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ON THE BOUNDARY SPECTRUM IN BANACH ALGEBRAS

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We investigate some properties of the set $S_\partial(a) = \{\lambda \in \mathbb{C} : \lambda - a \in \partial S\}$ (which we call the boundary spectrum of a) where ∂S denotes the topological boundary of the set S of all non-invertible elements of a Banach algebra A, and where a is an element of A.

1. INTRODUCTION AND PRELIMINARIES

Let A be a complex Banach algebra with unit 1. We shall denote the spectrum of an element a in A by $\sigma(a)$ and the spectral radius of a in A by $r(a)$ (or by $\sigma(a,A)$ and $r(a,A)$ respectively, if the particular Banach algebra needs to be emphasized). The distance from an element $\alpha \in \mathbb{C}$ to a subset E of $\mathbb{C}$ will be denoted by $d(\alpha, E)$, and $\delta(a)$ (or $\delta(a,A)$, if necessary) will indicate the distance $d\big(0,\sigma(a)\big)$ from 0 to the spectrum of a. If $\lambda \in \mathbb{C}$, then we shall write λ for the element $\lambda 1$ in A. We recall that if $\alpha \notin \sigma(a)$, then $d\big(\alpha,\sigma(a)\big) = 1/\Big(r\big((\alpha - a)^{-1}\big)\Big)$ ([**1**, Theorem 3.3.5]).

If E is a subset of a metric space $\mathcal{X}$, then $\partial_{\mathcal{X}} E$ denotes the topological boundary of E and $\mathrm{int}_{\mathcal{X}} E$ the topological interior of E relative to $\mathcal{X}$. For an $r > 0$ and an element x in $\mathcal{X}$, the notation $B_{\mathcal{X}}(x,\varepsilon)$ will be used to denote the open ball relative to $\mathcal{X}$ with centre x and radius ε. (If the choice of a metric space $\mathcal{X}$ is clear, the subscript $\mathcal{X}$ will be dropped.)

In this paper we consider the set $S_\partial(a) = \{\lambda \in \mathbb{C} : \lambda - a \in \partial S\}$ (or $S_\partial(a,A)$, if the particular Banach algebra needs to be emphasized) where S (or S_A, if necessary) denotes the set of all non-invertible elements of A. Some properties of this set are investigated: in particular the relationship between $S_\partial(a,A)$ and $S_\partial(a,B)$ where B is a closed subalgebra of a Banach algebra A such that B contains the unit of A, and the relationship between $S_\partial(a,A)$ and $S_\partial(Ta,B)$ where B is another Banach algebra and $T : A \to B$ a homomorphism. Finally, some results involving the boundary spectrum $S_\partial(a)$ of a positive element a in an ordered Banach algebra are obtained.

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2. Boundary spectrum

Let A be a complex Banach algebra with unit 1 and let S be the set of all non-invertible elements of A. Then S is a closed subset of A. Define, for $a \in A$, the set $S_\partial(a)$ in the complex plane as follows:

$$S_\partial(a) = \{\lambda \in \mathbb{C} : \lambda - a \in \partial S\}$$

We shall call this set the *boundary spectrum* of a in A. Also define, for $a \in A$,

$$r_1(a) = \sup\{|\lambda| : \lambda \in \partial\sigma(a)\}$$

and

$$r_2(a) = \sup\{|\lambda| : \lambda \in S_\partial(a)\}.$$

Proposition 2.1. *Let A be a Banach algebra and $a \in A$. Then $\partial\sigma(a) \subseteq S_\partial(a) \subseteq \sigma(a)$; and therefore $r_1(a) = r_2(a) = r(a)$, and if $\alpha \notin \sigma(a)$, then $d(\alpha, \partial\sigma(a)) = d(\alpha, S_\partial(a)) = d(\alpha, \sigma(a))$.*

Proof: To prove that $\partial\sigma(a) \subseteq S_\partial(a)$, let $\lambda \in \partial\sigma(a)$ and $\varepsilon > 0$. Then there exist a $\lambda_1 \in B(\lambda, \varepsilon) \cap \sigma(a)$ and a $\lambda_2 \in B(\lambda, \varepsilon) \cap (\mathbb{C}\backslash\sigma(a))$. If $b_1 = \lambda_1 - a$ and $b_2 = \lambda_2 - a$, then $b_1 \in S$, $b_2 \notin S$ and $b_1, b_2 \in B(\lambda - a, \varepsilon)$. Therefore $\lambda - a \in \partial S$, so that $\lambda \in S_\partial(a)$. This proves that $\partial\sigma(a) \subseteq S_\partial(a)$, and since S is closed, $\partial S \subseteq S$, so that $S_\partial(a) \subseteq \sigma(a)$. ∎

It follows from Proposition 2.1 that, for every $a \in A$, the set $S_\partial(a)$ is non-empty. Since ∂S is closed, $S_\partial(a)$ is closed, and since $S_\partial(a)$ is contained in the spectrum of a, it is bounded as well; in fact, $S_\partial(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leqslant r(a)\}$. Therefore $S_\partial(a)$ is a compact set.

