CONTRIBUTIONS TO NONCOMMUTATIVE IDEAL THEORY

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Introduction. The well-known results of Krull concerning the minimal prime divisors and the radical of an ideal in a commutative ring have been extended to the noncommutative case in a recent paper [**5**] by N. H. McCoy. In that paper systematic use was made of the concept of an m-system, a set M of elements of the ring such that if $a \in M$ and $b \in M$ then $axb \in M$ for some element x of the ring. The m-system plays the same role in the noncommutative case that the multiplicatively closed system plays in the theory of Krull. For example, an ideal in a noncommutative ring is prime if and only if its complement is an m-system. What follows is an attempt based on the methods of McCoy to extend more of the Krull-Noether theory of commutative rings to the noncommutative case. Different treatments of the noncommutative case have previously been published by Krull [**2**], and Fitting [**1**]. Since the point of view of the present paper, however, is considerably different from that of either of these previous ones, little or no use has been made of their results. The results and methods of McCoy [**5**], on the other hand, have been used extensively.

The concept of an isolated component ideal (Krull [**3**] and [**4**]) leads in the noncommutative case to upper and lower right (or left) isolated component ideals each of which retains some of the properties of the isolated component ideals of the commutative case. These upper and lower components and the relations between them are investigated in §§ 2, 3 and 4. The results of these sections follow without any assumptions of finite chain conditions. The effect of descending and ascending chain conditions is considered in §5 and the latter is assumed in the remainder of the paper. Right primary ideals are defined in a manner which ensures, in the presence of either chain condition, that the radical of a right primary ideal is prime. The term *radical* is used throughout in the sense of McCoy [**5**]. Examples are given which show that not every ideal is representable as the intersection of a finite number of right primary ideals but any ideal which is so representable has a short representation and for short representations the same uniqueness theorems hold as in the commutative case. Thus in any two short representations of an ideal $\mathfrak{a}$ as the intersection of right primary ideals the number of primary components is the same and the radicals of these coincide in some order. Moreover, the isolated primary components are uniquely determined and must occur in any such representation of $\mathfrak{a}$.

1. Definitions and basic concepts. Let R be an arbitrary noncommutative ring. An ideal $\mathfrak{p}$ in R is prime if $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ implies either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, where $\mathfrak{a}$

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and $\mathfrak{b}$ are any ideals of R. It has been shown by McCoy [5] that an ideal $\mathfrak{p}$ is prime if and only if, for any elements a, b of R, $aRb \subseteq \mathfrak{p}$ implies that either a or b belongs to $\mathfrak{p}$.

DEFINITION 1.1. *A set M of elements of R is called an m-system if for any two elements a and b of M, there exists an element x of R such that $axb \in M$. The null set is also defined to be an m-system* (McCoy [5]).

It is clear from the above remark and from the definition that an ideal is prime if and only if its complement in R is an m-system.

DEFINITION 1.2. *An element a of R is said to be right prime to an ideal $\mathfrak{a}$ if $xRa \subseteq \mathfrak{a}$ implies that $x \in \mathfrak{a}$. An ideal $\mathfrak{b}$ is right prime to $\mathfrak{a}$ if it contains an element which is right prime to $\mathfrak{a}$.*

Elements and ideals left prime to $\mathfrak{a}$ can be defined in the obvious way but the left hand definitions and theorems will usually be omitted.

DEFINITION 1.3. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in R, the ideal consisting of all elements x of R such that $xRb \subseteq \mathfrak{a}$ for all b in $\mathfrak{b}$ is called the right ideal quotient of $\mathfrak{a}$ by $\mathfrak{b}$ and is denoted by $\mathfrak{a}\mathfrak{b}^{-1}$. Similarly $\mathfrak{b}^{-1}\mathfrak{a}$ consists of all x in R such that $bRx \subseteq \mathfrak{a}$ for all b in $\mathfrak{b}$.*

It is obvious that $\mathfrak{a}\mathfrak{b}^{-1}$ and $\mathfrak{b}^{-1}\mathfrak{a}$ always contain $\mathfrak{a}$ and that if $\mathfrak{b}$ is right prime to $\mathfrak{a}$ then $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{a}$.

DEFINITION 1.4. *If M is a non-null m-system, a set N of elements of R is called a right n-system associated with M (briefly a right M-n-system) if N contains M and if for every m in M and every n in N there exists an element x of R such that $nxm \in N$. If M is the null set the only right M-n-system is, by definition, the null set itself.*

We note that every m-system is a right (or left) n-system associated with itself. Moreover, the set-theoretic union of a finite or infinite number of right n-systems all of which are associated with the same m-system M, is again a right n-system associated with M. However it may also be associated with a larger m-system, properly containing M. As an illustration of this let R be the ring of integers, p any prime, and M the m-system consisting of all integers prime to p. Let N_i be the set of all integers which are not divisible by p^i $(i = 2, 3, 4, \ldots)$. Each N_i is an M-n-system. The union of all N_i is the set of all non-zero integers and is itself an m-system $\bar{M}$. It is therefore an $\bar{M}$-n-system where $\bar{M} \supset M$. We remark also that if, in a commutative ring, $\mathfrak{q}$ is a primary ideal and $\mathfrak{p}$ its associated prime, then $M = C(\mathfrak{p})$ (complement of $\mathfrak{p}$ in R) is an m-system and $N = C(\mathfrak{q})$ is an M-n-system.

2. Upper isolated component ideals.

DEFINITION 2.1. *If $\mathfrak{a}$ is any ideal in R and M is an m-system which does not meet $\mathfrak{a}$ (i.e. has no elements in common with $\mathfrak{a}$), the right upper M-component of $\mathfrak{a}$ is defined to be the set of all elements x of R having the property that every right M-n-system which contains x meets $\mathfrak{a}$.*

The right upper M-component of $\mathfrak{a}$ will be denoted by $\mathfrak{u}(\mathfrak{a}, M)$. In order to show that $\mathfrak{u}(\mathfrak{a}, M)$ is an ideal we shall require the following lemmas.

LEMMA 1. *If* $\mathfrak{a}$ *is any ideal and* M *an m-system which does not meet* $\mathfrak{a}$ *then there exists a maximal right* M-n-*system* N *which does not meet* $\mathfrak{a}$ *and* N *is uniquely determined by* M *and* $\mathfrak{a}$.

Proof. There exists at least one right M-n-system which does not meet $\mathfrak{a}$, namely M itself. The union N of all such right M-n-systems therefore satisfies the requirements of the lemma.

LEMMA 2. *Let* M *be any m-system and* N *any right* M-n-*system. Let* $\mathfrak{a}$ *be an ideal which does not meet* N. *Then* $\mathfrak{a}$ *is contained in a maximal ideal* $\mathfrak{q}^*$ *which does not meet* N *and* $\mathfrak{q}^*$ *has the property that if* $aRb \subseteq \mathfrak{q}^*$ *and* $\mathfrak{b} \in M$, *then* $a \in \mathfrak{q}^*$.

