For each colour set N, we consider the chaotic category EN with object set N and morphism spaces $(EN)\binom{k}{n} = *$ for all $k, n \in N$. We consider the category $\mathscr{D}_\infty \otimes EN$.*

Lemma 5.11. *For the proof of Theorem 5.9, we can without loss of generality assume a map $\pi\colon \mathscr{O} \to \mathscr{D}_\infty \otimes EN$ such that the diagram*

$$
\begin{tikzcd}
\mathscr{B} \otimes N \arrow[r] \arrow[d] & \mathscr{B} \otimes \mathbf{I} \arrow[d] \arrow[ddr, bend left] & \\
\mathscr{D}_1 \otimes N \arrow[r] \arrow[ddrr, phantom] \arrow[drr, bend right] & \mathscr{O} \arrow[dr, "\pi"] & \\
 & & \mathscr{D}_\infty \otimes EN
\end{tikzcd}
$$

commutes, where all arrows apart from π are either given or induced by the canonical maps $\mathscr{B} \to \mathscr{D}_1 \to \mathscr{D}_\infty$ and $N \to \mathbf{I} \to EN$.

Proof. The commutativity of the square is part of the general setting: recall that we assumed that $\mathscr{B}\otimes\mathbf{I} \to \mathscr{O}$ is a map of *based* operads, and $\mathscr{O}$ is canonically based as an operad under $\mathscr{D}_1$. To establish the map π, we replace $\mathscr{O}$ by a slightly larger operad: if we consider the product operad $\mathscr{O}' := \mathscr{O}\times(\mathscr{D}_\infty\otimes EN)$, together with:

- the diagonal inclusion $\mathscr{D}_1 \otimes N \to \mathscr{O}'$,
- the diagonal inclusion $\mathscr{B} \otimes \mathbf{I} \to \mathscr{O}'$,
- the second projection $\mathscr{O}' \to \mathscr{D}_\infty \otimes EN$,

then the above diagram clearly commutes with $\mathscr{O}$ instead of $\mathscr{O}'$. Moreover, note that *each* operation space of $\mathscr{D}_\infty \otimes EN$ is contractible: hence $\mathscr{O}'$ is again admissibly graded with $\pi_0(\mathscr{O}'\binom{}{n}) = \pi_0(\mathscr{O}\binom{}{n})$ and $\mathscr{O}'$ is again an operad with homological stability, satisfying $\mathscr{O}'\binom{}{n} = \mathscr{O}\binom{}{n}$.

Finally, the first projection $\mathscr{O}' \to \mathscr{O}$ induces a morphism of monads $\mathbb{O}' \Rightarrow \mathbb{O}$ and hence a map $\tilde{F}^{\mathscr{O}'}_{\mathscr{B}\otimes\mathbf{I}}(\boldsymbol{X}) \to U^{\mathscr{O}}_{\mathscr{O}'}\tilde{F}^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(\boldsymbol{X})$ of $\mathscr{O}'$-algebras, which is in particular a map of A_∞-algebras. If we denote by $\mathbb{I}$ the monad for $\mathscr{B} \otimes \mathbf{I}$, then we can easily see that, since $\mathscr{O}$ is $\mathfrak{S}$-free, each X_n is well-based, and every space is Hausdorff, the simplicial map $B_\bullet(\mathbb{O}', \mathbb{I}, \boldsymbol{X}) \to B_\bullet(\mathbb{O}, \mathbb{I}, \boldsymbol{X})$ is levelwise an equivalence. Second, both simplicial spaces are proper in the sense of [17, §11], by using that the inclusions of the identities are cofibrations. Therefore, by [16, Thm. A.4], the induced map on the geometric realisations $\tilde{F}^{\mathscr{O}'}_{\mathscr{B}\otimes\mathbf{I}}(\boldsymbol{X})$ and $\tilde{F}^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(\boldsymbol{X})$ is again an equivalence, whence their group completions are equivalent as loop spaces. □

Using the lemma, we obtain, as in the monochromatic case, two maps:

1. The map $\boldsymbol{X} \to *$ to $* = (*)_{n\in N}$ induces a map $\tilde{F}^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(\boldsymbol{X}) \to \tilde{F}^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(*)$ of $\mathscr{O}$-algebras, which is in particular a map of levelwise A_∞-algebras.
2. The morphism $\mathscr{O} \to \mathscr{D}_\infty \otimes EN$ induces a map $\tilde{F}^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(\boldsymbol{X}) \to \tilde{F}^{\mathscr{D}_\infty\otimes EN}_{\mathscr{B}\otimes\mathbf{I}}(\boldsymbol{X})$ of levelwise A_∞-algebras.

The two targets can be identified with the following spaces:

Lemma 5.12. *For each $n \in N$, we have equivalences of A_∞-algebras*

$$
\begin{aligned}
\tilde{F}^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(*)_n &\simeq \mathscr{O}\tbinom{}{n},\\
\tilde{F}^{\mathscr{D}_\infty\otimes EN}_{\mathscr{B}\otimes\mathbf{I}}(\boldsymbol{X})_n &\simeq F^{\mathscr{D}_\infty}_{\mathscr{B}}(\mathrm{hocolim}_{\mathbf{I}}(X_\bullet)).
\end{aligned}
$$

Proof. For the first equivalence, we note that since $(\mathscr{B} \otimes \mathbf{I})\binom{}{n} = *$, we have $F^{\mathscr{B}\otimes\mathbf{I}}_{\mathscr{B}\otimes N}(*)_n = *$. Now consider the natural map

$$
\tilde{F}^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(*) = \tilde{F}^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(F^{\mathscr{B}\otimes\mathbf{I}}_{\mathscr{B}\otimes N}(*)) \to F^{\mathscr{O}}_{\mathscr{B}\otimes\mathbf{I}}(F^{\mathscr{B}\otimes\mathbf{I}}_{\mathscr{B}\otimes N}(*)) = F^{\mathscr{O}}_{\mathscr{B}\otimes N}(*) = \mathscr{O}\tbinom{}{n}.
$$

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This map arises from the augmentation $B_\bullet := B_\bullet(\mathbb{O}, \mathbb{I}, \mathbb{I}(*)) \to B_{-1} := \mathbb{O}(*)$, which has a (colour-wise) extra degeneracy $s_{-1}\colon B_p \to B_{p+1}$ from the unit of $\mathbb{I}$ and hence is an equivalence by [23, Cor. 4.5.2].

For the second equivalence, we start with the general observation that for a sequence $\mathcal{Q} \to \mathcal{P} \to \mathcal{O}$ of N-coloured operads, we have a levelwise equivalence of $\mathcal{O}$-algebras among the derived algebras $\tilde{F}^{\mathcal{O}}_{\mathcal{Q}}(X) \simeq \tilde{F}^{\mathcal{O}}_{\mathcal{P}}(\tilde{F}^{\mathcal{P}}_{\mathcal{Q}}(X))$: by construction, the left side is the geometric realisation of the bisimplicial space with $B_{p,q} = \mathbb{O}\mathbb{P}^{p+1}\mathbb{Q}^q X$. If we first realise each $B_{p,\bullet}$, then we obtain a simplicial space $\tilde{B}_\bullet$ with $\tilde{B}_p = |B_\bullet(\mathbb{O}\mathbb{P}^{p+1}, \mathbb{Q}, X)|$. Again, we have an augmentation map $\tilde{B}_\bullet \to \tilde{B}_{-1} := |B_\bullet(\mathbb{O}, \mathbb{Q}, X)|$, which admits an extra degeneracy by the unit of $\mathbb{P}$, whence the induced map $|B_{\bullet,\bullet}| \simeq |\tilde{B}_\bullet| \to \tilde{B}_{-1}$ is an equivalence, as desired. In our case, we obtain for each $n \in N$ an equivalence of $\mathcal{D}_\infty$-algebras

$$\tilde{F}^{\mathcal{D}_\infty \otimes EN}_{\mathcal{B}\otimes\mathbf{I}}(X)_n \simeq \tilde{F}^{\mathcal{D}_\infty \otimes EN}_{\mathcal{B}\otimes EN}(\tilde{F}^{\mathcal{B}\otimes EN}_{\mathcal{B}\otimes\mathbf{I}}(X))_n \simeq F^{\mathcal{D}_\infty}_{\mathcal{B}}(\mathrm{hocolim}_{\mathbf{I}}(X_\bullet)),$$

where for the last equivalence, we use that $\tilde{F}^{\mathcal{B}\otimes EN}_{\mathcal{B}\otimes\mathbf{I}}(X)$ is, when regarded as a functor $EN \to \mathbf{Top}_*$, the constant diagram with value $\mathrm{hocolim}_{\mathbf{I}}(X)$, and $\tilde{F}^{\mathcal{D}_\infty \otimes EN}_{\mathcal{B}\otimes EN}$ is equivalent to the postcomposition with $F^{\mathcal{D}_\infty}_{\mathcal{B}}$, using that the natural map $\tilde{F}^{\mathcal{D}_\infty}_{\mathcal{B}}(X) \to F^{\mathcal{D}_\infty}_{\mathcal{B}}(X)$ is an equivalence for each based space X, as the underlying simplicial space is constant. □

Putting everything together, we get, for each $n \in N$, a map of A_∞-algebras

$$\tilde{F}^{\mathcal{O}}_{\mathcal{B}\otimes\mathbf{I}}(X) \to \mathcal{O}\binom{}{n} \times F^{\mathcal{D}_\infty}_{\mathcal{B}}(\mathrm{hocolim}_{\mathbf{I}}(X_\bullet)),$$

which, after group completion, gives us the map from Theorem 5.9. In the next subsection, we will show that it is an equivalence and, by doing so, prove the theorem.

5.3. *Proof of the splitting theorem*

For the proof of Theorem 5.9, we denote the monads associated with $\mathcal{O}$, respectively $\mathcal{D}_\infty \otimes EN$ by $\mathbb{O}$, respectively $\mathbb{D}$, and we also write $\mathbb{O}(X)_n$, respectively $\mathbb{D}(X)_n$ for the n^{th} level.