In general, $\partial\sigma(a) \neq S_\partial(a)$. We proceed to illustrate this with an example.

Example 2.2. ([**1**, Remark 1, p.56]) Let $l^2(\mathbb{Z})$ be the Hilbert space of all bilateral square-summable sequences and $\{e_n : n \in \mathbb{Z}\}$ the orthonormal basis where, for each integer n, the vector e_n is $(\ldots, \xi_{-1}, (\xi_0), \xi_1, \ldots)$, where $\xi_n = 1$ and $\xi_i = 0$ for all integers i different from n. (In this case, the term in round brackets indicates the one corresponding to index zero.) Let $T, R : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be the weighted shifts

$$Te_n = \begin{cases} 0 & \text{if } n = -1 \\ e_{n+1} & \text{if } n \neq -1 \end{cases}$$

and

$$Re_n = \begin{cases} e_0 & \text{if } n = -1 \\ 0 & \text{if } n \neq -1. \end{cases}$$

Then $0 \in \sigma(T)$ and $\sigma(T + \lambda R)$ is contained in the unit circle for all $\lambda \neq 0$.

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Moreover, if $0 < |\lambda| < 1$, then λ is an eigenvalue of T. Indeed, if $\xi_j = 0$ for all $j \in \mathbb{N}\cup\{0\}$ and $\xi_{-j} = \lambda^{j-1}$ for all $j \in \mathbb{N}$, then the (non-zero) element $(\dots, \xi_{-1}, (\xi_0), \xi_1, \dots)$ of $l^2(\mathbb{Z})$ is in the kernel of $T - \lambda I$.

(The above facts about the operators T and R also follow from ([**3**, Problem 84]).)

EXAMPLE 2.3. Let $l^2(\mathbb{Z})$ be the Hilbert space of all bilateral square-summable sequences, A the Banach algebra $\mathcal{L}\big(l^2(\mathbb{Z})\big)$ of all bounded linear operators on $l^2(\mathbb{Z})$ and T the element of A defined in Example 2.2. Then $\partial\sigma(T)$ is properly contained in $S_\partial(T)$.

PROOF: Let (λ_n) be a sequence different from zero converging to zero, R the operator defined in Example 2.2 and $T_n = T + \lambda_n R$ ($n \in \mathbb{N}$). Then $\|T_n - T\| \to 0$. Moreover, by Example 2.2, each T_n is invertible and T is not invertible. Hence $T \in \partial S$. Therefore $-T \in \partial S$, and so $0 \in S_\partial(T)$. However, Example 2.2 together with the remark thereafter imply that 0 is an interior point of $\sigma(T)$, and so $0 \notin \partial\sigma(T)$. ∎

We recall the following well-known property of boundary points of the set of invertible (or non-invertible) elements:

THEOREM 2.4. ([**10**, Theorem 2.5, p. 397]) *Let A be a Banach algebra and $a \in A$. If $a \in \partial S$, then a is a topological divisor of zero.*

From the above theorem we immediately obtain the following property of the boundary spectrum of a:

COROLLARY 2.5. *Let A be a Banach algebra and $a \in A$. If $\lambda \in S_\partial(a)$, then $\lambda - a$ is a topological divisor of zero.*

LEMMA 2.6. *Let A be a Banach algebra, $a \in \partial S$ and d an invertible element. Then $ad \in \partial S$ and $da \in \partial S$.*

PROOF: If $a \in \partial S$ and d is invertible, then for each $\varepsilon > 0$ there exist elements $c_1 \in S\cap B\Big(a, (\varepsilon/\|d\|)\Big)$ and $c_2 \in (A\backslash S)\cap B\Big(a, (\varepsilon/\|d\|)\Big)$. It follows that $c_1 d \in S\cap B(ad, \varepsilon)$ and $c_2 d \in (A\backslash S)\cap B(ad, \varepsilon)$. Hence $ad \in \partial S$, and similarly $da \in \partial S$. ∎

It follows from Lemma 2.6 that $a \in \partial S$ if and only if $\lambda a \in \partial S$, for all $\lambda \neq 0$.