Proof. Since the union of any linearly ordered set of ideals which do not meet N is an ideal which does not meet N, the existence of $\mathfrak{q}^*$ follows from Zorn's Lemma [6, p. 101].

Now suppose a is an element of R which does not belong to $\mathfrak{q}^*$. Then $(a, \mathfrak{q}^*)$ properly contains $\mathfrak{q}^*$ and hence by the maximal property of $\mathfrak{q}^*$ must contain an element n of N. Thus

$$n = ia + ra + ar' + \sum_{i,j} r_i a r_j + q$$

where i is an integer, r, r', r_i, r_j are elements of R and $q \in \mathfrak{q}^*$. Now if $b \in M$ there exists an element x of R such that $nx\text{b} \in N$ where

$$nxb = iaxb + raxb + ar'xb + \sum_{i,j} r_i a r_j xb + qxb.$$

But if $aRb \subseteq \mathfrak{q}^*$ every element in the sum on the right hand side of this equation belongs to $\mathfrak{q}^*$, and nxb belongs to both N and $\mathfrak{q}^*$, contrary to the definition of $\mathfrak{q}^*$. Hence if $aRb \subseteq \mathfrak{q}^*$ and $b \in M$ we must have $a \in \mathfrak{q}^*$, as required.

Lemma 2 states that every element of M is right prime to $\mathfrak{q}^*$. It will be convenient to refer to this property by saying that $\mathfrak{q}^*$ has property (A) relative to M. We remark also that if an ideal has property (A) relative to M then its complement in R is a right M-n-system and conversely.

LEMMA 3. *Let* $\mathfrak{a}$ *be an ideal and* M *an m-system which does not meet* $\mathfrak{a}$. *A set* $\mathfrak{q}$ *of elements of* R *is a minimal ideal containing* $\mathfrak{a}$ *and having property* (A) *relative to* M *if and only if* $C(\mathfrak{q})$ *is a maximal right* M-n-*system which does not meet* $\mathfrak{a}$.

Proof. (i) First suppose $C(\mathfrak{q})$ is a maximal right M-n-system which does not meet $\mathfrak{a}$. By Lemma 2, $\mathfrak{a}$ is contained in a maximal ideal $\mathfrak{q}^*$ which does not meet $C(\mathfrak{q})$. Moreover, $\mathfrak{q}^*$ has property (A) relative to M and hence $C(\mathfrak{q}^*)$ is a right M-n-system which does not meet $\mathfrak{a}$. Since $\mathfrak{q}^*$ does not meet $C(\mathfrak{q})$ we have $C(\mathfrak{q}) \subseteq C(\mathfrak{q}^*)$ and hence, from the maximal property of $C(\mathfrak{q})$, it follows that $C(\mathfrak{q}) = C(\mathfrak{q}^*)$ and $\mathfrak{q} = \mathfrak{q}^*$. Thus $\mathfrak{q}$ is an ideal with property (A) relative to M. Finally $\mathfrak{q}$ is a minimal such ideal, for if $\mathfrak{q} \supset \mathfrak{q}' \supseteq \mathfrak{a}$ where $\mathfrak{q}'$ has property (A) relative to M then $C(\mathfrak{q}')$ is a right M-n-system which does not meet $\mathfrak{a}$ and properly contains $C(\mathfrak{q})$, contrary to the maximal property of $C(\mathfrak{q})$.

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(ii) Conversely, suppose $\mathfrak{q}$ is a minimal ideal containing $\mathfrak{a}$ and having property (A) relative to M. Then $C(\mathfrak{q})$ is a right M-n-system which does not meet $\mathfrak{a}$, and by Lemma 1 is contained in a maximal such right M-n-system N. Hence by (i) proved above $C(N)$ is a minimal ideal containing $\mathfrak{a}$ and having property (A) relative to M and since $C(\mathfrak{q}) \subseteq N$, $\mathfrak{q} \supseteq C(N)$ and by the minimal property of $\mathfrak{q}$, $\mathfrak{q} = C(N)$ whence $C(\mathfrak{q})$ is a maximal right M-n-system which does not meet $\mathfrak{a}$. This completes the proof.

THEOREM 1. *The right upper isolated M-component* $\mathfrak{u}(\mathfrak{a}, M)$ *of* $\mathfrak{a}$ *is an ideal. Its complement in R is the uniquely determined maximal right M-n-system which does not meet* $\mathfrak{a}$, *and* $\mathfrak{u}(\mathfrak{a}, M)$ *itself is the crosscut of all ideals containing* $\mathfrak{a}$ *which have property* (A) *relative to M.*

Proof. Let M be an m-system which does not meet $\mathfrak{a}$ and let N be the maximal right M-n-system not meeting $\mathfrak{a}$ whose existence is assured by Lemma 1. By Lemma 3, $\mathfrak{q} = C(N)$ is a minimal ideal containing $\mathfrak{a}$ and having property (A) relative to M. Since the crosscut of any set of ideals containing $\mathfrak{a}$ and having property (A) again has property (A) it follows that there is a unique minimal such ideal which must be equal to $\mathfrak{q}$ and hence $\mathfrak{q}$ is the crosscut of all ideals containing $\mathfrak{a}$ which have property (A) relative to M. It remains to prove that $\mathfrak{q} = \mathfrak{u}(\mathfrak{a}, M)$.

First, $\mathfrak{q} \subseteq \mathfrak{u}(\mathfrak{a}, M)$. For if $x \in \mathfrak{q}$ then x does not belong to N, the maximal right M-n-system which does not meet $\mathfrak{a}$. Hence every M-n-system which contains x meets $\mathfrak{a}$ and $x \in \mathfrak{u}(\mathfrak{a}, M)$. On the other hand, $\mathfrak{u}(\mathfrak{a}, M) \subseteq \mathfrak{q}$. For if $x \in \mathfrak{u}(\mathfrak{a}, M)$ then x cannot belong to N and must belong to $\mathfrak{q}$. Hence $\mathfrak{u}(\mathfrak{a}, M) = \mathfrak{q}$ and the theorem is proved.

COROLLARY 1. *If* $\mathfrak{a} \supseteq \mathfrak{b}$ *and M is an m-system which does not meet* $\mathfrak{a}$ *then* $\mathfrak{u}(\mathfrak{a}, M) \supseteq \mathfrak{u}(\mathfrak{b}, M)$.

COROLLARY 2. *If* M_1, M_2, *are m-systems which do not meet* $\mathfrak{a}$ *and if* $M_1 \supseteq M_2$ *then* $\mathfrak{u}(\mathfrak{a}, M_1) \supseteq \mathfrak{u}(\mathfrak{a}, M_2)$.

Proof. Every M_1-n-system is also an M_2-n-system and hence the maximal M_1-n-system which does not meet $\mathfrak{a}$ is contained in the maximal such M_2-n-system. Taking complements,

$$\mathfrak{u}(\mathfrak{a}, M_1) \supseteq \mathfrak{u}(\mathfrak{a}, M_2).$$

If $\mathfrak{p}$ is a prime ideal which divides $\mathfrak{a}$ and if $M = C(\mathfrak{p})$, then $\mathfrak{u}(\mathfrak{a}, M)$ will also be referred to as the right upper $\mathfrak{p}$-component of $\mathfrak{a}$ and will also be denoted, when convenient, by $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$.

