Canad. Math. Bull. Vol. **36** (4), 1993 pp. 407–413

A WEAK HADAMARD SMOOTH RENORMING OF $L_1(\Omega, \mu)$

JONATHAN M. BORWEIN AND SIMON FITZPATRICK

ABSTRACT. We show that $L_1(\mu)$ has a weak Hadamard differentiable renorm (*i.e.* differentiable away from the origin uniformly on all weakly compact sets) if and only if μ is sigma finite. As a consequence several powerful recent differentiability theorems apply to subspaces of L_1.

1. **Introduction.** Let X be a real Banach space, and let X^* be the continuous linear functionals on X, equipped with the usual norm $\|y\| := \sup\{\langle x, y\rangle : \|x\| \leq 1\}$. We recall that a function $f: X \longrightarrow \mathbb{R}$ is *weak Hadamard differentiable* at a point x if the Gateaux derivative exists at x and is uniform on all weakly compact sets. Equivalently,

$$\lim_{n\to\infty} \frac{f(x + t_n h_n) - f(x)}{t_n} = \nabla f(x)(h) \tag{1.1}$$

whenever $h_n \longrightarrow h$ weakly, and $t_n \longrightarrow 0$. (See [BP] and [Ph].) Clearly any point of Fréchet differentiability is a point of weak Hadamard differentiability. The converse holds in the following setting:

THEOREM 1.1 ([BF]). *Let X be a Banach space and let $f: X \longrightarrow R$ be convex and continuous. Suppose X contains no copy of $\ell_1(\mathbb{N})$. Then f is Fréchet differentiable at x if and only if f is weak Hadamard differentiable at x. In particular any equivalent weak Hadamard norm on X is actually a Fréchet norm and X is necessarily Asplund.*

In [BF] it was shown that if X contains a copy of $\ell_1(\mathbb{N})$ then there is a convex continuous function with a point of weak Hadamard differentiability which is not a point of Fréchet differentiability (see also [Bo2], [Or]). In [Bo2] it was also shown that $C([0, 1])$ has no weak Hadamard renorm. Indeed:

THEOREM 1.2 ([Bo2]). *Let X be a $C(\Omega)$, with Ω a compact Hausdorff space. If X has a weak Hadamard renorm, then X is an Asplund space.*

It is the purpose of this note to show that $L_1(\mu) = L_1(\Omega, \Sigma, \mu)$ has a weak Hadamard differentiable renorm (*i.e.* Gateaux differentiable away from the origin uniformly on all weakly compact sets) if and only if μ is sigma finite. As a consequence several powerful recent "bornological" differentiability theorems ([BP], [Pr], [PPN]) apply in the weak

The first author was partially supported by NSERC grant OGP005116.

Received by the editors March 5, 1992.

AMS subject classification: Primary: 46A17, 46B22; secondary: 46B03, 46B15.

Key words and phrases: Asplund spaces, Mackey convergence, weak Hadamard derivatives, renorms, Dunford-Pettis property, locally Mackey rotund, bornological derivatives.

Hadamard sense to subspaces of L_1. Previously these theorems have really only found application in the Gateaux and Fréchet senses. We observe that as a consequence of Theorems 1.1 and 1.2 there are many separable (or WCG) spaces that do not admit such renorms: *e.g.*, $C([0,1])$ and any separable X not containing $\ell_1(\mathbb{N})$ whose dual is non separable.

2. **Norms on $L_1(\mu)$.** We write θ and $B(X)$ or B for the origin and closed unit ball of the Banach space X respectively. As is standard, we write f_x for any (sub)gradient of the norm at x. We begin by giving a sufficient condition for Mackey convergence of a sequence in the dual of a sigma-finite $L_1(\mu)$:

LEMMA 2.1. *Suppose that $\langle y_n\rangle$ converges to θ in mean (or only in measure) in $L_1(\mu)$ and suppose that* $\sup_{n\in\mathbb{N}}\|y_n\|_\infty<\infty$. *Then $\langle y_n\rangle$ converges to θ in the Mackey topology,* $\tau\big(L_\infty(\mu),L_1(\mu)\big)$.