PROPOSITION 2.7. *Let a be an invertible element of a Banach algebra A. Then $S_\partial(a^{-1}) = \big(S_\partial(a)\big)^{-1}$.*

PROOF: For any $\lambda \neq 0$ and any invertible element $a \in A$ we have $\lambda - a^{-1} = \lambda\big(a - (1/\lambda)\big)a^{-1}$. So if $\lambda \in S_\partial(a^{-1})$, then $\lambda\big(a-(1/\lambda)\big)a^{-1} \in \partial S$. It follows from Lemma 2.6 that $a - (1/\lambda) \in \partial S$, so that $1/\lambda \in S_\partial(a)$. We have proved that $S_\partial(a^{-1}) \subseteq \big(S_\partial(a)\big)^{-1}$ for all invertible elements a, and therefore also $\big(S_\partial(a)\big)^{-1} \subseteq S_\partial(a^{-1})$ for all invertible a. ∎

Further mapping properties of S_∂ will be investigated in a future paper.

Let B be a closed subalgebra of a Banach algebra A such that B contains the unit element 1 of A. It is well known that if $a \in B$, then $\partial\sigma(a, B) \subseteq \partial\sigma(a, A)$ ([**1**, Theorem

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3.2.13]). We shall show that $S_\partial(a,B) \subseteq S_\partial(a,A)$ holds as well. In order to do this, we need the following results, some of which are interesting in their own right.

THEOREM 2.8. ([1, Theorem 3.2.13 (i)]) *Let B be a closed subalgebra of a Banach algebra A such that B contains the unit element 1 of A. Then $B \backslash S_B$ is the union of all components of $B \cap (A \backslash S_A)$ containing points of $B \backslash S_B$.*

LEMMA 2.9. *Let B be a closed subalgebra of a Banach algebra A such that B contains the unit element 1 of A. If E is a subset of A, then $\partial_B E \subseteq \partial_A E$.*

THEOREM 2.10. *Let B be a closed subalgebra of a Banach algebra A such that B contains the unit element 1 of A. Then S_B is the union of S_A and all the components of $B \cap (A \backslash S_A)$ containing points of S_B.*

PROOF: Clearly $S_A \subseteq S_B$. If $x \in S_B$ and $x \notin S_A$, then $x \in B \cap (A \backslash S_A)$, so that x is contained in a component of $B \cap (A \backslash S_A)$ which contains points of S_B. Hence S_B is contained in the union of S_A and all the components of $B \cap (A \backslash S_A)$ containing points of S_B.

Conversely, let Ω be a component of $B \cap (A \backslash S_A)$ which contains points of S_B. If $\Omega \nsubseteq S_B$, then Ω is a component of $B \cap (A \backslash S_A)$ which contains a point of $B \backslash S_B$. Theorem 2.8 implies that $\Omega \subseteq B \backslash S_B$, which contradicts the fact that Ω contains points of S_B. Hence $\Omega \subseteq S_B$. ∎

The following result was proved in [2], using the fact that boundary points of the set of invertible elements of a Banach algebra are topological divisors of zero (see Theorem 2.4) and therefore permanently singular. We provide an alternative proof.

THEOREM 2.11. ([2, Corollary 18, p. 14]) *Let B be a closed subalgebra of a Banach algebra A such that B contains the unit element 1 of A. Then $\partial_B S_B \subseteq \partial_A S_A$.*

PROOF: To prove that $\partial_B S_B \subseteq \partial_B S_A$, suppose that $x \notin \partial_B S_A$. If $x \notin B$, then $x \notin \partial_B S_B$, so suppose that $x \in B$. Then there exists an $\varepsilon > 0$ such that either (i) $B_B(x,\varepsilon) \subseteq S_A$ or (ii) $B_B(x,\varepsilon) \subseteq B \backslash S_A$. Since $S_A \subseteq S_B$, case (i) implies that $B_B(x,\varepsilon) \subseteq S_B$, so that $x \notin \partial_B S_B$, so suppose that $B_B(x,\varepsilon)$ is contained in a component Ω of $B \cap (A \backslash S_A)$. If Ω contains points of S_B, then by Theorem 2.10, Ω is contained in S_B, so that $x \notin \partial_B S_B$. If Ω contains no points of S_B, then $\Omega \subseteq B \backslash S_B$, so that once again, $x \notin \partial_B S_B$.