Since permuting and blocking inputs preserve the degree of the operations, we get, for each colour $n \in N$ and each degree $\delta \in \pi_0(\mathcal{O}\binom{}{n})$, a functor $\mathcal{O}\binom{-}{n}^\delta\colon (N \wr \mathbf{Inj})^{\mathrm{op}} \to \mathbf{Top}$. This gives rise to a decomposition

$$\mathbb{O}(X)_n = \coprod_\delta \mathbb{O}(X)^\delta_n \quad \text{with} \quad \mathbb{O}(X)^\delta_n := \int^{K \in N\wr\mathbf{Inj}} \mathcal{O}\binom{K}{n}^\delta \times X(K).$$

We denote by $\tilde{x}_n = [\tilde{e}_n; ()] \in \mathbb{O}(X)^{e_n}_n$ the image of the propagator and define, in analogy with Construction 5.4,

$$\mathbb{O}(X)^\infty_n := \mathrm{hocolim}\left(\mathbb{O}(X)^0_n \xrightarrow{-\curlyvee\tilde{x}_n} \mathbb{O}(X)^{e_n}_n \xrightarrow{-\curlyvee\tilde{x}_n} \mathbb{O}(X)^{2e_n}_n \xrightarrow{-\curlyvee\tilde{x}_n} \cdots\right).$$

Again, we have two relevant maps:

1. The map $f\colon \mathbb{O}(X) \to \mathbb{O}(*)$ decomposes into maps $f^\delta_n\colon \mathbb{O}(X)^\delta_n \to \mathbb{O}(*)^\delta_n$, which are compatible with stabilisations. Thus, we get a map between the mapping telescopes $f^\infty_n\colon \mathbb{O}(X)^\infty_n \to \mathbb{O}(*)^\infty_n$.
2. The map of $\mathcal{O}$-algebras $\eta\colon \mathbb{O}(X) \to \mathbb{D}(X)$ restricts to maps of spaces $\eta^\delta_n\colon \mathbb{O}(X)^\delta_n \to \mathbb{D}(X)_n$, and the triangle

$$\begin{tikzcd} \mathbb{O}(X)^\delta_n \arrow[rr, "-\curlyvee\tilde{x}_n"] \arrow[dr, "\eta^\delta_n"'] & & \mathbb{O}(X)^{\delta\curlyvee e_n}_n \arrow[dl, "\eta^{\delta\curlyvee e_n}_n"] \\ & \mathbb{D}(X)_n & \end{tikzcd}$$

is H-commutative, whence we obtain a map from the mapping telescope $\eta^\infty_n\colon \mathbb{O}(X)^\infty_n \to \mathbb{D}(X)_n$.

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Lemma 5.13 (Key Lemma). *For each colour $n \in N$, the product map*

$$(f_n^\infty, \eta_n^\infty)\colon \mathbb{O}_n^\infty(\boldsymbol{X}) \to \mathbb{O}(*)_n^\infty \times \mathbb{D}(\boldsymbol{X})_n$$

is a homology equivalence.

Let us first prove Theorem 5.9 using the Key Lemma 5.13.

Proof of Theorem 5.9. Let us abbreviate $\tilde{\mathbb{O}} := \tilde{F}^{\mathcal{O}}_{\mathcal{B}\otimes\mathbf{I}}$ and $\tilde{\mathbb{D}} := \tilde{F}^{\mathcal{D}_\infty\otimes EN}_{\mathcal{B}\otimes\mathbf{I}}$. Then we have to show that the map $(f,\eta)\colon \tilde{\mathbb{O}}(\boldsymbol{X}) \to \tilde{\mathbb{O}}(*) \times \tilde{\mathbb{D}}(\boldsymbol{X})$ from the previous section is levelwise an equivalence.

To this aim, we study the stabilisations for $\tilde{\mathbb{O}}$: as before, restricting the operation spaces of $\mathcal{O}$ gives rise to a grading of each level $B_\bullet(\mathbb{O},\mathbb{I},\boldsymbol{X})_n$, and we denote the components by $B_\bullet(\mathbb{O},\mathbb{I},\boldsymbol{X})_n^\delta$ and their realisation by $\tilde{\mathbb{O}}(\boldsymbol{X})_n^\delta$. Adding the propagator gives rise to maps $B_\bullet(\mathbb{O},\mathbb{I},\boldsymbol{X})_n^\delta \to B_\bullet(\mathbb{O},\mathbb{I},\boldsymbol{X})_n^{\delta\curlyvee e_n}$ of simplicial spaces and thus also to maps $\tilde{\mathbb{O}}(\boldsymbol{X})_n^\delta \to \tilde{\mathbb{O}}(\boldsymbol{X})_n^{\delta\curlyvee e_n}$. We denote the colimits by $B_\bullet(\mathbb{O},\mathbb{I},\boldsymbol{X})_n^\infty$ and $\tilde{\mathbb{O}}(\boldsymbol{X})_n^\infty$; then clearly $\tilde{\mathbb{O}}(\boldsymbol{X})_n^\infty \simeq |B_\bullet(\mathbb{O},\mathbb{I},\boldsymbol{X})_n^\infty|$.

Again, we obtain a product map $(f_n^\infty,\eta_n^\infty)\colon \tilde{\mathbb{O}}(\boldsymbol{X})_n^\infty \to \tilde{\mathbb{O}}(*)_n^\infty \times \tilde{\mathbb{D}}(\boldsymbol{X})_n$, and we claim that it is a homology equivalence: since we have seen already that the simplicial spaces are proper, we can invoke the spectral sequence for the geometric realisation from [24, Prop. A1], whence it is enough to see that, for each dimension $p \geqslant 0$ and each $n \in N$, the map $B_p(\mathbb{O},\mathbb{I},\boldsymbol{X})_n^\infty \to B_p(\mathbb{O},\mathbb{I},*)_n^\infty \times B_p(\mathbb{D},\mathbb{I},\boldsymbol{X})_n$ is a homology equivalence. As we have $B_p(\mathbb{O},\mathbb{I},*)_n^\infty = \mathbb{O}(*)_n^\infty$, the map in question is exactly the map from the Key Lemma 5.13 for the sequence $\mathbb{I}^p\boldsymbol{X}$. This shows the subclaim.

The rest of the proof is a combinatorially enhanced variation of the first part of the proof of [1, Thm. 5.4] that uses the classical group completion theorem: let us denote by $\boldsymbol{e}_n \in \pi_0(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)$ the component of the 0-simplex $(\tilde{e}_n,[\mathfrak{v};\varnothing])$: that is, the propagator and the unit, and by $\boldsymbol{x}_n \in \pi_0(\tilde{\mathbb{O}}(\boldsymbol{X})_n)$ the component of the 0-simplex $\tilde{x}_n$. By a classical telescope argument, the subclaim implies that the map

$$H_\bullet(f_n,\eta_n)\colon H_\bullet(\tilde{\mathbb{O}}(\boldsymbol{X})_n)[\boldsymbol{x}_n^{-1}] \to H_\bullet(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)[\boldsymbol{e}_n^{-1}]$$

that is induced by the map of Pontrjagin rings is an isomorphism.

Now recall that all input blocking maps $\beta_n^K : \mathcal{O}\binom{K}{n} \to \pi_0(\mathcal{O}\binom{}{n})$ are assumed to be surjective. Then $(f_n,\eta_n)_*\colon \pi_0(\tilde{\mathbb{O}}(\boldsymbol{X})_n) \to \pi_0(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)$ is surjective as well, so under the above map, the multiplicative submonoid $\pi_0(\tilde{\mathbb{O}}(\boldsymbol{X})_n)$ is sent surjectively onto the the submonoid $\pi_0(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)$. Therefore, we can localise further, with respect to the multiplicative submonoids of *all* path components on both sides, still obtaining an isomorphism. We get a diagram

$$\begin{tikzcd}
H_\bullet(\tilde{\mathbb{O}}(\boldsymbol{X})_n)[\boldsymbol{x}_n^{-1}] \arrow[r, "H_\bullet(f_n,\eta_n)", "\cong"'] \arrow[d] & H_\bullet(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)[\boldsymbol{e}_n^{-1}] \arrow[d] \\
H_\bullet(\tilde{\mathbb{O}}(\boldsymbol{X})_n)[\pi_0^{-1}] \arrow[r, "H_\bullet(f_n,\eta_n)", "\cong"'] \arrow[d, equal] & H_\bullet(\tilde{\mathbb{O}}(*)_n \times \tilde{\mathbb{D}}(\boldsymbol{X})_n)[\pi_0^{-1}] \arrow[d, equal] \\
H_\bullet(\Omega B\tilde{\mathbb{O}}(\boldsymbol{X})_n) \arrow[r, "H_\bullet(\Omega B(f_n,\eta_n))", "\cong"'] & H_\bullet(\Omega B\tilde{\mathbb{O}}(*)_n \times \Omega B\tilde{\mathbb{D}}(\boldsymbol{X})_n),
\end{tikzcd}$$

where the vertical isomorphisms between the second and the third row follow from the group completion theorem [18] for $\mathcal{D}_1$-algebras. This shows that $\Omega B(f_n,\eta_n)$ is a homology equivalence of loop spaces and thus a weak equivalence. □

The pending proof of Key Lemma 5.13 requires some further preparation. Recall from Definition 4.3 that for each tuple K, we denote by $N[K] \subseteq N \wr \mathbf{Inj}$ the full subgroupoid spanned by all objects of the form τ^*K. For an input profile K and an output $n \in N$, recall the stable operation spaces $\mathcal{O}\binom{K}{n}^\infty$. Since input permutation and precomposition commutes with stabilisation, these spaces assemble, for each $n \in N$ and each tuple K, into a functor $\mathcal{O}\binom{-}{n}^\infty\colon N[K]^{\mathrm{op}} \to \mathbf{Top}$.

Now let $\boldsymbol{Q}\colon N[K] \to \mathbf{Top}$ be any functor. Again, we have two maps: first, for each tuple $L = \tau^* K$ and each $n \in N$, we have the stable input block map $\mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L) \to \mathcal{O}\binom{}{n}^\infty$, which ignores the factor $\boldsymbol{Q}(L)$. These maps define a natural transformation of functors from $\mathcal{O}\binom{-}{n}^\infty \times \boldsymbol{Q}(-)$ to the constant functor $N[K]^{\mathrm{op}} \times N[K] \to \mathbf{Top}$ with value $\mathcal{O}\binom{}{n}^\infty$, so we get

$$\alpha_1\colon \int^L \mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L) \to \mathcal{O}\binom{}{n}^\infty.$$

Second, the morphism $\pi\colon \mathcal{O} \to \mathcal{D}_\infty \otimes EN$ gives, for each $n \in N$, rise to a natural transformation $\mathcal{O}^\infty\binom{-}{n} \Rightarrow (\mathcal{D}_\infty \otimes EN)\binom{}{n}$ of functors $N[K]^{\mathrm{op}} \to \mathbf{Top}$, so we obtain a map, where L ranges in $N[K]$,

$$\alpha_2\colon \int^L \mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L) \to \int^L (\mathcal{D}_\infty \otimes EN)\binom{L}{n} \times \boldsymbol{Q}(L) \cong \mathcal{D}_\infty \times_{\mathfrak{S}_r} \coprod_{L=\tau^* K} \boldsymbol{Q}(L).$$

The following lemma is a coloured version of [1, Lem. 5.2]:

Lemma 5.14. *The product map $\alpha_{\boldsymbol{Q}} := (\alpha_1, \alpha_2)$ is a homology equivalence:*

$$\alpha_{\boldsymbol{Q}}\colon \int^{L \in N[K]} \mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L) \to \mathcal{O}\binom{}{n}^\infty \times \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_{L \in N[K]} \boldsymbol{Q}(L).$$

Proof. Since $\mathcal{D}_\infty(r)$ is contractible, the sequence

$$\mathcal{O}\binom{L}{n}^\infty \longrightarrow \mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L) \xrightarrow{\pi\times\mathrm{id}} \mathcal{D}_\infty(r) \times \boldsymbol{Q}(L)$$

induces a split long exact sequence of homotopy groups for each $L = \tau^* K$ and each choice of basepoint. If we take for the total space and the base space the disjoint union over all such L, the common fibre for each component is $\mathcal{O}\binom{L}{n}^\infty \cong \mathcal{O}\binom{K}{n}^\infty$. If we moreover quotient by the free and compatible $\mathfrak{S}_r$-actions on total space and base space, we finally obtain a long exact sequence of homotopy groups that is induced by

$$\mathcal{O}\binom{K}{n}^\infty \to \int^L \mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L) \to \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_{L=\tau^* K} \boldsymbol{Q}(L).$$