3. Lower isolated component ideals. In this section we shall define a right lower isolated component of an ideal $\mathfrak{a}$, and we shall investigate its relationship to the upper isolated component discussed in the previous section.

DEFINITION 3.1. *If* $\mathfrak{a}$ *is any ideal in R and M any m-system which does not meet* $\mathfrak{a}$, *the right lower isolated component of* $\mathfrak{a}$ *coresponding to M, or briefly the right* *lower M-component of* $\mathfrak{a}$, *is defined to be the set of all elements x of R such that* $xRm \subseteq \mathfrak{a}$ *for some element m of M.*

The right lower M-component of $\mathfrak{a}$ will be denoted by $\mathfrak{l}(\mathfrak{a}, M)$. It is clear that $\mathfrak{l}(\mathfrak{a}, M)$ is an ideal, for if $x \in \mathfrak{l}(\mathfrak{a}, M)$ certainly, $-x$, and rx and xr belong to $\mathfrak{l}(\mathfrak{a}, M)$ for all r in R. Also if $x_1Rm_1 \subseteq \mathfrak{a}$ and $x_2Rm_2 \subseteq \mathfrak{a}$ where m_1, m_2 belong to M, then $m_1rm_2 \in M$ for some r in R and

$$(x_1 + x_2)Rm_1rm_2 \subseteq x_1Rm_1rm_2 + x_2Rm_2 \subseteq \mathfrak{a}.$$

Hence $x_1 + x_2 \in \mathfrak{l}(\mathfrak{a}, M)$.

THEOREM 2. *If $\mathfrak{a}$ is an ideal and M an m-system which does not meet $\mathfrak{a}$ then* $\mathfrak{u}(\mathfrak{a}, M) \supseteq \mathfrak{l}(\mathfrak{a}, M) \supseteq \mathfrak{a}$.

Proof. If $x \in \mathfrak{l}(\mathfrak{a}, M)$ then $xRm \subseteq \mathfrak{a}$ for some element m of M. Hence every right M-n-system which contains x meets $\mathfrak{a}$ and $x \in \mathfrak{u}(\mathfrak{a}, M)$. That $\mathfrak{l}(\mathfrak{a}, M) \supseteq \mathfrak{a}$ is obvious from the definition.

THEOREM 3.

(a) $\mathfrak{u}[\mathfrak{u}(\mathfrak{a}, M), M] = \mathfrak{u}(\mathfrak{a}, M)$,

(b) $\mathfrak{l}[\mathfrak{u}(\mathfrak{a}, M), M] = \mathfrak{u}(\mathfrak{a}, M)$,

(c) $\mathfrak{u}[\mathfrak{l}(\mathfrak{a}, M), M] = \mathfrak{u}(\mathfrak{a}, M)$.

Proof. (a) The complement in R of $\mathfrak{u}(\mathfrak{a}, M)$ is a right M-n-system N and hence is certainly the maximal such that does not meet $\mathfrak{u}(\mathfrak{a}, M)$. Hence by Theorem 1, $C(N) = \mathfrak{u}(\mathfrak{a}, M)$ is the right upper M-component of $\mathfrak{u}(\mathfrak{a}, M)$.

(b) The ideal $\mathfrak{l}[\mathfrak{u}(\mathfrak{a}, M), M]$ consists of all elements x of R such that $xRm \subseteq \mathfrak{u}(\mathfrak{a}, M)$ for some m in M. But since $\mathfrak{u}(\mathfrak{a}, M)$ has property (A) relative to M this implies that $x \in \mathfrak{u}(\mathfrak{a}, M)$. Hence $\mathfrak{l}[\mathfrak{u}(\mathfrak{a}, M), M] \subseteq \mathfrak{u}(\mathfrak{a}, M)$. Since, by Theorem 2, $\mathfrak{u}(\mathfrak{a}, M) \subseteq \mathfrak{l}[\mathfrak{u}(\mathfrak{a}, M), M]$, the equality follows.

(c) If $x \in \mathfrak{u}[\mathfrak{l}(\mathfrak{a}, M), M]$ then every right M-n-system N which contains x meets $\mathfrak{l}(\mathfrak{a}, M)$, that is, N contains an element n such that $nRm \subseteq \mathfrak{a}$ for some m in M. But since $n \in N$ and $m \in M$, $nrm \in N$ for some r in R. Hence N meets $\mathfrak{a}$ and $x \in \mathfrak{u}(\mathfrak{a}, M)$ and we have $\mathfrak{u}[\mathfrak{l}(\mathfrak{a}, M), M] \subseteq \mathfrak{u}(\mathfrak{a}, M)$. But since $\mathfrak{a} \subseteq \mathfrak{l}(\mathfrak{a}, M)$, by Corollary 1, Theorem 1, $\mathfrak{u}(\mathfrak{a}, M) \subseteq \mathfrak{u}[\mathfrak{l}(\mathfrak{a}, M), M]$ and the equality follows.

DEFINITION 3.2. *For all ordinal numbers α we define the ideal $\mathfrak{l}^{\alpha}(\mathfrak{a}, M)$ by induction as follows: $\mathfrak{l}^1(\mathfrak{a}, M) = \mathfrak{l}(\mathfrak{a}, M)$. If α is not a limit ordinal, $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{l}[\mathfrak{l}^{\alpha-1}(\mathfrak{a}, M), M]$, while if α is a limit ordinal, $\mathfrak{l}^{\alpha}(\mathfrak{a}, M)$ is the union of all $\mathfrak{l}^{\sigma}(\mathfrak{a}, M)$ for which $\sigma < \alpha$.*

It is clear that $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) \supseteq \mathfrak{l}^{\sigma}(\mathfrak{a}, M)$ if $\sigma < \alpha$.

THEOREM 4. *For all ordinal numbers α*, $\mathfrak{u}(\mathfrak{a}, M) \supseteq \mathfrak{l}^{\alpha}(\mathfrak{a}, M)$.

Proof. By Theorem 2 the result is known for $\alpha = 1$. We assume the theorem for all ordinals less than α and proceed by induction.

Case 1. If α is not a limit ordinal and so has an immediate predecessor $\alpha - 1$ we have

$$\begin{aligned}\mathfrak{l}^{\alpha}(\mathfrak{a}, M) &= \mathfrak{l}[\mathfrak{l}^{\alpha-1}(\mathfrak{a}, M), M] \\ &\subseteq \mathfrak{u}[\mathfrak{l}^{\alpha-1}(\mathfrak{a}, M), M] && \text{by Theorem 2,} \\ &\subseteq \mathfrak{u}[\mathfrak{u}(\mathfrak{a}, M), M] && \text{by Corollary 1, Theorem 1,} \\ &= \mathfrak{u}(\mathfrak{a}, M) && \text{by Theorem 3(a).}\end{aligned}$$

Case 2. If α is a limit ordinal $\mathfrak{l}^{\alpha}(\mathfrak{a}, M)$ is the union of all $\mathfrak{l}^{\sigma}(\mathfrak{a}, M)$ for $\sigma < \alpha$. Hence if $x \in \mathfrak{l}^{\alpha}(\mathfrak{a}, M)$ then $x \in \mathfrak{l}^{\sigma}(\mathfrak{a}, M)$ for $\sigma < \alpha$ and $x \in \mathfrak{u}(\mathfrak{a}, M)$ by the induction assumption, and hence $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) \subseteq \mathfrak{u}(\mathfrak{a}, M)$.