We have proved that $\partial_B S_B \subseteq \partial_B S_A$. Together with Lemma 2.9 the result follows. ∎

COROLLARY 2.12. *Let B be a closed subalgebra of a Banach algebra A such that B contains the unit element 1 of A. If $a \in B$, then $S_\partial(a,B) \subseteq S_\partial(a,A)$.*

PROOF: If $\lambda \in S_\partial(a,B)$, then $\lambda - a \in \partial_B S_B$. It follows from Theorem 2.11 that $\lambda - a \in \partial_A S_A$, so that $\lambda \in S_\partial(a,A)$. ∎

Now we consider the situation where A and B are Banach algebras (with B not necessarily a subalgebra of A) and $T : A \to B$ a homomorphism, and investigate the relationship between $S_\partial(a, A)$ and $S_\partial(Ta, B)$, where $a \in A$. We first establish some properties involving TS_A and S_B, and $T(\partial_A S_A)$ and $\partial_B S_B$. The proof of the next lemma is trivial:

Lemma 2.13. *Let A and B be Banach algebras and $T : A \to B$ a homomorphism. Then the following hold:*

1. $T^{-1}S_B \subseteq S_A$.
2. *If T is surjective, then $S_B \subseteq TS_A$.*
3. *If T is bijective, then $T^{-1}S_B = S_A$ and $TS_A = S_B$.*

Theorem 2.14. *Let A and B be Banach algebras and $T : A \to B$ a continuous isomorphism. Then $T(\partial_A S_A) = \partial_B S_B$.*

Proof: If $x \in \partial_A S_A$, then there exist sequences (x_n) in S_A and (y_n) in $A \backslash S_A$ such that $x_n \to x$ and $y_n \to x$. It follows from Lemma 2.13 (3) that $Ty_n \in B \backslash S_B$ and $Tx_n \in S_B$. Since T is continuous, $Tx_n \to Tx$ and $Ty_n \to Tx$. Hence $Tx \in \partial_B S_B$.

Conversely, if $y \in \partial_B S_B$, say $y = Tx$ with $x \in A$, then there exist sequences (z_n) in S_B and (w_n) in $B \backslash S_B$ such that $z_n \to y$ and $w_n \to y$. It follows from Lemma 2.13 (3) that $z_n = Tx_n$ with $x_n \in S_A$ and that $w_n \in B \backslash TS_A$, so that $w_n = Tu_n$ with $u_n \in A \backslash S_A$. Since T is bijective, linear and bounded, T^{-1} exists and is linear and bounded (by the Bounded Inverse Theorem), which implies that $x_n \to x$ and $u_n \to x$. Since (x_n) is in S_A and (u_n) is in $A \backslash S_A$, it follows that $x \in \partial_A S_A$. ∎

In the following result $\ker T$ will denote the kernel of T.

Theorem 2.15. *Let A and B be Banach algebras, $T : A \to B$ a continuous isomorphism and $a \in A$. Then*

$$S_\partial(a, A) = S_\partial(Ta, B) = \bigcup_{b \in \ker T} S_\partial(a + b, A).$$

Proof: If $\lambda \in S_\partial(a, A)$, then $\lambda - a \in \partial_A S_A$, so that Theorem 2.14 implies that $\lambda - Ta = T(\lambda - a) \in \partial_B S_B$, and so $\lambda \in S_\partial(Ta, B)$.

If $\lambda \in S_\partial(a + b, A)$ for some $b \in \ker T$, then $\lambda - Ta = T(\lambda - a - b) \in \partial_B S_B$, by Theorem 2.14, so that $\lambda \in S_\partial(Ta, B)$.

We have proved that

$$S_\partial(a, A) \subseteq S_\partial(Ta, B) \text{ and } \bigcup_{b \in \ker T} S_\partial(a + b, A) \subseteq S_\partial(Ta, B).$$

If $\lambda \in S_\partial(Ta, B)$, then $T(\lambda - a) = \lambda - Ta \in \partial_B S_B$, so that Theorem 2.14 implies that $T(\lambda - a) \in T(\partial_A S_A)$. The injectivity of T implies that $\lambda - a \in \partial_A S_A$, so that $\lambda \in S_\partial(a, A)$. Since $0 \in \ker T$, we obtain the following inclusions:

$$S_\partial(Ta, B) \subseteq S_\partial(a, A) \subseteq \bigcup_{b \in \ker T} S_\partial(a + b, A)$$

Hence the results follow. ∎

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3. Applications in ordered Banach algebras

In this section we investigate certain results in ordered Banach algebras involving the boundary spectrum. From ([**9**, Section 3]) we recall that an *algebra cone* C of a complex Banach algebra A with unit 1 is a subset of A containing 1 which is closed under the following operations: addition, positive scalar multiplication, and multiplication. If A has an algebra cone C, then A, or more specifically (A, C), is called an *ordered Banach algebra* (OBA). If, in addition, $C \cap -C = \{0\}$, then C is called *proper*.