Now the product map $\alpha_{\boldsymbol{Q}}$ is the composition of the two middle vertical maps in the following (3×3)-diagram, where we abbreviate $\mathfrak{S} := \mathfrak{S}_r$,

$$\begin{tikzcd}
\mathcal{O}\binom{K}{n}^\infty \arrow[r] \arrow[d, equal] & \int^L \mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L) \arrow[r] \arrow[d, "\int^L (\mathrm{id},\pi)\times\mathrm{id}"] & \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L) \arrow[d, equal] \\
\mathcal{O}\binom{K}{n}^\infty \arrow[r] \arrow[d, "\beta_n^K"'] & \int^L \left(\mathcal{O}\binom{L}{n}^\infty \times (\mathcal{D}_\infty \otimes EN)\binom{L}{n}\right) \times \boldsymbol{Q}(L) \arrow[r] \arrow[d, "\int^L (\beta_n^L\times\mathrm{id})\times\mathrm{id}"] & \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L) \arrow[d, equal] \\
\mathcal{O}\binom{}{n}^\infty \arrow[r] & \mathcal{O}\binom{}{n}^\infty \times \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L) \arrow[r] & \mathcal{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L).
\end{tikzcd}$$

Here the top-left square commutes up to homotopy, and all other squares commute strictly. We have already seen that the top row induces a long exact sequence on homotopy groups, and the second row is clearly a fibration. By the 5-lemma, the first middle vertical map is a weak equivalence. Similarly, we know that both the second and third rows are fibrations, so we obtain a morphism between the associated

Serre spectral sequences in homology. Since $\mathcal{O}$ is an operad with homological stability, the map β_n^K between the fibres is a homology equivalence, so by a standard comparison argument [27, 5.2.12], the second middle vertical map is also a homology equivalence. □

Now we have everything together to prove the Key Lemma 5.13.

Proof of the Key Lemma 5.13. Recall that, after identifying the stable spaces $\mathbb{O}(*)_n^\infty$ with $\mathcal{O}\binom{}{n}^\infty$, our aim is to show that the map

$$q := (f_n^\infty, \eta_n^\infty)\colon \mathbb{O}(\boldsymbol{X})_n^\infty \to \mathcal{O}\binom{}{n}^\infty \times \mathbb{D}(\boldsymbol{X})_n$$

induces isomorphisms on homology. If we denote by $(N \wr \mathbf{Inj})_{\leqslant r}$ the full subcategory of $N \wr \mathbf{Inj}$ with objects tuples of length at most r, then the two sides of q are exhaustively filtered by $F_{-1} = F'_{-1} = \varnothing$ and

$$F_r := \int^{K\in (N\wr \mathbf{Inj})_{\leqslant r}} \mathcal{O}\binom{K}{n}^\infty \times \boldsymbol{X}(K)$$

$$\text{respectively} \quad F'_r := \mathcal{O}\binom{}{n}^\infty \times \int^{K\in (N\wr \mathbf{Inj})_{\leqslant r}} (\mathscr{D}_\infty \otimes EN)\binom{K}{n} \times \boldsymbol{X}(K),$$

and the map q is filtration-preserving. Let us fix a system $S_r \subseteq N^r$ of representatives for unordered tuples and set $\boldsymbol{Q}(K) := X_{k_1} \wedge \cdots \wedge X_{k_r}$. Then the filtration quotients are of the form

$$F_r/F_{r-1} \cong \bigvee_{K\in S_r} \int^{L\in N[K]} \mathcal{O}\binom{L}{n}_+^\infty \wedge \boldsymbol{Q}(L)$$

$$F'_r/F'_{r-1} \cong \bigvee_{K\in S_r} \mathcal{O}\binom{}{n}_+^\infty \wedge \mathscr{D}_\infty(r)_+ \wedge_{\mathfrak{S}_r} \bigvee_{L=\tau^*K} \boldsymbol{Q}(L),$$

and the map $q_r\colon F_r/F_{r-1} \to F'_r/F'_{r-1}$ between the filtration quotients splits as a bouquet $q_r = \bigvee_{K\in S_r} q_K$. We show that each q_K is a homology equivalence; then it follows that also the map q_r between the filtration quotients is a homology equivalence, so by applying a comparison argument [27, 5.2.12] to the morphism of spectral sequences assigned to the filtration-preserving map q, we get that q itself is a homology equivalence.

In order to see that each q_K is indeed a homology equivalence, we use that $\boldsymbol{X}$ is well-based and obtain that the induced maps

$$\int^{L\in N[K]} \mathcal{O}\binom{L}{n}^\infty \to \int^{L\in N[K]} \mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L)$$

$$\text{and} \quad \mathscr{D}_\infty(r) \times_{\mathfrak{S}_r} [K] \to \mathscr{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_{L=\tau^*K} \boldsymbol{Q}(L)$$

are cofibrations, where we write $[K] := \{\tau^*K;\ \tau \in \mathfrak{S}_r\}$. If we write $\alpha_{\boldsymbol{Q}}$ for the product map from Lemma 5.14 and α_0 for the analogous one for the trivial family $* = (*)_{n\in N}$, then we obtain a morphism of cofibre sequences (written vertically for space reasons)

$$\begin{tikzcd}
\int^L \mathcal{O}\binom{L}{n}^\infty \arrow[r, "\alpha_0"] \arrow[d] & \mathcal{O}\binom{}{n}^\infty \times \mathscr{D}_\infty(r) \times_{\mathfrak{S}_r} [K] \arrow[d] \\
\int^L \mathcal{O}\binom{L}{n}^\infty \times \boldsymbol{Q}(L) \arrow[r, "\alpha_{\boldsymbol{Q}}"] \arrow[d] & \mathcal{O}\binom{}{n}^\infty \times \mathscr{D}_\infty(r) \times_{\mathfrak{S}_r} \coprod_L \boldsymbol{Q}(L) \arrow[d] \\
\int^L \mathcal{O}\binom{L}{n}_+^\infty \wedge \boldsymbol{Q}(L) \arrow[r, "q_K"] & \mathcal{O}\binom{}{n}_+^\infty \wedge \mathscr{D}_\infty(r)_+ \wedge_{\mathfrak{S}_r} \bigvee_L \boldsymbol{Q}(L),
\end{tikzcd}$$

where L ranges in $N[K]$. By Lemma 5.14, α_0 and $\alpha_{\mathbf{Q}}$ induce isomorphisms in homology, so by the 5-lemma applied to the long exact sequence associated to the cofibre sequence, we obtain that also q_K is a homology equivalence. □

6. $\Lambda\mathfrak{M}_{*,1}$ as a relatively free algebra

In this section, we combine the insights from the previous sections: we translate the results of Section 2 and Section 3, which are expressed in terms of groups, into the analogous results in terms of classifying spaces.

This will lead to Theorem 6.5, expressing $\Lambda\mathfrak{M}_{*,1}$ as the colour-1 part of a relatively free $\mathscr{M}$-algebra, with generators an algebra over the family of twisted tori, which depends on ∂-irreducible mapping classes. Using the results of Section 4 and Section 5, we deduce the identification from Theorem 1.1.

6.1. Recollections from Section 2 and Section 4

Recall Definition 2.14 and Definition 2.19, let $\mathcal{S}$ be a surface of type $\Sigma_{g,n}$ for some $g \geqslant 0$ and $n \geqslant 1$, and note that a mapping class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ is ∂-irreducible if and only if the cut locus of φ is equal to $[\partial\mathcal{S}]$: that is, it is the collection of oriented isotopy classes of all boundary curves of $\mathcal{S}$. Note in particular the following special cases:

- if $\mathcal{S}$ is a cylinder, $[\partial\mathcal{S}]$ contains two isotopy classes, as the two curves of $\partial\mathcal{S}$ are not isotopic as oriented curves; the mapping class $\mathbb{1} \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ has empty cut locus, and every other mapping class is ∂-irreducible and has cut locus equal to $[\partial\mathcal{S}]$;
- if $\mathcal{S}$ is a disc, $[\partial\mathcal{S}]$ contains one isotopy class (of a null-homotopic curve); the unique mapping class $\mathbb{1} \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ has empty cut locus and was declared not to be ∂-irreducible.

Alternatively, we saw that φ is ∂-irreducible if and only if the white part $W \subset \mathcal{S}$ deformation retracts onto $\partial^{\mathrm{out}}W = \partial\mathcal{S}$. Being ∂-irreducible is an invariant of conjugacy classes in $\Gamma(\mathcal{S}, \partial\mathcal{S})$, and in fact even of conjugacy classes of $\Gamma(\mathcal{S})$ that are contained in the normal subgroup $\Gamma(\mathcal{S}, \partial\mathcal{S})$.

Recall Construction 4.13. We have a similar action of $R_n = T^n \rtimes \mathfrak{S}_n$ on the space $\mathscr{M}\binom{}{n} = \mathbf{M}_\partial\binom{0}{n}$ by postcomposition, regarding $R_n \subset \mathbf{M}_\partial\binom{n}{n}$. By taking this action pointwise over S^1, we obtain an action of R_n on $\Lambda\mathscr{M}\binom{}{n}$.

Notation 6.1. For $g \geqslant 0$ and $n \geqslant 1$, we denote by $\mathscr{M}_{g,n} \subset \mathscr{M}\binom{}{n} = \mathbf{M}_\partial\binom{0}{n}$ the subspace corresponding to conformal classes $(W, \tilde{\vartheta})$ with W of type $\Sigma_{g,n}$

We have a homotopy equivalence $\mathscr{M}_{g,n} \to \mathfrak{M}_{g,n}$ given by sending $(W, \tilde{\vartheta})$ to (W, ϑ), where ϑ is obtained by restricting $\tilde{\vartheta}$ to (the preimage of) ∂W. We have therefore a homotopy equivalence

$$\Lambda\mathscr{M}_{g,n} \simeq \Lambda\mathfrak{M}_{g,n} \simeq \coprod_{[\varphi]\in\mathrm{Conj}(\Gamma_{g,n})} BZ(\varphi, \Gamma_{g,n}).$$

In the following, we will mostly replace the spaces $\Lambda\mathfrak{M}_{g,n}$ by the spaces $\Lambda\mathscr{M}_{g,n}$. In particular, we denote

$$\Lambda\mathscr{M}_{*,1} := \coprod_{g\geqslant 0} \Lambda\mathscr{M}_{g,1} \simeq \Lambda\mathfrak{M}_{*,1},$$

where the last equivalence is an equivalence of $\mathscr{D}_2$-algebras.

6.2. Action of $\mathfrak{S}_n$ on $\Lambda\mathscr{M}_{g,n}$

In this subsection, we analyse the action of $\mathfrak{S}_n$ on the set of components of $\Lambda\mathscr{M}_{g,n}$ and classify the orbits of this action.