THEOREM 5. *For any ordinal number* α, $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{l}^{\alpha+1}(\mathfrak{a}, M)$ *if and only if* $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{u}(\mathfrak{a}, M)$.

Proof. (i) If $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{u}(\mathfrak{a}, M)$ then $\mathfrak{l}^{\alpha+1}(\mathfrak{a}, M) = \mathfrak{l}[\mathfrak{u}(\mathfrak{a}, M), M] = \mathfrak{u}(\mathfrak{a}, M)$ by Theorem 3(b).

(ii) If $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{l}^{\alpha+1}(\mathfrak{a}, M)$, let x be any element of $\mathfrak{l}^{\alpha+1}(\mathfrak{a}, M)$ so that $xRm \subseteq \mathfrak{l}^{\alpha}(\mathfrak{a}, M)$ for some element m of M. But under the assumption $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{l}^{\alpha+1}(\mathfrak{a}, M)$ the condition $xRm \subseteq \mathfrak{l}^{\alpha}(\mathfrak{a}, M)$ implies $x \in \mathfrak{l}^{\alpha}(\mathfrak{a}, M)$. Hence $\mathfrak{l}^{\alpha}(\mathfrak{a}, M)$ has property (A) relative to M and since $\mathfrak{u}(\mathfrak{a}, M)$ is the minimal ideal having this property we have $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{u}(\mathfrak{a}, M)$.

COROLLARY 1. *There exists an ordinal number* α, *finite or transfinite, such that* $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{u}(\mathfrak{a}, M)$.

Since the $\mathfrak{l}^{\sigma}(\mathfrak{a}, M)$ are well ordered and the union of every subset of them is again an $\mathfrak{l}^{\sigma}(\mathfrak{a}, M)$, by Zorn's lemma they are all contained in a maximal one, $\mathfrak{l}^{\alpha}(\mathfrak{a}, M)$. Necessarily $\mathfrak{l}^{\alpha}(\mathfrak{a}, M) = \mathfrak{l}^{\alpha+1}(\mathfrak{a}, M) = \mathfrak{u}(\mathfrak{a}, M)$ by the theorem.

COROLLARY 2. *If the ascending chain condition holds in the residue class ring* $R/\mathfrak{a}$ *then* $\mathfrak{l}^{n}(\mathfrak{a}, M) = \mathfrak{u}(\mathfrak{a}, M)$ *for some finite* n.

COROLLARY 3. *If the ascending chain condition holds in* $R/\mathfrak{a}$ *and if* $x \in \mathfrak{u}(\mathfrak{a}, M)$ *then for every element* r *of* R *there exists an element* m_r *of* M *such that* $xrm_r \in \mathfrak{a}$. *The element* m_r *is independent of* r *if and only if* $\mathfrak{l}(\mathfrak{a}, M) = \mathfrak{u}(\mathfrak{a}, M)$.

Proof. By Corollary 1, $\mathfrak{l}^{n}(\mathfrak{a}, M) = \mathfrak{u}(\mathfrak{a}, M)$. Hence, if $x \in \mathfrak{u}(\mathfrak{a}, M)$ there exists an element m of M such that $xRm \subseteq \mathfrak{l}^{n-1}(\mathfrak{a}, M)$. That is, for every r_1 in R there is an element $m(r_1)$ of M such that

$$xr_1mRm(r_1) \subseteq \mathfrak{l}^{n-2}(\mathfrak{a}, M).$$

Hence for every r_2 in R there is an element $m(r_2)$ such that

$$xr_1mr_2m(r_1)Rm(r_2) \subseteq \mathfrak{l}^{n-3}(\mathfrak{a}, M).$$

Carrying on in this way we find

$$xr_1mr_2m(r_1)r_3m(r_2) \dots r_{n-1}m(r_{n-2})Rm(r_{n-1}) \subseteq \mathfrak{a}.$$

Now r_2 can be chosen so that $mr_2m(r_1) \in M$; r_3 so that $mr_2m(r_1)r_3m(r_2) \in M$ and so on. Finally, choose r_n so that $mr_2m(r_1)r_3m(r_2) \dots (r_{n-2})r_nm(r_{n-1}) \in M$ and the result follows.

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Finally, if for all x in $\mathfrak{u}(\mathfrak{a}, M)$, $xrm \in \mathfrak{a}$ where m is independent of r then $xRm \subseteq \mathfrak{a}$ and $x \in \mathfrak{l}(\mathfrak{a}, M)$ and $\mathfrak{u}(\mathfrak{a}, M) = \mathfrak{l}(\mathfrak{a}, M)$. Conversely, if $\mathfrak{u}(\mathfrak{a}, M) = \mathfrak{l}(\mathfrak{a}, M)$ then there exists an m independent of r and the proof of the corollary is complete.

4. The commutative case. We shall now investigate the relationship of $\mathfrak{l}(\mathfrak{a}, M)$ and $\mathfrak{u}(\mathfrak{a}, M)$ to the isolated component ideals defined by Krull [**4**, p. 16] in a commutative ring.

THEOREM 6. *If $\mathfrak{a}$ is an ideal in a commutative ring R, and M an m-system which does not meet $\mathfrak{a}$, the set $\mathfrak{a}(M)$ of all elements x of R for which $xm \in \mathfrak{a}$ for some element m of M, is an ideal.*

Proof. If $xm_1 \in \mathfrak{a}$ and $ym_2 \in \mathfrak{a}$ where m_1 and m_2 are elements of M, then if r is chosen so that $m_1rm_2 \in M$ we have

$$(x - y)m_1rm_2 = xm_1rm_2 - ym_2m_1r \in \mathfrak{a}.$$

and therefore $x - y \in \mathfrak{a}(M)$. Since obviously $cx \in \mathfrak{a}(M)$ for all c in R, $\mathfrak{a}(M)$ is an ideal.

DEFINITION 4.1. *The ideal $\mathfrak{a}(M)$ defined in Theorem 6 is called the isolated M-component of $\mathfrak{a}$.*

The isolated component ideal of Krull was defined exactly as in Definition 4.1 except that M was restricted to be a multiplicatively closed system. Since every multiplicatively closed system is an m-system [**5**] our definition of $\mathfrak{a}(M)$ coincides with that of Krull whenever the latter applies, that is, whenever M is multiplicatively closed. That $\mathfrak{u}(\mathfrak{a}, M)$ and $\mathfrak{l}(\mathfrak{a}, \mathrm{M})$ may both be considered as generalizations of $\mathfrak{a}(M)$ to the noncommutative case may now be seen from the following result.