An algebra cone C of A induces an *ordering* "$\leqslant$" on A in the following way:

$$a \leqslant b \text{ if and only if } b - a \in C$$

(where $a, b \in A$). This ordering is reflexive and transitive. Furthermore, C is proper if and only if the ordering has the additional property of being antisymmetric. Considering the ordering that C induces we find that $C = \{a \in A : a \geqslant 0\}$ and therefore we call the elements of C *positive*.

An algebra cone C of A is called *closed* if it is a closed subset of A. Furthermore, C is said to be *normal* if there exists a constant $\alpha > 0$ such that it follows from $0 \leqslant a \leqslant b$ in A that $\|a\| \leqslant \alpha\|b\|$. It is well known that if C is normal, then C is proper. If C has the property that if $a \in C$ and a is invertible, then $a^{-1} \in C$, then C is said to be *inverse-closed*. If B is a Banach algebra such that $1 \in B \subseteq A$, then $C \cap B$ is an algebra cone of B, and hence $(B, C \cap B)$ is an OBA.

In [**9, 8**], and later in [**4, 5, 6, 7**], some spectral theory of positive elements in ordered Banach algebras was developed. In particular, we recall the following results:

Theorem 3.1. ([**9**, Theorem 4.1(1)]) *Let (A, C) be an OBA with C normal. If $a, b \in A$ such that $0 \leqslant a \leqslant b$, then $r(a) \leqslant r(b)$.*

We refer to the above property by saying that the spectral radius in (A, C) is *monotone.*

Theorem 3.2. ([**9**, Theorem 5.2]) *Let (A, C) be an OBA with C closed and such that the spectral radius in (A, C) is monotone. If $a \in C$, then $r(a) \in \sigma(a)$.*

Using the boundary spectrum we obtain the following (slightly stronger) analogues of Theorem 3.2 and ([**6**, Theorem 3.3]):

Proposition 3.3. *Let (A, C) be an OBA with C closed and such that the spectral radius in (A, C) is monotone. If $a \in C$, then $r(a) \in S_\partial(a)$.*

Proof: If $a \in C$, then by Theorem 3.2 $r(a) \in \sigma(a)$. Hence $r(a) \in \partial\sigma(a)$ and so $r(a) \in S_\partial(a)$. ∎

Proposition 3.4. *Let (A, C) be an OBA with C closed and inverse-closed, and such that the spectral radius in (A, C) is monotone. If a is an invertible element of C, then $\delta(a) \in S_\partial(a)$.*

PROOF: If $a \in C$ and a is invertible, then $a^{-1} \in C$, since C is inverse-closed. Proposition 3.3 implies that $r(a^{-1}) \in S_\partial(a^{-1})$. Hence $r(a^{-1}) = 1/\lambda_0$ for some $\lambda_0 \in S_\partial(a)$, by Proposition 2.7. Since $r(a^{-1}) = 1/\big(\delta(a)\big)$, the result follows. ∎

In the following result B is a subalgebra of A but not necessarily closed in A.

THEOREM 3.5. *Let (A, C) be an OBA and B a Banach algebra with $1 \in B \subseteq A$.*

1. *Suppose that the spectral radius in (A, C) is monotone. If $0 \leqslant a \leqslant b$ with $a, b \in B$ and either $\partial\sigma(a, B) = \partial\sigma(a, A)$ or $S_\partial(a, B) = S_\partial(a, A)$, then $r(a, B) \leqslant r(b, B)$.*
2. *Suppose that the spectral radius in $(B, C \cap B)$ is monotone. If $0 \leqslant a \leqslant b$ with $a, b \in B$ and either $\partial\sigma(b, B) = \partial\sigma(b, A)$ or $S_\partial(b, B) = S_\partial(b, A)$, then $r(a, A) \leqslant r(b, A)$.*

PROOF:

1. Since B is a subalgebra of A, we have that $\sigma(b, A) \subseteq \sigma(b, B)$, so that $r(b, A) \leqslant r(b, B)$. The monotonicity of the spectral radius in (A, C) implies that $r(a, A) \leqslant r(b, A)$. Finally, the assumption that either $\partial\sigma(a, B) = \partial\sigma(a, A)$ or $S_\partial(a, B) = S_\partial(a, A)$ yields $r(a, B) = r(a, A)$, by Proposition 2.1. Combining the results, it follows that $r(a, B) \leqslant r(b, B)$.
2. Similarly as in (1), the fact that B is a subalgebra of A, the monotonicity of the spectral radius in $(B, C \cap B)$ and the additional assumption imply, respectively, that $r(a, A) \leqslant r(a, B)$, $r(a, B) \leqslant r(b, B)$ and $r(b, B) = r(b, A)$, which yield the result. ∎

We note that Theorem 3.5 (2) is a stronger version of ([**9**, Proposition 4.5]).

For our next result we need the following lemma and theorem:

LEMMA 3.6. ([**7**, Lemma 4.1]) *Let A be a Banach algebra, $x, y \in A$ and $\alpha \in \mathbb{C}$. If $\alpha - x$ is invertible and $r\big((\alpha - x)^{-1}(x - y)\big) < 1$, then $\alpha - y$ is invertible.*

THEOREM 3.7. ([**7**, Proof of Theorem 4.2]) *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. If $y \in C$ such that $x \leqslant y$ and either $xy \leqslant yx$ or $yx \leqslant xy$, and α is a positive real number such that $\alpha > r(x)$, then*

$$r\big((\alpha - x)^{-1}(y - x)\big) \leqslant r\big((\alpha - x)^{-1}\big)r(y - x).$$

Now let (A, C) be an OBA. Define, for each $x \in C$, an analogue $A'(x)$ of the set $A(x)$ (defined in ([**7**, Section 4])) as follows:

$$A'(x) = \{y \in A : x \leqslant y, \quad xy \leqslant yx \text{ or } yx \leqslant xy \quad \text{and} \quad d\big(r(y), S_\partial(x)\big) \geqslant d\big(\alpha, S_\partial(x)\big) \text{ for all } \alpha \in S_\partial(y)\}$$

Then $x \in A'(x)$, $A'(x) \subseteq C$ and $A'(0) = C$. Finally, the following theorem is a complementary result to ([**7**, Theorem 4.2]):

Theorem 3.8. *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then $S_\partial(y) \subseteq S_\partial(x) + r(x-y)$ for all $y \in A'(x)$.*

Proof: Let $y \in A'(x)$. Then $0 \leqslant x \leqslant y$, so that $r(x) \leqslant r(y)$, by Theorem 3.1. If $r(x) = r(y)$, then $d(r(y), S_\partial(x)) = 0$, by Proposition 3.3, so that, by the assumption, $d(\alpha, S_\partial(x)) = 0$ for all $\alpha \in S_\partial(y)$. This implies that $d(\alpha, S_\partial(x)) \leqslant r(x-y)$ for all $\alpha \in S_\partial(y)$, so that $S_\partial(y) \subseteq S_\partial(x) + r(x-y)$.

So suppose that $r(x) < r(y)$, and suppose there exists an $\alpha \in S_\partial(y)$ such that $d(\alpha, S_\partial(x)) > r(x-y)$. Proposition 3.3 implies that $r(y) \in S_\partial(y)$ and hence, by the assumption, we may take $\alpha \in \mathbb{R}^+$ with $\alpha > r(x)$. Since $\alpha \notin \sigma(x)$, it follows from Proposition 2.1 that $d(\alpha, S_\partial(x)) = d(\alpha, \sigma(x))$, so that $d(\alpha, S_\partial(x)) = 1/\Big(r\big((\alpha - x)^{-1}\big)\Big)$. Therefore $r\big((\alpha-x)^{-1}\big)r(x-y) < 1$ with $\alpha \in \mathbb{R}^+$ and $\alpha > r(x)$.

It follows from Theorem 3.7 that $r\big((\alpha - x)^{-1}(y-x)\big) < 1$, so that $\alpha \notin \sigma(y)$, by Lemma 3.6. Hence $\alpha \notin S_\partial(y)$ — a contradiction. Therefore $d(\alpha, S_\partial(x)) \leqslant r(x-y)$ for all $\alpha \in S_\partial(y)$, so that $S_\partial(y) \subseteq S_\partial(x) + r(x-y)$. ∎

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