PROOF. Let $\varepsilon>0$ be given and fix a weakly compact set W in $L_1(\mu)$. Select $M>\sup_{w\in W}\|w\|_1\vee\sup_{n\in\mathbb{N}}\|y_n\|_\infty$. By the Dunford-Pettis criterion for weak compactness in $L_1(\mu)$ [Di2] there is $\varepsilon>\delta>0$ such that

$$\sup_{w\in W}\int_B |w(t)|\,d\mu(t)<\varepsilon \text{ whenever } \mu(B)<\delta<\varepsilon. \tag{2.1}$$

Pick N in $\mathbb{N}$ so that $\mu(\{t:|y_n(t)|\geq\delta\}\leq\delta$ for $n\geq N$. Define $B_n:=\{t:|y_n(t)|\geq\delta\}$ in Σ and set $A_n:=\Omega\setminus B_n$. Then for $w\in W$ and $n>N$

$$\begin{aligned}\Big|\int_\Omega y_n(t)w(t)\,d\mu\Big| &\leq \Big|\int_{A_n} y_n(t)w(t)\,d\mu\Big|+\Big|\int_{B_n} y_n(t)w(t)\,d\mu\Big|\\ &\leq \delta\int_{A_n}|w(t)|\,d\mu+M\int_{B_n}|w(t)|\,d\mu\\ &\leq \delta\int_{\Omega}|w(t)|\,d\mu+M\int_{B_n}|w(t)|\,d\mu.\end{aligned}$$

Thus for $n>N$

$$\Big|\int_\Omega y_n(t)w(t)\,d\mu\Big|\leq M\Big(\delta+\int_{B_n}|w(t)|\,d\mu\Big)<2M\varepsilon$$

on using (2.1). As ε is arbitrary $\langle y_n\rangle$ converges to θ in the Mackey topology. (Note that there is no loss of generality in considering null sequences.) ∎

Given a Hausdorff topology T, we say that a norm on X is *locally T rotund* (LTR) if whenever $\langle x_n\rangle$ and x lie in the unit ball $B(X)$

$$\lim_{n\to\infty}\Big\|\frac{x_n+x}{2}\Big\|=1\Rightarrow T\text{-}\lim_{n\to\infty}x_n=x. \tag{2.2}$$

Observe that any LTR norm is strictly convex. In particular, we say that a dual norm is *locally Mackey rotund* (LMR) if this holds in the Mackey topology $\tau(X^*,X)$. Correspondingly, a dual norm is *locally weak* rotund* if this holds in $\sigma(X^*,X)$ and is *locally uniformly rotund* if this holds in the strong topology $\beta(X^*,X)$. This last case recaptures the standard definition, [Da], [Di1].

LEMMA 2.2. *Let $\| \ \|$ on X be such that the dual norm, $\| \ \|_*$, is a locally Mackey rotund dual norm. Then $\| \ \|$ is weak Hadamard differentiable on X (away from θ).*

PROOF. Since $\| \ \|_*$ is LMR it is strictly convex, and so $\| \ \|$ is smooth (Gateaux). Let $\langle h_n \rangle$ converge weakly to h in X, and let $\langle t_n \rangle$ converge to 0 from above. We apply (1.1) to the norm. By the Mean Value theorem

$$(2.3) \qquad \lim_{n\to\infty} \frac{\|x - t_n h_n\| - \|x\|}{t_n} - f_x(h) = \lim_{n\to\infty} [f_{x_n}(h_n) - f_x(h_n)]$$

for some $\langle x_n \rangle$ converging to x in norm. However, $f_{x_n}(x) + f_x(x) \longrightarrow 2$ since the gradient is norm-weak* continuous. Thus $\|\frac{f_{x_n}+f_x}{2}\|_* \longrightarrow 1$ as each support functional has unit norm. Since the dual norm is LMR we deduce that $f_{x_n} \longrightarrow f_x$ in the Mackey topology and that the error term in (2.3), $f_{x_n}(h_n) - f_x(h_n)$, tends to zero as is required. ∎

The converse to Lemma 2.2 is certainly false since the dual of a Fréchet norm need not be (LUR) and in an Asplund setting (LUR) and (LMR) coincide [Bo2].