THEOREM 7. *If $\mathfrak{a}$ is any ideal in a commutative ring R, and M is an m-system which does not meet $\mathfrak{a}$, then $\mathfrak{u}(\mathfrak{a}, M) = \mathfrak{l}(\mathfrak{a}, M) = \mathfrak{a}(M)$.*

Proof. If $x \in \mathfrak{a}(M)$ then $xm \in \mathfrak{a}$ for some element m of M. Hence, since R is commutative, $xRm \subseteq \mathfrak{a}$. Therefore $x \in \mathfrak{l}(\mathfrak{a}, M)$, and $\mathfrak{a}(M) \subseteq \mathfrak{l}(\mathfrak{a}, M)$.

Now if $x \in \mathfrak{u}(\mathfrak{a}, M)$, every M-n-system which contains x meets $\mathfrak{a}$. But the set of elements $N = \{x, M, xm\}$ containing x, M, and all elements xm where $\mathrm{m} \in M$, is an M-n-system containing x. Hence N meets $\mathfrak{a}$ and since M does not meet $\mathfrak{a}$ it follows that $xm \in \mathfrak{a}$ for some element m of M. Therefore $x \in \mathfrak{a}(M)$ and we have now $\mathfrak{u}(\mathfrak{a}, M) \subseteq \mathfrak{a}(M) \subseteq \mathfrak{l}(\mathfrak{a}, M)$. But by Theorem 2, $\mathfrak{l}(\mathfrak{a}, M) \subseteq \mathfrak{u}(\mathfrak{a}, M)$ and the theorem follows.

5. Chain conditions. For most of what follows it will be necessary to assume that the ring R satisfies the ascending chain condition for two sided ideals. Before imposing this restriction, however, we shall develop some consequences of the following weak form of the descending chain condition.

CONDITION A. *For every ideal* $\mathfrak{a}$ *which is not prime, the ring* $R/\mathfrak{a}$ *satisfies the descending chain condition for two sided ideals.*

THEOREM 8. *If* $\mathfrak{s}$ *is a minimal proper divisor of* $\mathfrak{a}$ *then* $\mathfrak{s}^{-1}\mathfrak{a}$ *is a prime divisor of* $\mathfrak{a}$ *and is not right prime to* $\mathfrak{a}$.

Proof. If $\mathfrak{s}^{-1}\mathfrak{a} = R$, it is prime, and since $\mathfrak{s}R(\mathfrak{s}^{-1}\mathfrak{a}) \subseteq \mathfrak{a}$ and $\mathfrak{s}$ is a proper divisor of $\mathfrak{a}$ it follows that $\mathfrak{s}^{-1}\mathfrak{a}$ is nrp to $\mathfrak{a}$. ("nrp" means "not right prime.") If $\mathfrak{s}^{-1}\mathfrak{a} \neq R$, suppose $xRy \subseteq \mathfrak{s}^{-1}\mathfrak{a}$ where x is not in $\mathfrak{s}^{-1}\mathfrak{a}$. Then for every element s of $\mathfrak{s}$, $sRxRy \subseteq \mathfrak{a}$, but for some element s' of $\mathfrak{s}$, $s'Rx$ not $\subseteq \mathfrak{a}$. Choose r in R so that $s'rx$ is not in $\mathfrak{a}$ and form the ideal $(s'rx, \mathfrak{a})$ which properly contains $\mathfrak{a}$ but is contained in $\mathfrak{s}$. From the minimal property of $\mathfrak{s}$ we have therefore $\mathfrak{s} = (s'rx, \mathfrak{a})$ and every element s of $\mathfrak{s}$ has the form

$$s = a + is'rx + r_1 s'rx + s'rxr_2 + \sum_{i,j} r_i s'rxr_j,$$

where $a \in \mathfrak{a}$, i is an integer and r_1, r_2, r_i, r_j are elements of R. Since $s'RxRy \subseteq \mathfrak{a}$ it is clear from the form of the above expression for s that $sRy \subseteq \mathfrak{a}$ and hence $y \in \mathfrak{s}^{-1}\mathfrak{a}$. Thus if x is not in $\mathfrak{s}^{-1}\mathfrak{a}$ and $xRy \subseteq \mathfrak{s}^{-1}\mathfrak{a}$ then $y \in \mathfrak{s}^{-1}\mathfrak{a}$ and hence $\mathfrak{s}^{-1}\mathfrak{a}$ is prime. Since $\mathfrak{s}$ contains an element s not in $\mathfrak{a}$ and since $sR(\mathfrak{s}^{-1}\mathfrak{a}) \subseteq \mathfrak{a}$ it is clear that $\mathfrak{s}^{-1}\mathfrak{a}$ is nrp to $\mathfrak{a}$.

COROLLARY. *If condition* A *holds in* R *then every ideal* $\mathfrak{a} \neq R$ *has a minimal prime divisor which is not right prime to* $\mathfrak{a}$.

For condition A ensures the existence of a minimal ideal containing $\mathfrak{a}$ and hence a prime $\mathfrak{p}$ which is nrp to $\mathfrak{a}$. This prime must contain a minimal prime divisor of $\mathfrak{a}$ which will also be nrp to $\mathfrak{a}$.

THEOREM 9. *If the ascending chain condition holds for two sided ideals in* R *then every ideal* $\mathfrak{c}$ *in* R *has at most a finite number of minimal prime divisors* [2].

Proof. If $\mathfrak{c}$ is a prime ideal the theorem is obvious. If $\mathfrak{c}$ is not prime there exist elements a_1, and b_1 of R which do not belong to $\mathfrak{c}$ such that $a_1Rb_1 \subseteq \mathfrak{c}$. Hence if $\mathfrak{c}$ is contained in an infinite number of minimal primes $\mathfrak{p}_i$ either a_1 or b_1 must belong to an infinite number of these. Suppose it is a_1 and let $\mathfrak{a}_1 = (a_1, \mathfrak{c})$. Then $\mathfrak{a}_1$ is a proper divisor of $\mathfrak{c}$, $\mathfrak{p}_i \supseteq \mathfrak{a}_1$ for an infinite number of primes $\mathfrak{p}_i$, and each of these $\mathfrak{p}_i$ is a minimal prime divisor of $\mathfrak{a}_1$. Hence $\mathfrak{a}_1$ cannot be prime. Therefore if $\mathfrak{c}$ has an infinite number of minimal prime divisors it has a proper divisor with the same property and continuation of this argument leads to a contradiction of the ascending chain condition in R.