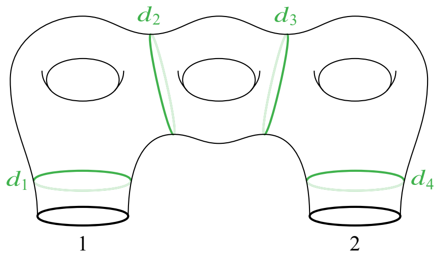

Figure 10. *If we denote by $D_i := D_{d_i}$ the Dehn twist along d_i, then the mapping classes $D_1D_2D_4$ and $D_1D_3D_4$ are both ∂-irreducible in $\Gamma_{3,2}$ and not conjugate to each other, but they are conjugate in the extended mapping class group $\Gamma_{3,(2)}$.*

Notation 6.2. For a mapping class $\varphi \in \Gamma_{g,n}$, we denote by $\Lambda\mathfrak{M}_{g,n}(\varphi)$ the connected component of $\Lambda\mathfrak{M}_{g,n}$ corresponding to $BZ(\varphi, \Gamma_{g,n})$. It contains all free loops $\lambda\colon S^1 \to \mathfrak{M}_{g,n}$ with the following property: if $\mathcal{S}$ is a Riemann surface of type $\Sigma_{g,n}$ representing $\lambda(1)$ and if $\varphi' \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ is the monodromy of λ, then there is a diffeomorphism $\Xi\colon \mathcal{S} \to \Sigma_{g,n}$ preserving the parametrisation *and the ordering* of the boundary components, with $(\varphi')^{\Xi} = \varphi \in \Gamma_{g,n}$. We denote by $\Lambda\mathscr{M}_{g,n}(\varphi)$ the corresponding component of $\Lambda\mathscr{M}_{g,n}$.

Note that $\Lambda\mathscr{M}_{g,n}(\varphi) \subset \Lambda\mathscr{M}_{g,n}$ is invariant under the action of $T^n \subset R_n$ but not necessarily under the action of $\mathfrak{S}_n$. We denote by $\mathfrak{S}_n \cdot \Lambda\mathscr{M}_{g,n}(\varphi) \subset \Lambda\mathscr{M}_{g,n}$ the orbit of $\mathscr{M}_{g,n}(\varphi)$ under the action of $\mathfrak{S}_n$ or, equivalently, under the action of R_n. We have

$$\mathfrak{S}_n \cdot \Lambda\mathscr{M}_{g,n}(\varphi) = \bigcup_{\varphi'} \Lambda\mathscr{M}_{g,n}(\varphi'),$$

where φ' ranges over all conjugates of φ in the extended mapping class group $\Gamma_{g,(n)}$. Note that these conjugates still lie in the subgroup $\Gamma_{g,n} \subset \Gamma_{g,(n)}$, which is normal. Note also that $\Lambda\mathscr{M}_{g,n}(\varphi') = \Lambda\mathscr{M}_{g,n}(\varphi)$ if and only if φ and φ' are conjugate not only in $\Gamma_{g,(n)}$, but also in $\Gamma_{g,n}$: see Figure 10 for an example that shows the difference.

The subspace of $\mathfrak{M}_{g,n}$ corresponding to $\mathfrak{S}_n \cdot \Lambda\mathfrak{M}_{g,n}(\varphi) \subset \Lambda\mathfrak{M}_{g,n}$ can be described as follows: it contains all free loops $\lambda\colon S^1 \to \mathfrak{M}_{g,n}$ with the same property as in Notation 6.2, but where Ξ is only required to preserve the parametrisation, and not necessarily the order, of the boundary components.

Lemma 6.3. *Let $\varphi \in \Gamma_{g,n}$, and let $\mathfrak{H} \subset \mathfrak{S}_n$ be the image of $Z(\varphi, \Gamma_{g,(n)})$ under the natural map $\Gamma_{g,(n)} \to \mathfrak{S}_n$. Then there is a bijection of $\mathfrak{S}_n$-sets*

$$\pi_0\big(\mathfrak{S}_n \cdot \Lambda\mathscr{M}_{g,n}(\varphi)\big) \cong \mathfrak{S}_n/\mathfrak{H}.$$

Proof. The action of $\mathfrak{S}_n$ on $\pi_0(\Lambda\mathscr{M}_{g,n})$ can be described as follows: given $\sigma \in \mathfrak{S}_n$, we choose a mapping class $\xi \in \Gamma_{g,(n)}$, which is sent to σ under the natural map $\Gamma_{g,(n)} \to \mathfrak{S}_n$, and a representative Ξ of ξ; then the component $\pi_0(\Lambda\mathscr{M}_{g,n}(\varphi'))$ is sent by σ to the component $\pi_0(\Lambda\mathscr{M}_{g,n}((\varphi')^{\Xi}))$.

The action of $\mathfrak{S}_n$ on $\pi_0(\mathfrak{S}_n \cdot \Lambda\mathscr{M}_{g,n}(\varphi))$ is transitive, so it suffices to check that the stabiliser of the component $\Lambda\mathscr{M}_{g,n}(\varphi)$ is the subgroup $\mathfrak{H}$. First, note that an element $\sigma \in \mathfrak{H}$ can be lifted to a class $\xi \in \Gamma_{g,(n)}$ that commutes with φ; this implies that $\mathfrak{H}$ is contained in the stabiliser of $\Lambda\mathscr{M}_{g,n}(\varphi)$. Vice versa, if $\sigma \in \mathfrak{S}_n$ belongs to the stabiliser of $\Lambda\mathscr{M}_{g,n}(\varphi)$, then we can choose a lift $\xi \in \Gamma_{g,(n)}$ of σ and a representative Ξ such that $\Lambda\mathscr{M}_{g,n}(\varphi^{\Xi})$ is equal to $\Lambda\mathscr{M}_{g,n}(\varphi)$); this implies that φ^{Ξ} is conjugate to φ in $\Gamma_{g,n}$: that is, there is a mapping class $\bar{\xi} \in \Gamma_{g,n}$ and a representative $\bar{\Xi}$ with $\varphi^{\Xi} = \varphi^{\bar{\Xi}}$. As a consequence $\xi^{-1}\bar{\xi} \in \Gamma_{g,(n)}$ commutes with φ, and since $\xi^{-1}\bar{\xi}$ also projects to σ along the natural map $\Gamma_{g,(n)} \to \mathfrak{S}_n$, we conclude that $\sigma \in \mathfrak{H}$. ◻

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6.3. Relative generators for the $\mathcal{M}|_1$-algebra $\Lambda\mathcal{M}_{*,1}$

Definition 6.4. For all $n \geqslant 1$ and $g \geqslant 0$, we define the space

$$\mathfrak{C}_{g,n} := \coprod_{\substack{[\varphi]\in\mathrm{Conj}(\Gamma_{g,n})\\ \partial\text{-irreducible}}} \Lambda\mathcal{M}_{g,n}(\varphi).$$

The action of R_n on $\Lambda\mathcal{M}_{g,n}$ restricts to an action on those path components that constitute $\mathfrak{C}_{g,n}$. We furthermore set, for all $n \geqslant 1$,

$$\mathfrak{C}_n := \coprod_{g\geqslant 0} \mathfrak{C}_{g,n}$$

and obtain a $\boldsymbol{R}$-algebra $\mathfrak{C} := (\mathfrak{C}_n)_{n\geqslant 1}$, where $\boldsymbol{R} = (R_n)_{n\geqslant 1}$.

To see that the action of R_n on $\Lambda\mathfrak{M}_{g,n}$ indeed restricts to an action on $\mathfrak{C}_{g,n}$, we note that, for every ∂-irreducible mapping class $\varphi \in \Gamma_{g,n}$, the entire orbit $\mathfrak{S}_n \cdot \Lambda\mathcal{M}_{g,n}(\varphi)$ is contained in $\mathfrak{C}_{g,n}$. Note also that, though $\mathfrak{S}_n \cdot \Lambda\mathcal{M}_{g,n}(\varphi)$ may be disconnected, by Lemma 6.3 the action of R_n is transitive on the connected components of $\mathfrak{S}_n \cdot \Lambda\mathcal{M}_{g,n}(\varphi)$.

We now want to look at the relatively free $\mathcal{M}$-algebra $F^{\mathcal{M}}_{\boldsymbol{R}}(\mathfrak{C})$. Here we see, levelwise, that for each $n \geqslant 1$, we have

$$F^{\mathcal{M}}_{\boldsymbol{R}}(\mathfrak{C})_n \cong \int^{(k_1,\dots,k_r)\in\mathbb{N}_{\geqslant 1}\wr\boldsymbol{\Sigma}} \mathcal{M}\binom{k_1,\dots,k_r}{n} \times_{R_{k_1}\times\cdots\times R_{k_r}} \left(\mathfrak{C}_{k_1}\times\cdots\times\mathfrak{C}_{k_r}\right).$$

Theorem 6.5. *There is an equivalence of $\mathcal{M}|_1$-algebras*

$$F^{\mathcal{M}}_{\mathcal{B}\otimes\boldsymbol{R}}(\mathfrak{C}_+)_1 = F^{\mathcal{M}}_{\boldsymbol{R}}(\mathfrak{C})_1 \simeq \Lambda\mathcal{M}_{*,1}$$

The proof of Theorem 6.5 occupies the remainder of this section.

Recall the initial $\mathcal{M}$-algebra $(\mathcal{M}\binom{}{n})_{n\geqslant 1}$. Note that $\Lambda\mathcal{M}\binom{}{n}$ has several connected components, and the ones corresponding to connected surfaces form precisely the subspace $\Lambda\mathcal{M}_{*,n} := \coprod_{g\geqslant 0}\Lambda\mathcal{M}_{g,n}$. We therefore have an inclusion of $\boldsymbol{R}$-algebras $\mathfrak{C} \subset (\Lambda\mathcal{M}\binom{}{n})_{n\geqslant 1}$. This inclusion is adjoint to a morphism of $\mathcal{M}$-algebras $F^{\mathcal{M}}_{\boldsymbol{R}}(\mathfrak{C}) \to (\Lambda\mathcal{M}\binom{}{n})_{n\geqslant 1}$, which can be restricted to a morphism of $\mathcal{M}|_1$-algebras $\kappa\colon F^{\mathcal{M}}_{\boldsymbol{R}}(\mathfrak{C})_1 \to \Lambda\mathcal{M}\binom{}{1} = \Lambda\mathcal{M}_{*,1}$. It suffices to prove that κ is a weak equivalence at the level of spaces. The right-hand side can be decomposed into its connected components as

$$\Lambda\mathcal{M}_{*,1} = \coprod_{\substack{g\geqslant 0\\ [\varphi]\in\mathrm{Conj}(\Gamma_{g,1})}} \Lambda\mathcal{M}_{g,1}(\varphi).$$

Fix $g \geqslant 0$ and a mapping class $\varphi \in \Gamma_{g,1}$. Decompose $\Sigma_{g,1}$ along a system of curves $c_1,\dots,c_h$ representing the cut locus of φ, and let W and Y denote the white and the yellow regions. We use Notation 3.2 and write $Y = \coprod_{i=1}^r \coprod_{j=1}^{s_i} Y_{i,j}$. Each component $Y_{i,j}$ of Y is of type Σ_{g_i,n_i}, and the restriction $\varphi_{i,j} \in \Gamma(Y_{i,j},\partial Y_{i,j})$ of φ is conjugated by a suitable diffeomorphism $\Xi_{i,j}\colon Y_{i,j} \to \Sigma_{g_i,n_i}$ to $\bar{\varphi}_i \in \Gamma_{g_i,n_i}$.

