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A NOTE ON BANACH SPACES CONTAINING ℓ_1

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ABSTRACT. If Y is a subspace of a Banach space X, either Y contains an isomorphic copy of ℓ_1 or each weak Cauchy sequence in X/Y has a subsequence that is the image under the quotient mapping of a weak Cauchy sequence in X. If X is weakly sequentially complete and Y is reflexive, X/Y is weakly sequentially complete. Related structural results are given.

It has recently been shown that if a Banach space X contains a bounded sequence (x_n) having no weak Cauchy subsequence, then (x_n) has a subsequence that is basic and equivalent to the unit vector basis of ℓ_1. A proof of this theorem for real scalars can be found in [5], while a proof for the complex case is given in [2]. The purpose of this note is to show how this result can be applied in the study of the structure of Banach spaces. The letters X, Y, Z will denote Banach spaces over the real or complex scalars. The term "subspace" will mean "closed, infinite-dimensional, linear subspace." Recall that X is weakly sequentially complete if every weak Cauchy sequence in X converges weakly to an element of X. X has the Schur property if weak and norm convergence of sequences coincide in X. If X has the Schur property, then a weak Cauchy sequence in X is norm convergent.

If (x_n) is a sequence in X, $[x_n]$ denotes its closed linear span in X. For terminology and properties related to Schauder bases, the reader is referred to [6]. If (x_n) and (y_n) are bases for X and Y, respectively, they are said to be equivalent provided $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges. This is equivalent to the existence of an isomorphism T of X onto Y satisfying $Tx_n = y_n$ for all n.

THEOREM 1. *Let Y be a subspace of X and let $Q:X \to X/Y$ denote the quotient mapping. Then either Y contains an isomorphic copy of ℓ_1 or each weak Cauchy sequence in X/Y has a subsequence that is the image under Q of a weak Cauchy sequence in X.*

Proof. Assume that (u_n) is a weak Cauchy sequence in X/Y such that no

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subsequence of (u_n) is the image under Q of a weak Cauchy sequence in X. Choose a bounded sequence (x_n) in X such that $Q(x_n)=u_n$ for all n. Then (x_n) has no weak Cauchy subsequence. By passing to a subsequence, we may assume (x_n) is a basic sequence that is equivalent to the unit vector basis of ℓ_1. Let $v'_n=u_{2n}-u_{2n-1}$ for all n. Since $v'_n\to 0$ weakly, there is an increasing sequence (p_n) and a sequence $v_n\in co\{v'_k:p_n\le k\le p_{n+1}-1\}$, $n=1,2,\ldots$, such that $\|v_n\|\to 0$ (cf. [3, p. 422]). Let (w_n) denote the corresponding sequence of convex combinations far out in $(x_{2n}-x_{2n-1})$. Note that $(x_{2n}-x_{2n-1})$ is also equivalent to the unit vector basis (e_n) of ℓ_1, say under an isomorphism T carrying $x_{2n}-x_{2n-1}$ into e_n. Then T carries (w_n) into a block basis (z_n) of (e_n), where each z_n is a convex combination of e_k, for $p_n\le k\le p_{n+1}-1$. Therefore, $\|z_n\|=1$ for all n, and it follows by Lemma 1 of [4] that (z_n), and hence (w_n), is equivalent to (e_n). Moreover, $Q(w_n)=v_n$ for all n. If (w_n^*) denotes the sequence of coefficient functionals in $[w_n]^*$, we have $M=\sup_n\|w_n^*\|<\infty$. Since $d(w_n, Y)\to 0$, there exists an increasing sequence (n_k) of positive integers and a sequence $(y_k)\subset Y$ such that $\|w_{n_k}-y_k\|<M^{-1}2^{-k-1}$ for all k. Then $\sum_{k=1}^{\infty}\|w_{n_k}^*\|\,\|w_{n_k}-y_k\|<1$ which, by [1], implies that (w_{n_k}) and (y_k) are equivalent basic sequences. Thus (y_k) is equivalent to the unit vector basis of ℓ_1, completing the proof.

COROLLARY 2. *If X is a weakly sequentially complete Banach space and Y is a reflexive subspace of X, then X/Y is weakly sequentially complete.*

The criterion for a Banach space to contain ℓ_1 can be used to give the following general property of linear operators.

THEOREM 3. *Let $T:X\to Y$ be a continuous linear operator. Then either T maps bounded sequences into sequences having a weak Cauchy subsequence or there exists a subspace Z of X that is isomorphic to ℓ_1 for which $T|_Z$ is an isomorphism.*

Proof. If (x_n) is a bounded sequence in X such that (Tx_n) has no weak Cauchy subsequence, there exists a basic subsequence (Tx_{n_k}) equivalent to the unit vector basis of ℓ_1. Thus there is a constant $M>0$ such that for all m and scalars $a_1,\ldots,a_m$, we have

$$M\sum_{k=1}^{m}|a_k|\le\left\|\sum_{k=1}^{m}a_kTx_{n_k}\right\|\le\|T\|\sup_n\|x_n\|\sum_{k=1}^{m}|a_k|.$$

If $Z=[x_{n_k}]$, it follows that Z is isomorphic to ℓ_1 and that $T|_Z$ is an isomorphism.

COROLLARY 4. *Let Y be a Banach space with the Schur property and let $T:X\to Y$ be a continuous linear operator. Then either T is compact or there exists a subspace Z of X that is isomorphic to ℓ_1 for which $T|_Z$ is an isomorphism.*

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