THEOREM 10. *If the ascending chain condition holds for two sided ideals in* R *and* $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ *are the minimal prime divisors of an ideal* $\mathfrak{c}$ *then*

$$\mathfrak{p}_{i_1}R\mathfrak{p}_{i_2}R \ldots R\mathfrak{p}_{i_m} \subseteq \mathfrak{c}$$

where $i_1, i_2, \ldots, i_m$ *is some finite permutation of the integers* $1, 2, \ldots, n$ *with repetitions allowed.*

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Proof. The theorem is trivially true if $\mathfrak{c}$ is prime. If $\mathfrak{c}$ is not prime, there exist elements a and b not in $\mathfrak{c}$ such that $aRb \subseteq \mathfrak{c} \subset \mathfrak{p}_i$ $(i = 1, 2, \ldots, n)$. Hence for each i either $a \in \mathfrak{p}_i$ or $b \in \mathfrak{p}_i$. Form the ideals $\mathfrak{a}_1 = (a, \mathfrak{c})$ and $\mathfrak{b}_1 = (b, \mathfrak{c})$ both of which are proper divisors of $\mathfrak{c}$. Let $\mathfrak{p}'_1, \mathfrak{p}'_2, \ldots, \mathfrak{p}'_r$ be the minimal prime divisors of $\mathfrak{a}_1$ and $\mathfrak{p}''_1, \mathfrak{p}''_2, \ldots, \mathfrak{p}''_r$ be those of $\mathfrak{b}_1$. Now suppose that both $\mathfrak{a}_1$ and $\mathfrak{b}_1$ have the property that we wish to prove of $\mathfrak{c}$, so that $\mathfrak{p}'_{j_1}R\mathfrak{p}'_{j_2}R \ldots R\mathfrak{p}'_{j_q} \subseteq \mathfrak{a}_1$ and $\mathfrak{p}''_{k_1}R\mathfrak{p}''_{k_2}R \ldots R\mathfrak{p}''_{k_t} \subseteq \mathfrak{b}_1$ and hence, since $a_1Rb_1 \subseteq \mathfrak{c}$,

$$\mathfrak{p}'_{j_1}R\mathfrak{p}'_{j_2}R \ldots R\mathfrak{p}'_{j_q}R\mathfrak{p}''_{k_1}R \ldots R\mathfrak{p}''_{k_t} \subseteq \mathfrak{c}.$$

Now each $\mathfrak{p}'_j$ and $\mathfrak{p}''_k$, being a prime divisor of $\mathfrak{c}$, contains a minimal prime of $\mathfrak{c}$, and hence $\mathfrak{p}_{i_1}R\mathfrak{p}_{i_2}R \ldots R\mathfrak{p}_{i_m} \subseteq \mathfrak{c}$ where $\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_m}$ are minimal primes of $\mathfrak{c}$. Hence if the theorem is false for $\mathfrak{c}$ it is false for a proper divisor of $\mathfrak{c}$, and a continuation of this argument leads to a contradiction of the ascending chain condition in R.

COROLLARY. *If the ascending chain condition holds for two sided ideals in R then every ideal $\mathfrak{a} \neq R$ has a minimal prime divisor which is not right prime to $\mathfrak{a}$.*

Proof. If $\mathfrak{a}$ is prime but $\neq R$ then $\mathfrak{a}$ itself is the required minimal prime. If $\mathfrak{a}$ is not prime, by Theorem 10 we have

$$\mathfrak{p}_1R\mathfrak{p}_2R \ldots R\mathfrak{p}_s \subseteq \mathfrak{a} \tag{1}$$

where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$ are (not necessarily distinct) minimal primes of $\mathfrak{a}$ and $s > 1$. Hence there exists a shortest product of the form (1) which belongs to $\mathfrak{a}$; that is, there exists an $s > 1$ such that (1) holds but

$$\mathfrak{p}_1R\mathfrak{p}_2R \ldots R\mathfrak{p}_{s-1} \text{ not} \subseteq \mathfrak{a}.$$

It follows that $\mathfrak{p}_s$ is nrp to $\mathfrak{a}$.

6. Primary ideals. In this section we shall require the results of [5] concerning the radical of an ideal. The radical $\mathfrak{r}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ is defined as the set of all elements x of R such that every m-system containing x meets $\mathfrak{a}$. McCoy has shown that $\mathfrak{r}(\mathfrak{a})$ is an ideal and is equal to the intersection of all minimal prime divisors of $\mathfrak{a}$.

DEFINITION 6.1. *An ideal $\mathfrak{q}$ is said to be right primary if all elements not in $\mathfrak{r}(\mathfrak{q})$ are right prime to $\mathfrak{q}$.*

Thus $\mathfrak{q}$ is right primary if the conditions $aRb \subseteq \mathfrak{q}$ and $b \notin \mathfrak{r}(\mathfrak{q})$ together imply $a \in \mathfrak{q}$.

THEOREM 11. *If either Condition* A *or the ascending chain condition holds in R then the radical of a right primary ideal is prime.*

Proof. Suppose $\mathfrak{q}$ is right primary. By the corollaries to Theorems 8 and 10, if $\mathfrak{q} \neq R$ it has a minimal prime divisor $\mathfrak{p}$ which is nrp to $\mathfrak{q}$. Hence for every element p of $\mathfrak{p}$ we have $xRp \subseteq \mathfrak{q}$ for some x not in $\mathfrak{q}$. Since $\mathfrak{q}$ is right primary

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this implies that $p \in \mathfrak{r}(\mathfrak{q})$ and hence $\mathfrak{p} \subseteq \mathfrak{r}(\mathfrak{q})$. But since $\mathfrak{r}(\mathfrak{q})$ is the intersection of the minimal primes of $\mathfrak{q}$ we have $\mathfrak{r}(\mathfrak{q}) \subseteq \mathfrak{p}$ and the theorem follows. If $\mathfrak{q} = R$ then $\mathfrak{r}(\mathfrak{q}) = R$ and the theorem holds in this case too.

In rings which satisfy no finite chain conditions it seems possible that right primary ideals may exist whose radical is not prime. Such an ideal $\mathfrak{q}$, if it exists, must be such that all its minimal prime divisors are right prime to $\mathfrak{q}$ and no product of the form $\mathfrak{p}_1 R \mathfrak{p}_2 R \ldots R \mathfrak{p}_n$, where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ are (not necessarily distinct) minimal prime divisors of $\mathfrak{q}$, can belong to $\mathfrak{q}$. Since, in a commutative ring, every minimal prime divisor of $\mathfrak{q}$ is nrp to $\mathfrak{q}$ [6, p. 112] our definition of a (right) primary ideal implies a prime radical in the commutative case even without chain conditions. In fact, in a commutative ring it reduces to the usual definition of a primary ideal by virtue of [6, p. 182, Theorem 59].

7. Ideals expressible as the intersection of right primary ideals. In this section we shall consider ideals which can be represented as the intersection of a finite number of right primary ideals and shall find what characteristics of such a representation are uniquely determined by the ideal in question. It will be assumed throughout the remainder of the paper that the ascending chain condition holds for the two sided ideals of R. A representation

$$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r \tag{2}$$

of an ideal $\mathfrak{a}$ as the intersection of right primary ideals, $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ will be called an irredundant representation if no one of the ideals $\mathfrak{q}_i$ contains the intersection of the remaining ones.

THEOREM 12. *If* (2) *is an irredundant representation of an ideal* $\mathfrak{a}$ *as the intersection of right primary ideals* $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$, *then an element* x *is right prime to* $\mathfrak{a}$ *if and only if* $x \in C(\mathfrak{p}_i)$ *for* $i = 1, 2, \ldots, r$, *where* $\mathfrak{p}_i$ *is the radical of* $\mathfrak{q}_i$.