Note that W is a connected surface: each component of W touches some component of $\partial^{\mathrm{out}}W$, and there is precisely one outgoing boundary component, since $\partial^{\mathrm{out}}W = \partial\Sigma_{g,1} \cong S^1$. We denote by $\mathcal{M}(W) \subset \mathcal{M}\binom{s_1\times n_1,\dots,s_r\times n_r}{1}$ the component of the surface type of W, with one outgoing boundary component and $h = \sum_i s_i \cdot n_i$ incoming boundary curves partitioned as follows: for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s_i$ the curves $\partial Y_{i,j} \subset \partial^{\mathrm{in}}W$ form a piece of the partition, corresponding to an input of colour n_i. The total number of inputs is thus $s_1+\cdots+s_r$. Consider now the following subspace, which a priori

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is a union of connected components of $F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_1$

$$\begin{aligned}F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_\varphi &:= \mathscr{M}(W)\times_{\prod_i R^{s_i}_{n_i}\rtimes\mathfrak{S}_{s_i}}\prod_{i=1}^{r}\big(\mathfrak{S}_{n_i}\cdot\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}\\ &= \mathscr{M}(W)\times_{T^h\rtimes\prod_i(\mathfrak{S}^{s_i}_{n_i}\rtimes\mathfrak{S}_{s_i})}\prod_{i=1}^{r}\big(\mathfrak{S}_{n_i}\cdot\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}\subset F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_1.\end{aligned}$$

Lemma 6.6. *The space $F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_\varphi$ is connected.*

Proof. We observe that $\mathscr{M}(W)$ is connected; therefore, in order to prove that $F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_\varphi$ is connected, it suffices to check that $T^h\rtimes\prod_{i=1}^r(\mathfrak{S}^{s_i}_{n_i}\rtimes\mathfrak{S}_{s_i})$ acts transitively on the connected components of $\prod_{i=1}^r\big(\mathfrak{S}_{n_i}\cdot\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}$.

This reduces to checking that for all $1\leqslant i\leqslant r$, the group $(\mathfrak{S}^{s_i}_{n_i}\rtimes\mathfrak{S}_{s_i})$ acts transitively on the components of $\big(\mathfrak{S}_{n_i}\cdot\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}$, and to this aim it suffices to check that $\mathfrak{S}_{n_i}$ acts transitively on the components of $\mathfrak{S}_{n_i}\cdot\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)$, which is clear by definition. □

Using Notation 3.7, the proof of Lemma 6.6 shows in fact that there is a homeomorphism

$$F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_\varphi\cong\mathscr{M}(W)\times_{T^h\rtimes\prod_i(\mathfrak{H}^{s_i}_i\rtimes\mathfrak{S}_{s_i})}\prod_{i=1}^{r}\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i},$$

using that $\mathfrak{H}_i\subseteq\mathfrak{S}_{n_i}$ is the stabiliser of the path component $\Lambda\mathscr{M}_{g,n}(\bar\varphi_i)\subseteq\mathfrak{C}_{g_i,n_i}$. The following proposition directly implies Theorem 6.5, since $F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_\varphi$ and $\Lambda\mathscr{M}_{g,1}(\varphi)$ are the connected components of $F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_1$ and $\Lambda\mathscr{M}_{*,1}$.

Proposition 6.7. *The map κ restricts to a homotopy equivalence*

$$\kappa\colon F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_\varphi\to\Lambda\mathscr{M}_{g,1}(\varphi).$$

Proof. The space $\mathscr{M}(W)$ classifies the group $\Gamma(W,\partial W)$, whereas the product $\prod_i\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}$ is a classifying space for the group $\prod_i(Z(\bar\varphi_i,\Gamma_{g_i,n_i}))^{s_i}$. This implies that there is a homotopy equivalence

$$\mathscr{M}(W)\times\prod_{i=1}^{r}\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}\simeq B\left(\Gamma(W,\partial W)\times\prod_{i=1}^{r}(Z(\bar\varphi_i,\Gamma_{g_i,n_i}))^{s_i}\right).$$

For $\mathfrak{H}:=\prod_i(\mathfrak{H}^{s_i}_i\rtimes\mathfrak{S}_{s_i})\subseteq\mathfrak{S}_h$, the compact Lie group $T^h\rtimes\mathfrak{H}\subseteq R_n$ acts freely on $\mathscr{M}(W)$ by Lemma 4.14. Since $\mathscr{M}(W)\times\prod_i(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i))^{s_i}$ is a Tychonoff space, [12, Thm. A8viii] ensures that we obtain a principal fibre bundle

$$T^h\rtimes\mathfrak{H}\to\mathscr{M}(W)\times\prod_{i=1}^{r}\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}\to\mathscr{M}(W)\times_{T^h\rtimes\mathfrak{H}}\prod_{i=1}^{r}\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}.$$

We will split the fibre bundle in two stages, involving separately the factors $\mathfrak{H}$ and T^h of the fibre. First, consider the principal fibre bundle

$$\mathfrak{H}\to\mathscr{M}(W)\times\prod_{i}\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}\to\mathscr{M}(W)\times_{\mathfrak{H}}\prod_{i}\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i},$$

from which we conclude that $\mathscr{M}(W)\times_{\mathfrak{H}}\prod_i\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}$ is connected and aspherical.

In fact, $\mathscr{M}(W)\times_{\mathfrak{H}}\prod_i\big(\Lambda\mathscr{M}_{g_i,n_i}(\bar\varphi_i)\big)^{s_i}$ is a classifying space for the group $\tilde{Z}(\varphi)$ introduced in Definition 3.4: the space $\mathscr{M}(W)\simeq B\Gamma(W,\partial W)$ admits a free action of the finite group $\mathfrak{H}$, and the

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quotient $\mathscr{M}(W)/\mathfrak{H}$ is a classifying space for the extended mapping class group $\Gamma^{\mathfrak{H}}(W)$ (see Definition 2.6). Similarly, the homotopy quotient

$$\left(\prod_{i=1}^{r}\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}\right) /\!\!/ \mathfrak{H} = E\mathfrak{H}\times_{\mathfrak{H}}\left(\prod_{i=1}^{r}\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}\right)$$

is a classifying space for $Z(\varphi_Y,\Gamma(Y))$, which by Lemma 3.6 is isomorphic to $\prod_i(Z(\bar{\varphi}_i,\Gamma_{g_i,(n_i)})^{s_i}\rtimes\mathfrak{S}_{s_i})$. The balanced product $\mathscr{M}(W)\times_{\mathfrak{H}}\prod_i\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}$ is then homotopy equivalent to

$$\mathscr{M}(W)\times_{\mathfrak{H}}\left(\prod_{i=1}^{r}\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}\times E\mathfrak{H}\right),$$

which is a classifying space for $\tilde{Z}(\varphi)$. The last step is to consider the fibre bundle (which is no longer a principal bundle)

$$T^h\to\mathscr{M}(W)\times_{\mathfrak{H}}\prod_{i=1}^{r}\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}\to\mathscr{M}(W)\times_{T^h\rtimes\mathfrak{H}}\prod_{i=1}^{r}\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}.$$

Since the fibre and the total space are aspherical, the base space is also aspherical: note that the base is precisely $F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_{\varphi}$. The above fibre bundle shows that the fundamental group of $F^{\mathscr{M}}_{\boldsymbol{R}}(\mathfrak{C})_{\varphi}$ is a quotient of $\tilde{Z}(\varphi)$ by a normal subgroup $\mathbb{Z}^h$. Consider the gluing map

$$\tilde{\kappa}\colon\mathscr{M}(W)\times_{\mathfrak{H}}\prod_{i=1}^{r}\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}\to\Lambda\mathscr{M}_{g,1}(\varphi).$$

The map induced by $\tilde{\kappa}$ on fundamental groups is $\varepsilon\colon\tilde{Z}_{\partial}(\varphi)\to Z_{\partial}(\varphi)$, so by Proposition 3.8, we just have to identify the kernel of ε and the subgroup

$$\pi_1(T^h)\subset\pi_1\left(\mathscr{M}(W)\times_{\mathfrak{H}}\prod_{i=1}^{r}\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}\right).$$

The inclusion of the fibre $T^h\hookrightarrow\mathscr{M}(W)\times_{\mathfrak{H}}\prod_i\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}$ lifts to the covering space $\mathscr{M}(W)\times\prod_i\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}$ of the right-hand side and becomes part of the inclusion of the fibre $T^h\rtimes\mathfrak{H}\hookrightarrow\mathscr{M}(W)\times\prod_i\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}$ of the first principal bundle that we considered above. In particular, the inclusion $\pi_1(T^h)\hookrightarrow\tilde{Z}(\varphi)=\pi_1\left(\mathscr{M}(W)\times_{\mathfrak{H}}\prod_i\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}\right)$ has its image inside

$$\Gamma(W,\partial W)\times Z(\varphi_Y,\Gamma(Y,\partial Y))=\pi_1\left(\mathscr{M}(W)\times\prod_{i=1}^{r}\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}\right).$$

Recall T^h is included into $\mathscr{M}(W)\times\prod_i\left(\Lambda\mathscr{M}_{g_i,n_i}(\bar{\varphi}_i)\right)^{s_i}$ as an orbit of the *balanced* diagonal T^h-action on the two main factors of the latter space. The action of $(z_1,\dots,z_h)\in T^h$ is *on right* on the first factor, by precomposition with $(z_1,\dots,z_h)\in R_n\subset\mathbf{M}_{\partial}\binom{h}{h}$, and is *on left* on the second factor, by postcomposition with *the inverse of* $(z_1,\dots,z_h)\in R_n\subset\mathbf{M}_{\partial}\binom{h}{h}$.

At the level of fundamental groups, the i th generator of $\pi_1(T^h)\cong\mathbb{Z}^h$ is mapped to $(D_{c_i},D_{c_i}^{-1})\in\Gamma(W,\partial W)\times Z(\varphi_Y,\Gamma(Y,\partial Y))$. We saw that these h couples generate the kernel of ε as a free abelian group of rank h. □

Putting together Theorem 6.5 and Theorem 5.9, we obtain Theorem 1.1:

Proof of Theorem 1.1. The point-set requirements for our Setting 5.8 are clearly satisfied in the case $\mathscr{O}=\mathscr{M}$, $\mathbf{C}=\boldsymbol{R}$, and $\boldsymbol{X}=\mathfrak{C}_+$. Hence, we can apply Theorem 5.9 and obtain

$$\Omega B\tilde{F}^{\mathscr{M}}_{\mathscr{B}\otimes\boldsymbol{R}}(\mathfrak{C}_+)_n\simeq\Omega B\mathscr{M}\binom{}{n}\times\Omega^{\infty}\Sigma^{\infty}\mathrm{hocolim}_{\boldsymbol{R}}(\mathfrak{C}_+).$$