PROPOSITION 2.3. *Every $L_1(\mu)$ with μ finite admits an equivalent locally Mackey rotund dual norm on $L_\infty(\mu)$.*

PROOF. For m in $L_\infty(\mu)$, set $\|m\| := \sqrt{\|m\|_\infty^2 + \|m\|_2^2}$. Since $L_2(\mu)$ embeds in $L_1(\mu)$ it follows that $\| \ \|$ is weak* lower semicontinuous and so defines an equivalent dual norm. We verify that it is LMR. So suppose that $\langle m_n \rangle$ and m lie in $\{m : \|m\| \leq 1\}$, and $\lim_{n\to\infty} \|\frac{m_n+m}{2}\| = 1$. Then $\lim_{n\to\infty} \|m_n\| = \|m\| = 1$. As usual, define

$$\Delta(m_n, m, \| \ \|) := \frac{\|m_n\|^2 + \|m\|^2}{2} - \left\|\frac{m_n + m}{2}\right\|^2.$$

Then $\Delta(m_n, m, \| \ \|) \geq \Delta(m_n, m, \| \ \|_2)$ so as $L_2(\mu)$ is LUR we deduce that $\|m_n - m\|_2 \longrightarrow 0$ and so $m_n - m \longrightarrow \theta$ in $L_1(\mu)$. As $\langle m_n \rangle$ is uniformly bounded, Lemma 2.1 now applies. ∎

THEOREM 2.4. *$L_1(\mu)$ has a weak Hadamard differentiable renorm if and only if μ is sigma finite.*

PROOF. If the measure is not sigma finite then it is well known that $L_1(\mu)$ admits no smooth renorm, [Da, p. 161], and is not a Gateaux differentiability space. Indeed, by the Borwein-Preiss Theorem [BP, Ph], it suffices to show that the original norm is nowhere Gateaux differentiable. But, since the support of any member of $L_1(\mu)$ is sigma finite, it is always possible to construct two subgradients in $L_\infty(\mu)$ at every point of the standard unit sphere. (Here as throughout the literature we implicitly assume that measures have no infinite atoms!)

Suppose $L_1(\mu)$ is sigma finite. Then there is an isometric linear mapping of $L_1(\mu)$ onto $L_1(\mu*)$ for some finite measure $\mu*$ [La, p. 138]. Thus there is no loss of generality in assuming that μ is a finite measure. Let $\|m\| := \sqrt{\|m\|_\infty^2 + \|m\|_2^2}$. Now $\| \ \|$ defines an equivalent dual norm on $L_\infty(\mu)$. By Proposition 2.3, $\| \ \|$ is locally Mackey rotund. By Lemma 2.2 $\| \ \|$ is weak Hadamard differentiable on $L_1(\mu)$. ∎

We recall that a vector e in a Banach lattice is a *weak order unit* or Freudenthal unit when $e \wedge x = 0$ implies $x = 0$. The representation theory of abstract L spaces (AL spaces) [Da, p. 138] produces:

COROLLARY 2.5. *An abstract L space admits a weak Hadamard differentiable renorm if and only if it admits a Gateaux differentiable renorm as holds if and only if it possesses a weak order unit.*

REMARKS 2.6. (a) When μ is finite, the constructed norm on $L_1(\mu)$ is given by the infimal convolution $\|x\| := \inf_z \sqrt{\|z\|_1^2 + \|x-z\|_2^2}$. It is easy to check that the infimum is attained. It also follows that when μ is a probability measure, $\|\ \|$ and $\|\ \|_2$ coincide on $L_2(\mu)$.

(b) In terms of the duality map $J_{\|\ \|}$, which is the subgradient of $\frac{1}{2}\|\ \|^2$, we may explicitly compute that $x^* \in J_{\|\ \|}(x)$ if and only if $x - x^* \in J_{\|\ \|_1}(x)$. This in turn means that

$$x^* = x \wedge s \vee (-s) \text{ where } s \text{ uniquely solves } s = \|(x-s)^+\|_1 + \|(-x-s)^+\|_1.$$

Also $s = \|x^*\|_\infty$.

(c) If X is weakly compactly generated [Di1, Di2] (as are separable or reflexive spaces) there is a continuous linear mapping T of a reflexive space R densely into X [DFJP]. Then $\|x^*\| := \sqrt{\|x^*\|_*^2 + \|T^*x^*\|_R^2}$ (where $\|\ \|_R$ is LUR) defines a locally weak* rotund dual norm on X^*. Not every strictly convex dual norm is locally weak* rotund—even on $\ell_2(\mathbb{N})$.

(d) With some computation, Lemma 2.1 may also be used to show that every smooth point of the standard unit sphere in $L_1(\mu)$ is weak Hadamard smooth.