Proof. (i) If a is nrp to $\mathfrak{a}$ then for some element x which is not in $\mathfrak{a}$, $xRa \subseteq \mathfrak{a}$. But this implies $xRa \subseteq \mathfrak{q}_i$ for $i = 1, 2, \ldots, r$ while $x \notin \mathfrak{q}_j$ for at least one value of j. Hence, since $\mathfrak{q}_j$ is right primary, $a \in \mathfrak{p}_j$. It follows that if $a \in C(\mathfrak{p}_i)$ for all i then a is right prime to $\mathfrak{a}$.

(ii) Conversely, suppose that a is an element of at least one of the primes $\mathfrak{p}_i$ and let it be $\mathfrak{p}_1$. By Theorem 10 some product of the form

$$aRaR \ldots aRa$$

is contained in $\mathfrak{q}_1$. Since the representation $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r$ is irredundant we can choose an element b which is contained in $\mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \ldots \cap \mathfrak{q}_r$ but not in $\mathfrak{q}_1$. Then

$$bRaR \ldots aRa \subseteq \mathfrak{a}.$$

Suppose the shortest such product which is contained in $\mathfrak{a}$ has s factors a. Then $s \geqslant 1$ since $b \notin \mathfrak{a}$. If $s = 1$ then $bRa \subseteq \mathfrak{a}$ and therefore a is nrp to $\mathfrak{a}$. If $s > 1$ then the product $bRaR \ldots aRa$, with $s - 1$ factors a, contains an element b'

which does not belong to $\mathfrak{a}$, while $b'Ra \subseteq \mathfrak{a}$, and again, a is nrp to $\mathfrak{a}$. This completes the proof.

THEOREM 13. *The intersection of any finite number of right primary ideals all of which have the same radical $\mathfrak{p}$ is a right primary ideal with radical $\mathfrak{p}$.*

Proof. Let $\mathfrak{q}_1,\mathfrak{q}_2, \ldots, \mathfrak{q}_r$ be right primary ideals all having radical $\mathfrak{p}$ and let $\mathfrak{q}$ be their intersection. Since $\mathfrak{p}$ is the only nimimal prime divisor of $\mathfrak{q}_i$ we have by Theorem 10 that $\mathfrak{p}R\mathfrak{p}R \ldots \mathfrak{p}R\mathfrak{p}$ is contained in each $\mathfrak{q}_i$ and hence $\mathfrak{p}R\mathfrak{p}R \ldots \mathfrak{p}R\mathfrak{p} \subseteq \mathfrak{q}$. Therefore, if $\mathfrak{p}_1$ is any prime divisor of $\mathfrak{q}$ we have $\mathfrak{p}_1 \supseteq \mathfrak{p}R\mathfrak{p}R \ldots \mathfrak{p}R\mathfrak{p}$ and hence $\mathfrak{p}_1 \supseteq \mathfrak{p}$ by the definition of a prime ideal. It follows that $\mathfrak{p}$ is a unique minimal prime divisor of $\mathfrak{q}$ and therefore $\mathfrak{p} = \mathfrak{r}(\mathfrak{q})$. Moreover, if $aRb \subseteq \mathfrak{q}$ and $a \notin \mathfrak{q}$ then $aRb \subseteq \mathfrak{q}_i$ for each i while $a \notin \mathfrak{q}_j$ for at least one j. Since $\mathfrak{q}_j$ is right primary with radical $\mathfrak{p}$ this implies that $b \in \mathfrak{p} = \mathfrak{r}(\mathfrak{q})$ whence $\mathfrak{q}$ is right primary with radical $\mathfrak{p}$.

THEOREM 14. *An irredundant intersection of a finite number of right primary ideals not all of which have the same radical is not a right primary ideal.*

Proof. Let $\mathfrak{q}$ be an irredundant intersection of right primary ideals $\mathfrak{q}_1,\mathfrak{q}_2, \ldots, \mathfrak{q}_r$ whose radicals are $\mathfrak{p}_1,\mathfrak{p}_2, \ldots, \mathfrak{p}_r$. If $\mathfrak{q}$ is right primary it has a unique minimal prime divisor $\mathfrak{p}$ and all elements not in $\mathfrak{p}$ are right prime to $\mathfrak{q}$. But by Theorem 12, if x is right prime to $\mathfrak{q}$, x is not contained in any of the primes $\mathfrak{p}_1,\mathfrak{p}_2, \ldots, \mathfrak{p}_r$. Hence $C(\mathfrak{p}) \subseteq C(\mathfrak{p}_i)$ and $\mathfrak{p} \supseteq \mathfrak{p}_i$ for $i = 1, 2, \ldots, r$. Since $\mathfrak{p}$ is a minimal prime divisor of $\mathfrak{q}$ it follows that each $\mathfrak{p}_i$ is equal to $\mathfrak{p}$. Hence if $\mathfrak{p}_1,\mathfrak{p}_2, \ldots, \mathfrak{p}_r$ are not all equal $\mathfrak{q}$ is not right primary.

DEFINITION 7.1. *An irredundant representation* (2) *of* $\mathfrak{a}$ *will be called a short representation if none of the ideals obtained by taking the intersection of two or more of the ideals* $\mathfrak{q}_1,\mathfrak{q}_2, \ldots, \mathfrak{q}_r$ *are right primary.*

In view of Theorems 13 and 14 the irredundant representation (2) is a short representation of $\mathfrak{a}$ if and only if no two of the radicals of $\mathfrak{q}_1,\mathfrak{q}_2, \ldots \mathfrak{q}_r$ are equal.

THEOREM 15. *Let* $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n$ *be an irredundant representation of* $\mathfrak{a}$ *as the intersection of right primary ideals, and let* $\mathfrak{p}_i$ *be the radical of* $\mathfrak{q}_i$. *If* $\mathfrak{p}$ *is a prime ideal not equal to* R *which contains* $\mathfrak{p}_1,\mathfrak{p}_2, \ldots \mathfrak{p}_r$ *but does not contain* $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n$, *then* $\mathfrak{u}(\mathfrak{a},\mathfrak{p}) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r$.