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Consider the left side first: we claim that $\varphi\colon \tilde{F}^{\mathscr{M}}_{\mathscr{B}\otimes\boldsymbol{R}}(\mathfrak{C}_+) \to F^{\mathscr{M}}_{\mathscr{B}\otimes\boldsymbol{R}}(\mathfrak{C}_+)$, the map of $\mathscr{M}$-algebras from the derived relatively free algebra to the actual one is an equivalence: here we use that the basepoints are isolated, whence the map splits into $\coprod_{K\in S}\varphi_K$ for a system $S \subseteq \coprod_r \mathbb{N}^r_{\geqslant 1}$ of representatives of tuples with respect to coordinate permutation. If we denote again by $r(k) \geqslant 0$ the number of occurrences of k in the sequence K, then the compact Lie group $G := \prod_{k\geqslant 0} R_k \wr \mathfrak{S}_{r_k}$ acts on $Y := \mathscr{M}\binom{K}{n} \times \prod_i \mathfrak{C}_{k_i}$, and φ_K is exactly the map that compares the *homotopy* quotient of this action with the *actual* quotient. However, Y is a Hausdorff space and G acts freely on Y, since $\mathscr{M}$ is $\mathfrak{S}$-free and R_k acts freely on $\mathscr{M}$ by precomposition; see Lemma 4.14. In this situation, [12, Thm. A.7] tells us that the map comparing the homotopy quotient with the actual quotient is an equivalence. In particular, for $n = 1$, the left side is equivalent, as a loop space, to the group completion $\Omega B\Lambda\mathfrak{M}_{*,1}$ by Theorem 6.5. Let us now look at the right side: here we see that

$$\operatorname{hocolim}_{\boldsymbol{R}}(\mathfrak{C}_+) \simeq \bigvee_{k\geqslant 1} (\mathfrak{C}_n)_+ \mathbin{/\!\!/} R_k = \{*\} \sqcup \coprod_{k\geqslant 1} \mathfrak{C}_k \mathbin{/\!\!/} R_k,$$

where $/\!\!/$ denotes the homotopy quotient. If we focus again on the case $n = 1$, then we saw in Example 4.16 that the first level of the initial $\mathscr{M}$-algebra, $\mathscr{M}\binom{}{1}$, coincides with the old $\mathscr{M}|_1$-algebra $\coprod_g \mathfrak{M}_{g,1}$ whose group completion is accessible by the Madsen–Weiss theorem. Hence, we can replace the factor $\Omega B\mathscr{M}\binom{}{1}$ by $\Omega^\infty\mathbf{MTSO}(2)$. This proves the claim. □

A. General mapping spaces into $\mathfrak{M}_{*,1}$

In this first appendix, we briefly discuss a variation of Theorem 1.1 for a generic parametrising topological space X, highlighting the main enhancements that the proof requires.

We chose to restrict ourselves to the case $X = S^1$ throughout the main part of the article because this special context already presents all the relevant complexity of the problem, and we believe that it is more instructive to consider the special case first.

A.1. General parametrising spaces X

Our goal is to give a description of $\Omega B(\operatorname{map}(X, \mathfrak{M}_{*,1}))$ as an infinite loop space. We assume for simplicity that X has the homotopy type of a connected CW complex.

Let $G := \pi_1(X)$. For all $g \geqslant 0$ and $n \geqslant 1$, we replace the space $\mathfrak{M}_{g,n}$ with the homotopy equivalent space $\mathscr{M}_{g,n}$, which admits an action of the group R_n. Since $\mathscr{M}_{g,n}$ is aspherical, the space $\operatorname{map}(X, \mathscr{M}_{g,n})$ is homotopy equivalent to $\operatorname{map}(BG, \mathscr{M}_{g,n})$.

More precisely, let $\operatorname{Conj}(G \to \Gamma_{g,n})$ be the set of conjugacy classes of homomorphisms $G \to \Gamma_{g,n}$, where two such homomorphisms φ, φ' are conjugate if there exists $\xi \in \Gamma_{g,n}$, represented by Ξ, with $\varphi' = \varphi^\Xi$: then we can identify

$$\operatorname{map}(X, \mathscr{M}_{g,n}) = \coprod_{[\varphi]\in\operatorname{Conj}(G\to\Gamma_{g,n})} \operatorname{map}(X, \mathscr{M}_{g,n})(\varphi) \simeq \coprod_{[\varphi]\in\operatorname{Conj}(G\to\Gamma_{g,n})} BZ(\varphi, \Gamma_{g,n}),$$

where $Z(\varphi, \Gamma_{g,n})$ denotes the centraliser of the image of φ in $\Gamma_{g,n}$.

Definition A.1. A homomorphism $\varphi\colon G \to \Gamma_{g,n}$ is called *∂-irreducible* if it is not the trivial representation in $\Gamma_{0,1}$ and there is no isotopy class of an essential arc $\alpha \subset \Sigma_{g,n}$, which is fixed by the entire image of φ. Being ∂-irreducible is a conjugacy-invariant property of representations. We denote by

$$\mathfrak{C}_{g,n}(X) := \coprod_{\substack{[\varphi]\in\operatorname{Conj}(G\to\Gamma_{g,n})\\ \partial\text{-irreducible}}} \operatorname{map}(X, \mathscr{M}_{g,n})(\varphi).$$

For all $n \geqslant 1$, there is a natural action of the group R_n on $\mathfrak{C}_{g,n}(X)$, and the analogue of Theorem 1.1 is the following:

Theorem A.2. *In the above setting and with the above definitions, there is an equivalence of loop spaces*

$$\Omega B(\mathrm{map}(X, \mathfrak{M}_{*,1})) \simeq \Omega^\infty \mathbf{MTSO}(2) \times \Omega^\infty \Sigma^\infty_+ \coprod_{n\geqslant 1} \coprod_{g\geqslant 0} \mathfrak{C}_{g,n}(X) /\!\!/ R_n.$$

In order to prove Theorem A.2, the main obstacle is, at the very beginning, to generalise Proposition 2.11 from the context of a single diffeomorphism Φ, representing a single mapping class $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$, to the context of a representation $\varphi\colon G \to \Gamma(\mathcal{S}, \partial\mathcal{S})$. This is done as follows:

Proposition A.3. *Let $\alpha_0, \ldots, \alpha_k, \beta$ be a collection of essential arcs in a connected surface with non-empty boundary $\mathcal{S}$ satisfying the two properties listed in Proposition 2.11. Let U be a small neighbourhood of $\alpha_0 \cup \cdots \cup \alpha_k \cup \beta \cup \partial\mathcal{S} \subset \mathcal{S}$ in $\mathcal{S}$, let $\mathrm{Diff}(\mathcal{S}, U) \subset \mathrm{Diff}(\mathcal{S}, \partial\mathcal{S})$ be the group of diffeomorphisms of $\mathcal{S}$ fixing U pointwise, and let $\Gamma(\mathcal{S}, U)$ be the associated mapping class group.*

Then the canonical homomorphism of groups $\Gamma(\mathcal{S}, U) \to \Gamma(\mathcal{S}, \partial\mathcal{S})$, induced by the inclusion $\mathrm{Diff}(\mathcal{S}, U) \subset \mathrm{Diff}(\mathcal{S}, \partial\mathcal{S})$, is injective. Moreover, a homomorphism $\varphi\colon G \to \Gamma(\mathcal{S}, \partial\mathcal{S})$ lifts to $\Gamma(\mathcal{S}, U)$ if and only if for all $\gamma \in G$ the mapping class $\varphi(\gamma) \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ fixes up to isotopy each of the arcs $\alpha_0, \ldots, \alpha_k$ and β.

Proof. Note that U is a connected surface with at least three boundary components, and in particular U is neither a disc nor an annulus. We can then apply [7, Thm. 3.18] (see also the remark after the cited Theorem) and conclude that the homomorphism of mapping class groups $\Gamma(\mathcal{S}\setminus U, \partial(\mathcal{S}\setminus U)) \to \Gamma(\mathcal{S}, \partial\mathcal{S})$ is injective. Composing with the natural isomorphism $\Gamma(\mathcal{S}\setminus U, \partial(\mathcal{S}\setminus U)) \cong \Gamma(\mathcal{S}, U)$, we obtain precisely the canonical homomorphism of groups $\Gamma(\mathcal{S}, U) \to \Gamma(\mathcal{S}, \partial\mathcal{S})$, which hence is injective.

For the second claim, the injectivity of $\Gamma(\mathcal{S}\setminus U, \partial(\mathcal{S}\setminus U)) \to \Gamma(\mathcal{S}, U)$ implies that $\varphi\colon G \to \Gamma(\mathcal{S}, \partial\mathcal{S})$ lifts to $\Gamma(\mathcal{S}, U)$ if and only if, for all $\gamma \in G$, the mapping class $\varphi(\gamma) \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ lies in the image of the canonical homomorphism, and Proposition 2.11 ensures that this is equivalent to requiring, for all $\gamma \in G$, that the mapping class $\varphi(\gamma)$ fixes up to isotopy each of the arcs $\alpha_0, \ldots, \alpha_k$, and β. $\square$

For a homomorphism $\varphi\colon G \to \Gamma(\mathcal{S}, \partial\mathcal{S})$, we define the *fixed arc complex* of φ as in Definition 2.14, but we require vertices to be isotopy classes of arcs fixed by *every* mapping class in the image of φ: in fact the fixed arc complex of φ is the intersection of the arc complexes of $\varphi(\gamma)$ for γ ranging in G, where we consider all arc complexes as simplicial subcomplexes of the arc complex of $\mathbb{1} \in \Gamma(\mathcal{S}, \partial\mathcal{S})$.

The bound on the dimension of the fixed arc complex of φ in terms of $\chi(\mathcal{S})$ is proved in the same way. The white-yellow decomposition of $\mathcal{S}$ along the cut locus is defined in the same way as in Construction 2.18: by choosing a maximal simplex in the fixed-arc complex of the homomorphism φ. The proof of Lemma 2.20 can be extended to this generalised context as follows:

Lemma A.4. *Let $\varphi\colon G \to \Gamma(\mathcal{S}, \partial\mathcal{S})$, let $\alpha_0, \ldots, \alpha_k$ be arcs representing a maximal simplex in the cut locus of φ, and let β be another arc fixed up to isotopy by φ; then β can be isotoped relative to endpoints to an arc lying in a small neighbourhood U of $\alpha_0 \cup \cdots \cup \alpha_k \cup \partial\mathcal{S}$.*

Proof. We assume without loss of generality that β is in minimal position with respect to $\alpha_0, \ldots, \alpha_k$. By Proposition A.3, we can lift φ to a homomorphism $\tilde{\varphi}\colon G \to \Gamma(\mathcal{S}, U')$, where U' is a small neighbourhood of $\alpha_0 \cup \cdots \cup \alpha_k \cup \beta \cup \partial\mathcal{S}$. For each $\gamma \in G$, we can thus represent $\tilde{\varphi}(\gamma)$ by a diffeomorphism $\Phi_\gamma\colon \mathcal{S} \to \mathcal{S}$ fixing U pointwise.