(e) It is worth noting that Lemma 2.1 needs more than weak* convergence as hypothesis on $\langle y_n \rangle$. Indeed, in $L_1(0,1)$ with Lebesgue measure, we let

$$y_n = x_n = \sin(2n\pi x).$$

Then the Riemann-Lebesgue lemma shows that $\langle y_n \rangle \longrightarrow \theta$ weak* in $L_\infty(0,1)$ and so, *a fortiori*, $\langle x_n \rangle \longrightarrow \theta$ weakly in $L_1(0,1)$. However $\langle y_n, x_n \rangle = \left(\int_0^1 \sin^2(2n\pi x)\,dx\right) = 1/2$ and does not tend to zero. Thus $\langle y_n \rangle$ is not Mackey null. ∎

3. **Applications.** Let **B** denote any symmetric, spanning, bornology of bounded convex subsets of X. We also suppose that **B** is closed under positive multiples and that if B_1, B_2 lie in **B** then $B_1 \cup B_2$ lies in a member of **B**. This insures that the topology, $\mathbf{B}^0$, of uniform convergence on members of **B** is a well defined locally convex topology on X^*. In reality we are most interested in the following cases:

GATEAUX (G). **B** is all finite dimensional bounded convex sets and $\mathbf{B}^0$ is the weak* topology.

HADAMARD (H). **B** is all norm compact convex sets and $\mathbf{B}^0$ is the bounded weak* topology (which coincides with the weak* topology since X is complete).

WEAK HADAMARD (W). **B** is all weakly compact convex sets and $\mathbf{B}^0$ is the Mackey topology (which coincides with the norm topology when X is reflexive).

FRÉCHET (F). **B** is all bounded convex sets and $\mathbf{B}^0$ is the strong (*i.e.* norm) topology.

A function $f: X \longrightarrow [-\infty, \infty]$ is said to be **B**-*differentiable* at x if it is Gateaux differentiable uniformly on elements of **B**. Then **B**-subdifferentiability is defined similarly. (See [BP], [Ph] for details.)

We define X to be a **B**-*Asplund space* if every continuous convex function defined on an open set U is generically **B**-differentiable throughout U (that is the differentiability points contain a dense G_δ). We define X to be **B**-*differentiability space* if every continuous convex function defined on an open set U is densely **B**-differentiable throughout U. We define X to be a *Minkowski* **B**-*differentiability space* if every continuous sublinear function defined on X is densely **B**-differentiable throughout X. Finally, we say that a member x^* of a set C in X^* is *weak** $\mathbf{B}^0$-*exposed* by x in X if $x^*(x) = \sup\{c(x) : c \in C\}$ and whenever $\langle c_n \rangle \in C$ has $c_n(x) \longrightarrow x^*(x)$ it follows that $c_n \longrightarrow x^*$ in the topology $\mathbf{B}^0$.

It is shown in [Bo2] that in an Asplund space every Mackey convergent sequence in the dual is norm convergent. In particular, in an Asplund space weak* Mackey exposed points and weak* strongly exposed points coincide. We also observe that Gateaux and Hadamard differentiability coincide for Lipschitz functions. Thus any Gateaux smooth norm is Hadamard differentiable. Hadamard subdifferentiability of a non-Lipschitz function is, by contrast, stronger than Gateaux subdifferentiability.

Examination of the results of Chapter 6 in Phelps [Ph] will convince the reader that with minor adjustments in the proofs the following holds:

THEOREM 3.1. *The following are equivalent:*

(i) X is a **B**-*differentiability space;*
(ii) X is a Minkowski **B**-*differentiability space;*
(iii) $X \times \mathbb{R}$ is a **B**-*differentiability space;*
(iv) every weak compact convex subset of X^* is the weak* closed convex hull of its weak** $\mathbf{B}^0$-*exposed points.*

We now formulate our main application. Additional details of definitions can be found in [Bo1] [BFK], [BP], [DGZ], [Pr], [PPN].