The rest of the proof is the same as for Lemma 2.20: in particular, we use the representatives Φ_γ to check that β' and β'' are fixed up to isotopy by $\varphi(\gamma)$ for all $\gamma \in G$. $\square$

The proof of Proposition 2.21 can be repeated word by word in the generalised context, and the analogue of Lemma 2.22 is the following:

Lemma A.5. *Let $\psi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ be a mapping class, and let Ψ be a diffeomorphism representing ψ. Moreover, let $\varphi\colon G \to \Gamma(\mathcal{S}, \partial\mathcal{S})$ be a group homomorphism and let $[c_1, \ldots, c_h]$ be the cut locus of φ.*

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Then $[\Psi(c_1),\dots,\Psi(c_h)]$ *is the cut locus of the conjugate* $\psi\varphi\psi^{-1}$*, which is defined by* $(\psi\varphi\psi^{-1})(\gamma) := \psi\cdot\varphi(\gamma)\cdot\psi^{-1}$*. In particular, if the image of* φ *commutes with* ψ*, then* ψ *preserves the cut locus of* φ *as an unordered collection of isotopy classes of oriented simple closed curves.*

Proof. The proof is the same as for Lemma 2.22, with the exception that we first lift φ to a homomorphism $\tilde{\varphi}\colon G \to \Gamma(\mathcal{S}, U)$, where U is a small neighbourhood of $\alpha_0 \cup \cdots \cup \alpha_k \cup \partial\mathcal{S}$, and then we compare this morphism with the conjugate $\psi\tilde{\varphi}\psi^{-1}\colon G \to \Gamma(\mathcal{S}, \Psi(U))$. □

This shows that, given a homomorphism $\varphi\colon G \to \Gamma(\mathcal{S}, \partial\mathcal{S})$, we can associate with it a cut locus $c_1,\dots,c_h$ separating $\mathcal{S}$ into a white region W and a yellow region Y. We fix a parametrisation by S^1 of each curve in the cut locus as in Subsection 3.1.

We can also lift uniquely φ to a morphism $\tilde{\varphi}\colon G \to \Gamma(\mathcal{S}, W)$. For each component $P \subset Y$, we then have a morphism $\varphi_P\colon G \to \Gamma(P, \partial P)$ obtained by composing $\tilde{\varphi}$ with the restriction $\Gamma(\mathcal{S}, W) \to \Gamma(P, \partial P)$. Mimicking Definition 3.1, we say that two path components P and P' of Y are *similar for* φ if there is a a diffeomorphism $\Xi\colon P \to P'$ preserving the boundary parametrisations and conjugating the homomorphism φ_P to $\varphi_{P'}$. We write $Y = \coprod_{i=1}^{r}\coprod_{j=1}^{s_i} Y_{i,j}$ as in Notation 3.2, and introduce morphisms $\bar{\varphi}_{i,j}\colon G \to \Gamma_{g_i,n_i}$ conjugate to $\varphi_{Y_{i,j}}$ in an analogue way.

We denote by $\varphi_Y\colon G \to \Gamma(Y, \partial Y)$ the composition of $\tilde{\varphi}$ and the restriction map $\Gamma(\mathcal{S}, W) \to \Gamma(Y, \partial Y)$, and denote by $Z(\varphi_Y) \subset \Gamma(Y)$ the subgroup of elements that commute with the entire image of φ_Y. Definition 3.4 and Lemma 3.5 then carry over to the generalised context.

In Section 3, one only has to replace '$\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$' by 'homomorphism $\varphi\colon G \to \Gamma(\mathcal{S}, \partial\mathcal{S})$', as well as 'representative Φ of φ fixing U' by 'the unique lift of φ to a homomorphism $\tilde{\varphi}\colon G \to \Gamma(\mathcal{S}, U)$'.

Finally, the discussion of Section 6 generalises straightforwardly to the context of a generic parametrising space X, leading to Theorem A.2.

A.2. Functoriality in X

Both sides of the equivalence in Theorem A.2 depend on a space X, and the left-hand side can be regarded as an enriched, contravariant functor from the category of (connected) topological spaces to the category of loop spaces. In fact, each space $\Omega B(\mathrm{map}(X, \mathfrak{M}_{*,1}))$ admits a natural infinite loop space structure, and for a map of spaces $f\colon X \to X'$, the corresponding map

$$\Omega B(\mathrm{map}(f, \mathfrak{M}_{*,1}))\colon \Omega B(\mathrm{map}(X', \mathfrak{M}_{*,1})) \to \Omega B(\mathrm{map}(X, \mathfrak{M}_{*,1}))$$

is defined as the group completion of a map of $\mathscr{M}|_1$-algebras, and hence a map of infinite loop spaces. We can thus consider $\Omega B(\mathrm{map}(-, \mathfrak{M}_{*,1}))$ as a functor from connected topological spaces to infinite loop spaces. Now it would be interesting to describe this functor solely in terms of the right-hand side: that is, to upgrade the assignment $X \mapsto \Omega^\infty\mathbf{MTSO}(2) \times \Omega^\infty\Sigma^\infty_+ \coprod_{n\geqslant 1}\coprod_{g\geqslant 0}\mathfrak{C}_{g,n}(X) /\!\!/ R_n$ to an enriched functor from topological spaces to infinite loop spaces, such that the equivalences given by Theorem A.2, for varying X, assemble into a natural equivalence of functors. We will content ourselves with a weaker but explicit result about functoriality.

Definition A.6. A map of connected topological spaces $f\colon X \to X'$ induces a map of fundamental groups $f_*\colon \pi_1(X) \to \pi_1(X')$, which is defined up to conjugation and which, in turn, allows us to transform any homomorphism $\varphi\colon \pi_1(X') \to \Gamma_{g,n}$ into a homomorphism $f^*\varphi = \varphi \circ f_*\colon \pi_1(X) \to \Gamma_{g,n}$, for all $g \geqslant 0$ and $n \geqslant 1$.

We say that f is *∂-faithful* if $f^*\varphi$ is ∂-irreducible whenever φ is ∂-irreducible, for all $g \geqslant 0$ and $n \geqslant 1$. We denote by $\mathbf{Top}^\partial$ the (topologically enriched) category of connected topological spaces that are homotopy equivalent to a CW complex, with morphisms being ∂-faithful continuous maps.

Note that being ∂-faithful is a homotopy-invariant property of continuous maps: that is, morphism spaces in $\mathbf{Top}^\partial$ are obtained by selecting unions of connected components from the morphism spaces

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in **Top**. Recall Definition A.1: if $f\colon X \to X'$ is a ∂-faithful map, then we obtain for each $g \geqslant 0$ and $n \geqslant 1$ a map

$$\mathfrak{C}_{g,n}(f)\colon \mathfrak{C}_{g,n}(X') \to \mathfrak{C}_{g,n}(X).$$

Thus, we can consider $\mathfrak{C}_{g,n}$ as a contravariant functor from $\mathbf{Top}^{\partial}$ to **Top**, for all $g \geqslant 0$ and $n \geqslant 1$. Putting together all values of g and n, and taking the actions of the groups R_n into account, we can consider $\mathfrak{C}$ as a contravariant functor from $\mathbf{Top}^{\partial}$ to $\boldsymbol{R}$-**Alg**. We can then repeat the proof of Theorem 6.5 and obtain an equivalence of (contravariant, enriched) functors

$$F_{\boldsymbol{R}}^{\mathscr{M}}(\mathfrak{C}(-))_1 \simeq \mathrm{map}(-, \mathscr{M}_{*,1})\colon \mathbf{Top}^{\partial} \to \mathscr{M}|_1\text{-}\mathbf{Alg}.$$

Composing with the group completion functor, we then obtain the following enriched version of Theorem A.2.

Theorem A.7. *Let* Ω-**Top** *and* Ω^{∞}-**Top** *denote the (enriched) categories of loop spaces and infinite loop spaces, respectively. Then there is is a weak equivalence between the following two enriched contravariant functors from* $\mathbf{Top}^{\partial}$ *to* Ω-**Top***:*

- *the functor* $\Omega B(\mathrm{map}(-, \mathfrak{M}_{*,1}))$*;*
- *the composition of the contravariant functor*

$$\Omega^{\infty}\mathbf{MTSO}(2) \times \Omega^{\infty}\Sigma^{\infty}_{+} \coprod_{n\geqslant 1}\coprod_{g\geqslant 0} \mathfrak{C}_{g,n}(-) /\!\!/ R_n$$

from $\mathbf{Top}^{\partial}$ *to* Ω^{∞}-**Top** *and the covariant, forgetful functor from* Ω^{∞}-**Top** *to* Ω-**Top**.

The weak equivalence of functors assigns to $X \in \mathbf{Top}^{\partial}$ *the weak equivalence of loop spaces given by Theorem A.2.*

Example A.8. We briefly discuss an example showing why Theorem A.7 cannot be generalised to an analogue statement concerning the entire category **Top** of connected topological spaces with the homotopy type of a CW complex, together with *all* continuous maps: let X be a connected CW complex and let $f\colon * \to X$ be the inclusion of a point. We note that $\coprod_{n\geqslant 1}\coprod_{g\geqslant 0}\mathfrak{C}_{g,n}(*)$ is the empty space, whence $\Omega^{\infty}\Sigma^{\infty}_{+}\coprod_{n\geqslant 1}\coprod_{g\geqslant 0}\mathfrak{C}_{g,n}(*)/\!\!/R_n$ is just a point A straightforward generalisation of Theorem A.7 would predict that the following square commutes up to homotopy, where the horizontal isomorphisms are given by Theorem A.2:

$$\begin{tikzcd}
\Omega B(\mathrm{map}(X, \mathfrak{M}_{*,1})) \arrow[r, "\simeq"] \arrow[d, "\Omega B(\mathrm{map}(f, \mathfrak{M}_{*,1}))"] & \Omega^{\infty}\mathbf{MTSO}(2) \times \Omega^{\infty}\Sigma^{\infty}_{+}\coprod_{n\geqslant 1}\coprod_{g\geqslant 0}\mathfrak{C}_{g,n}(X) /\!\!/ R_n \arrow[d, "\mathrm{pr}_{\Omega^{\infty}\mathbf{MTSO}(2)}"] \\
\Omega B(\mathrm{map}(*, \mathfrak{M}_{*,1})) \arrow[r, "\simeq"] & \Omega^{\infty}\mathbf{MTSO}(2).
\end{tikzcd}$$

However, we can apply the functor $\pi_0(-)$, taking values in groups, as all terms in the previous diagram are loop spaces. Applying π_0 to the top row, we obtain the group $\mathbb{Z} \oplus \left(\bigoplus_{n\geqslant 1}\bigoplus_{g\geqslant 0}\mathbb{Z}^{\oplus \pi_0(\mathfrak{C}_{g,n}(X))}\right)$, whereas applying π_0 to the bottom row, we obtain the group $\mathbb{Z}$.

After applying π_0, of the left vertical map sends the first summand $\mathbb{Z}$ isomorphically onto $\mathbb{Z}$, and the standard generator of each further summand $\mathbb{Z}^{\oplus \pi_0(\mathfrak{C}_{g,n}(X))}$ to $g + n - 1 \in \mathbb{Z}$, while the right vertical map sends the first summand $\mathbb{Z}$ isomorphically onto $\mathbb{Z}$, and each further summand $\mathbb{Z}^{\oplus \pi_0(\mathfrak{C}_{g,n}(X))}$ constantly to 0.

Note that it suffices to take $X = S^1$ to have $\pi_0(\mathfrak{C}_{g,n}(X)) \neq \varnothing$ for some choice of g and n and thus to ensure that the previous example is not void: see Example 2.17.

As an application of Theorem A.7, let $k \in \mathbb{Z} \setminus \{0\}$, and consider the power map $(-)^k\colon S^1 \to S^1$, which induces multiplication by k on the fundamental group $\pi_1(S^1) \cong \mathbb{Z}$. The following lemma proves that $(-)^k$ is ∂-faithful; hence it induces a self map of the infinite loop space $\Omega^\infty\mathbf{MTSO}(2) \times \Omega^\infty\Sigma^\infty_+ \coprod_{n\geqslant 1}\coprod_{g\geqslant 0} \mathfrak{C}_{g,n} /\!\!/ R_n$ that can be described using Theorem A.7.