THEOREM 3.2. *Suppose that X admits an equivalent weak Hadamard renorm. Then*

(i) X is a weak Hadamard Asplund space;
(ii) every real valued locally Lipschitz function is densely weak Hadamard differentiable. Moreover, the Clarke derivative of f at x is the weak closed convex hull of weak* limits of weak Hadamard gradients:*

$$\partial f(x) = w^* \operatorname{co}\{w^* \lim \nabla_W f(x_n) \colon x_n \longrightarrow x\};$$

(iii) every real valued lower semicontinuous function is densely weak Hadamard subdifferentiable throughout its graph;
(iv) every maximal monotone mapping (every minimal weak cusco) from X to X^* is generically single-valued and norm-Mackey upper semicontinuous.*

In particular, all the above hold in any subspace of $L_1(\mu)$ when μ is sigma finite.

PROOF. (i) and (iv) follow from the main result in [PPN], (ii) from the corresponding result in [Pr], and (iii) from the result in [BP] or [DGZ]. ■

Unlike Asplund or weak Asplund spaces, weak Hadamard Asplund spaces are not preserved by quotients. This is related to the fact that while in a reflexive space the weak Hadamard bornology coincides with the Fréchet bornology, in a Schur space [Di, p. 212] it coincides with the Gateaux bornology.

EXAMPLE 3.3. Theorem 1.2 shows that $C[0,1]$ has no weak Hadamard renorm. Now, it is well known that every separable space is a quotient of $\ell_1(\mathbb{N})$ [LT]. But $\ell_1(\mathbb{N})$ is separable and so has a Hadamard differentiable renorm. Since $\ell_1(\mathbb{N})$ is Schur, (or by Theorem 2.4) this norm is necessarily weak Hadamard differentiable. Thus $\ell_1(\mathbb{N})$ is a weak Hadamard Asplund space whose quotient $C[0,1]$ is not even a weak Hadamard differentiability space. ■

REFERENCES

[Bo1] J. M. Borwein, *Minimal cuscos and subgradients of Lipschitz functions*. In: Fixed Point Theory and its Applications, (eds. J.-B. Baillon and M. Thera), Pitman Lecture Notes in Math., Longman, Essex, 1991, 57–82.

[Bo2] ———, *Asplund spaces are "sequentially reflexive"*, University of Waterloo, July, 1991, Research Report CORR, 91-14.

[BF] J. M. Borwein and M. Fabian, *On convex functions having points of Gateaux differentiability which are not points of Fréchet differentiability*, Can. J. Math., in press.

[BP] J. M. Borwein and D. Preiss, *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*, Trans. Amer. Math. Soc. **303**(1987), 517–527.

[Da] Mahlon M. Day, *Normed Linear Spaces*, 3rd ed., Springer-Verlag, New York, 1973.

[DFJP] W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski, *Factoring weakly compact operators*, J. Functional Analysis **17**(1974), 311–327.

[DGZ] R. Deville, G. Godefroy and V. Zizler, *Un principe variational utilisant des fonctions bossés*, C. R. Acad. Sci. Paris, (1991), 281–286.

[Di1] J. Diestel, *Geometry of Banach spaces*, Lecture notes in Math. (485), Springer-Verlag, Berlin, 1975.

[Di2] ———, *Sequences and series in Banach spaces*, Graduate texts in Math. (92), Springer-Verlag, Berlin, 1984.

[Gi] J. Giles, *Convex analysis with application in differentiation of convex functions*, Research notes in Math. (58), Pitman, London, 1982.

[La] H. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, Berlin, 1974.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: sequence spaces*, Springer-Verlag, Berlin, 1979.

[NP] I. Namioka and R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. **42**(1975), 735–750.

[Or] P. Orno, *On J. Borwein's concept of sequentially reflexive Banach spaces*, 1991, preprint.

[Ph] R. Phelps, *Convex functions, monotone operators, and differentiability*, Lecture notes in Math. (1364), Springer-Verlag, Berlin, 1989.

[Pr] D. Preiss, *Differentiability of Lipschitz functions on Banach spaces*, J. Functional Analysis **91**(1990), 312–345.

[PPN] D. Preiss, R. R. Phelps and I. Namioka, *Smooth Banach spaces, weak Asplund spaces, and monotone or USCO mappings*, Israel J. Math. **72**(1990), 257–279.

[St] C. Stegall, *Gateaux differentiation of functions on a certain class of Banach spaces*. In: Functional Analysis: Surveys and Recent Results III, North Holland, 1984, 35–46.

Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
N2L 3G1

Present address:
Department of Mathematics and Statistics
Simon Fraser University
Burnaby, British Columbia
V5A 1S6

Department of Mathematics
University of Western Australia
Nedlands, W.A. 6019
Australia