Lemma A.9. *Let $\mathcal{S}$ be a surface of type $\Sigma_{g,n}$ for some $g \geqslant 0$ and $n \geqslant 1$. Let $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ be a mapping class, let $k \in \mathbb{Z}\setminus\{0\}$, and let $\bar{\alpha}$ be an essential arc in $\mathcal{S}$. Suppose that the isotopy class $[\bar{\alpha}]$ of $\bar{\alpha}$ relative to its endpoints is in the fixed-arc complex of φ^k. Then $[\bar{\alpha}]$ is in the fixed-arc complex of φ.*

Proof. Since the fixed-arc complexes of φ and φ^{-1} are equal, we can assume without loss of generality that $k \geqslant 1$. The case $k = 1$ is tautological, so we henceforth assume $k \geqslant 2$.

Let $p, q \in \partial\mathcal{S}$ be the endpoints of $\bar{\alpha}$, and consider the set $\mathrm{Arc}(p, q)$ of isotopy classes $[\alpha]$ of essential arcs α in $\mathcal{S}$ from p to q. An essential arc α is a map $\alpha\colon [0, 1] \to \mathcal{S}$, but we will abuse notation and denote by α also the image of this map, which is a subset of $\mathcal{S}$.

Let α and β be two essential arcs from p to q. We say that α and α' are *transverse* if they have linearly independent velocities at each intersection point, including the endpoints p and q. Moreover, we say that α and α' are in *minimal position away from the endpoints* if the number of intersection points of α and α', excluding p and q, is minimal among all pairs of transverse arcs α' and β' with $\alpha \sim \alpha'$ and $\beta \sim \beta'$. The bigon criterion applies: α and β are in minimal position if and only if they do not form bigons.

In particular, if α and β are non-isotopic arcs from p to q in minimal position away from the endpoints, then we can compare the velocities of α and β at p: these are linearly independent vectors in $T_p\mathcal{S}$ pointing inside $\mathcal{S}$: we say that α *is on right of* β if the sequence $(\frac{d}{dt}\alpha(0), \frac{d}{dt}\beta(0))$ is a positive basis of $T_p\mathcal{S}$.

The Alexander method guarantees the following: let $\alpha, \alpha', \beta, \beta'$ be four essential arcs in $\mathcal{S}$ from p to q, with $\alpha \sim \alpha' \nsim \beta \sim \beta'$, and suppose that α and β, as well as α' and β', are in minimal position away from their endpoints. Then there is an isotopy of $\mathcal{S}$ bringing $\alpha \cup \beta$ to $\alpha' \cup \beta'$. In particular, α is on right of β if and only if α' is on right of β'. We can now put a total order on $\mathrm{Arc}(p, q)$: for two distinct isotopy classes $[\alpha]$ and $[\beta]$, we say that $[\alpha] \prec [\beta]$ if, for any two representatives α and β, which are in minimal position away from the endpoints, α is on right of β.

The action of $\Gamma(\mathcal{S}, \partial\mathcal{S})$ on $\mathrm{Arc}(p, q)$ preserves the total order $\prec$: for this, let Φ be a diffeomorphism representing $\varphi \in \Gamma(\mathcal{S}, \partial\mathcal{S})$ and assume that Φ fixes a neighbourhood of $p \in \partial\mathcal{S}$ pointwise. Moreover, let α and β represent classes $[\alpha], [\beta] \in \mathrm{Arc}(p, q)$, and assume that α and β are in minimal position away from the endpoints and that α is on right of β, thus witnessing $[\alpha] \prec [\beta]$: then $\Phi(\alpha)$ and $\Phi(\beta)$ are also in minimal position away from the endpoints, and $\Phi(\alpha)$ is on right of $\Phi(\beta)$, witnessing that $\varphi([\alpha]) = [\Phi(\alpha)] \prec [\Phi(\beta)] = \varphi([\beta])$.

Let now $[\bar{\alpha}]$ be as in the hypothesis of the theorem, and assume for the sake of contradiction that $[\bar{\alpha}] \neq \varphi([\bar{\alpha}]) \in \mathrm{Arc}(p, q)$: then without loss of generality, we can assume $[\bar{\alpha}] \prec \varphi([\bar{\alpha}])$. We then have a chain of inequalities

$$[\bar{\alpha}] \prec \varphi([\bar{\alpha}]) \prec \varphi^2([\bar{\alpha}]) \prec \cdots \prec \varphi^{k-1}([\bar{\alpha}]) \prec \varphi^k([\bar{\alpha}]),$$

hence $[\bar{\alpha}] \prec \varphi^k([\bar{\alpha}])$, contradicting the hypothesis $[\bar{\alpha}] = \varphi^k([\bar{\alpha}])$. □

B. Braid groups, symmetric groups and free loops

In this second appendix, we sketch two results that are parallel to the identification in Theorem 1.1.

The first pertains to automorphisms of surfaces that have no genus but instead punctures; we thus replace mapping class groups by braid groups. A particular model of the corresponding classifying space is given by the collection of unordered configuration spaces of the 2-dimensional disc.

In analogy to that, the collection of configuration spaces of the ∞-dimensional disc is a model for the classifying space of symmetric groups. We outline the corresponding result for that setting.

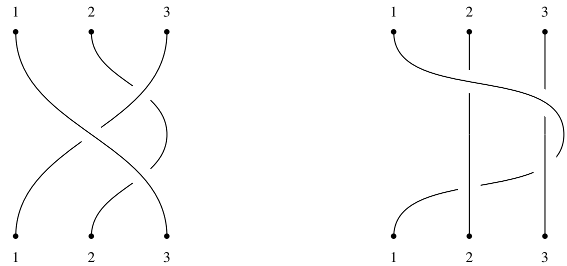

Figure 11. *The left braid is reducible as it is conjugate to a block sum of two braids, whereas the second one is not.*

B.1. Braid groups

First recall that the r^{th} braid group Br_r can be defined as the fundamental group of the unordered configuration space $C_r(\mathring{D}^2) := \tilde{C}_r(\mathring{D}^2)/\mathfrak{S}_r$ of the 2-dimensional disc, where

$$\tilde{C}_r(\mathring{D}^2) := \{(p_1, \dots, p_r) \in \mathring{D}^2;\ p_i \neq p_j \text{ for all } i \neq j\}.$$

The contractible topological group $\mathrm{Diff}_\partial(D^2)$ of diffeomorphisms of D^2 that are the identity in a neighbourhood of the boundary acts on $C_r(\mathring{D}^2)$; the stabiliser of a point $P := \{p_1, \dots, p_r\} \in \mathrm{C}_r(D^2)$ with respect to this this action is the subgroup $\mathrm{Diff}_\partial(D^2, P)$ of diffeomorphisms that preserve P as a set. The induced long exact sequence on homotopy groups of the resulting fibration

$$\mathrm{Diff}_\partial(D^2, P) \to \mathrm{Diff}_\partial(D^2) \to C_r(\mathring{D}^2)$$

thus implies that $\mathrm{Br}_r \cong \pi_0 \mathrm{Diff}_\partial(D^2, P)$, the mapping class group of a genus 0 surface with r (unordered) punctures. A classical argument by Fadell and Neuwirth [6] shows that the configuration spaces $\tilde{C}_r(\mathring{D}^2)$ and $C_r(\mathring{D}^2)$ are aspherical, which implies the well-known result $C_r(\mathring{D}^2) \simeq B\,\mathrm{Br}_r$.

We remark that Alexander's theory of arc systems developed in detail for surfaces in Section 2, Definitions 2.10, 2.14, 2.19, Propositions 2.11 and 2.21, Construction 2.18 and Lemma 2.20 has a canonical analogue if the surfaces with boundary considered before are replaced by a punctured disc: now the arcs are required to start and end in $\partial D^2 = S^1$ and their interior must lie in $\mathring{D}^2 \setminus \{p_1, \dots, p_r\}$. Most importantly, we can also define the notion of irreducibility of elements of braid groups as follows.

Definition B.1. 1. A braid $\gamma \in \mathrm{Br}_r$ is *reducible* if there exists an essential arc in $\mathring{D}^2 \setminus \{p_1, \dots, p_r\}$ that is fixed by γ; otherwise it is called *irreducible*. In other words, a braid γ is reducible if it is conjugate to a block sum of braids; see Figure 11.
2. Let $\mathfrak{I}_r \subseteq \Lambda C_r(\mathring{D}^2) = \Lambda B\mathrm{Br}_r$ denote the subspace of free loops whose corresponding conjugacy classes in Br_r consist of irreducible elements.

The genus-0 and colour-1 part of $\mathscr{M}$, which coincides with the framed little 2-disc operad $\mathscr{D}_2^{\mathrm{fr}}$, acts on the union $\Lambda \coprod_{r\geqslant 0} C_r(\mathring{D}^2)$; the Lie group $R_1 \cong S^1$ acts on $\mathfrak{I} := \coprod_{r\geqslant 1} \mathfrak{I}_r$, and analogous to Theorem 6.5 one obtains that

$$\Lambda \coprod_{r\geqslant 0} C_r(\mathring{D}^2) \simeq F_{S^1}^{\mathscr{D}_2^{\mathrm{fr}}}(\mathfrak{I}).$$

The free $\mathscr{D}_2^{\mathrm{fr}}$-algebra over a given S^1-space is, as a $\mathscr{D}_2$-algebra, equivalent to the free $\mathscr{D}_2$-algebra over the underlying space without the S^1-action as the operation spaces $\mathscr{D}_2^{\mathrm{fr}}(r)$ and $\mathscr{D}_2(r) \times (S^1)^r$ are equivalent as free right $(S^1)^r$-spaces, whence the coend construction just cancels the toric factor. We therefore obtain, after group completion, the following identification.

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Theorem B.2. *There is a homotopy equivalence*

$$\Omega B\Lambda \coprod_{n\geqslant 1} B\,\mathrm{Br}_r \simeq \Omega^2\Sigma^2_+ \coprod_{r\geqslant 1} \coprod_{\substack{[\gamma]\in\mathrm{Conj}(\mathrm{Br}_r)\\ \textit{irreducible}}} BZ(\gamma, \mathrm{Br}_r).$$

B.2. Symmetric groups

We conclude by considering free loop spaces of configuration spaces of an infinite-dimensional disc $\mathring{D}^\infty$. These are classifying spaces of symmetric groups: that is, $C_r(\mathring{D}^\infty) \simeq B\mathfrak{S}_r$. The analogous notion of irreducibility is even simpler: an element of $\mathfrak{S}_r$ (that is, a permutation) is irreducible if and only if it comprises a single cycle. In this case, its centraliser is cyclic of order r, and as above, one obtains that

$$\Lambda \coprod_{r\geqslant 0} B\mathfrak{S}_r \simeq F^{\mathscr{D}_\infty} \coprod_{k\geqslant 1} B(\mathbb{Z}/k).$$

After group completion, we obtain the analogue of Theorem B.2 for symmetric groups [21, Cor. 4.32].

Theorem B.3. *There is a homotopy equivalence*

$$\Omega B\Lambda \coprod_{r\geqslant 0} B\mathfrak{S}_r \simeq \Omega^\infty\Sigma^\infty_+ \coprod_{k\geqslant 1} B(\mathbb{Z}/k).$$

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