Proof. If $\mathfrak{p} \supseteq \mathfrak{p}_i$ then by Theorem 1, Corollary 2,

$$\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) \subseteq \mathfrak{u}(\mathfrak{a}, \mathfrak{p}_i).$$

But since $\mathfrak{q}_i$ has property (A) relative to the m-system $C(\mathfrak{p}_i)$, Theorem 1 shows that $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}_i) \subseteq \mathfrak{q}_i$. Hence

$$\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) \subseteq \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r. \tag{3}$$

Now if $r = n$, (3) gives $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) \subseteq \mathfrak{a}$ and since $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) \supseteq \mathfrak{a}$ we have $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) = \mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n$ and the result is proved in this case. If $r < n$, since $\mathfrak{p}$

does not contain $\mathfrak{p}_j$ for $j > r$, it follows that $\mathfrak{p}$ does not contain $\mathfrak{q}_j$ either; for since $\mathfrak{p}_j$ is the only minimal prime of $\mathfrak{q}_j$ it is contained in every prime which contains $\mathfrak{q}_j$. Hence there exist elements $m_1, m_2, \ldots m_{n-r}$ such that $m_i \in \mathfrak{q}_{r+i}$ but $m_i \notin \mathfrak{p}$ $(i = 1, 2, \ldots, n - r)$. Now since $m_1, m_2, \ldots, m_{n-r}$ all belong to the m-system $M = C(\mathfrak{p})$, there exist elements $x_1, x_2, \ldots, x_{n-r-1}$ such that the element $m = m_1 x_1 m_2 x_2 m_3 \ldots x_{n-r-1} m_{n-r}$ is contained in M. Also it is clear that $m \in \mathfrak{q}_{r+1} \cap \mathfrak{q}_{r+2} \cap \ldots \cap \mathfrak{q}_n$. Hence if $q \in \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r$ we have $qRm \subseteq \mathfrak{a}$ where $m \in M$ and therefore every right M-n-system which contains q meets $\mathfrak{a}$. Hence $q \in \mathfrak{u}(\mathfrak{a}, \mathfrak{p})$ and

$$\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r \subseteq \mathfrak{u}(\mathfrak{a}, \mathfrak{p}),$$

which with (3) gives the result stated in the theorem.

THEOREM 16. *If* (2) *is an irredundant representation of* $\mathfrak{a}$ *as the intersection of right primary ideals* $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r$ *with radicals* $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$, *then the minimal prime divisors of* $\mathfrak{a}$ *are exactly those primes which are minimal in the set* $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$.

Proof. For each i some product $\mathfrak{p}_i R \mathfrak{p}_i \ldots R \mathfrak{p}_i$ is contained in $\mathfrak{q}_i$. Taking products over $i = 1, 2, \ldots, r$,

$$\mathfrak{p}_{i_1} R \mathfrak{p}_{i_2} \ldots R \mathfrak{p}_{i_m} \subseteq \mathfrak{a}$$

where each $\mathfrak{p}_{i_j}$ is one of the primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$. Hence every prime which contains $\mathfrak{a}$ contains the above product and therefore must contain one of the primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$. Hence every minimal prime containing $\mathfrak{a}$ is a minimal prime of this set and conversely.

THEOREM 17. *If* (2) *is a short representation of* $\mathfrak{a}$ *as the intersection of right primary ideals* $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r$, *and if*[1] $\mathfrak{p} \neq R$ *is any minimal prime divisor of* $\mathfrak{a}$, *then* $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$ *is right primary and equal to one of the* $\mathfrak{q}_i$.

Proof. Since $\mathfrak{p}$ is a minimal prime divisor of $\mathfrak{a}$, by Theorem 16 it is the radical of one of the ideals $\mathfrak{q}_i$, say $\mathfrak{q}_j$. Since $\mathfrak{p}$ is minimal it cannot contain the radical of any ideal $\mathfrak{q}_i$ for $i \neq j$. Hence Theorem 15 gives $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) = \mathfrak{q}_j$.

COROLLARY 1. *If* $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ *(all different from* R*) are the minimal prime divisors of* $\mathfrak{a}$ *then in any short representation of* $\mathfrak{a}$ *as the intersection of a finite number of right primary ideals,* $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}_1), \ldots, \mathfrak{u}(\mathfrak{a}, \mathfrak{p}_m)$ *must occur among the right primary components.*

COROLLARY 2. *A necessary condition that an ideal* $\mathfrak{a}$ *be representable as the intersection of a finite number of right primary ideals is that* $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}_i)$ *be right primary for all minimal prime divisors* $\mathfrak{p}_i$ *of* $\mathfrak{a}$.

DEFINITION 7.2. *If* $\mathfrak{a}$ *is representable as the intersection of right primary ideals then the upper component ideals* $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$ *corresponding to the minimal prime divisors of* $\mathfrak{a}$ *are called the isolated right primary components of* $\mathfrak{a}$.

[1]The restriction $\mathfrak{p} \neq R$ excludes only the case in which $\mathfrak{a}$ is itself primary with radical R.

Thus the isolated right primary components of $\mathfrak{a}$ are right primary ideals which occur as components in every short representation of $\mathfrak{a}$ as the intersection of right primary ideals.

It is now easy to give examples of rings satisfying the ascending chain condition in which not all ideals are expressible as the intersection of a finite number of right primary ideals. Let R be the ring of all polynomials in two noncommutative indeterminates x and y with coefficients in a field K. Let $\mathfrak{a}$ be the ideal (xy) which has two minimal prime divisors $\mathfrak{p}_1 = (x)$ and $\mathfrak{p}_2 = (y)$, and is clearly not right primary. The radical of $\mathfrak{a}$ is $\mathfrak{p}_1 \cap \mathfrak{p}_2$ or (xy, yx). Now if $aRb \subseteq (xy)$ $b \notin (y)$ then $a \in (xy)$. Hence (xy) has property (A) relative to the m-system $C(\mathfrak{p}_2)$ and therefore $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}_2) = \mathfrak{a}$. Since $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}_2)$ is not right primary Theorem 17, Corollary 2, shows that $\mathfrak{a}$ is not the intersection of a finite number of right primary ideals.

Fitting's decomposition theorem [1] represents $\mathfrak{a}$ as the intersection of two "primary left ideals", namely,

$$(xy) = (x) \cap (y)_l$$

where (x) is the two sided ideal generated by x and $(y)_l$ is the left ideal generated by y. In the present paper, however, we consider only representations as intersections of two sided right primary ideals.

It can also be shown by examples that the necessary condition given in Theorem 17, Corollary 2, is not sufficient. Let R be the same ring as above and let $\mathfrak{a} = (x^2, xy)$. Then $\mathfrak{a}$ has a unique minimal prime divisor $\mathfrak{p} = (x)$ and $\mathfrak{r}(\mathfrak{a}) = (x)$. But $\mathfrak{a}$ is not right primary since $xRy \subseteq \mathfrak{a}$ while $x \notin \mathfrak{a}$ and $y \notin \mathfrak{r}(\mathfrak{a})$. Now $\mathfrak{l}(\mathfrak{a}, \mathfrak{p})$, the set of all elements r such that $rRm \subseteq \mathfrak{a}$ for some m in $C(\mathfrak{p})$, is easily seen to be equal to (x) and therefore $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) \supseteq (x)$. But (x) has property (A) relative to $C(\mathfrak{p})$ and therefore $\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) = (x)$. Since $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$ is right primary the necessary condition of Corollary 2 is satisfied. By Theorem 17, in any short representation of $\mathfrak{a}$ as the intersection of a finite number of right primary ideals, (x) must occur as one component. The other components must be sought among the other right primary divisors of $\mathfrak{a}$, namely, (x,y) (x^2,y), (x,y^n), (x^2,xy,y^n) and (x^2,xy,yx,y^n), $n \geqslant 2$. It is easy to verify that none of the possible finite intersections is equal to $\mathfrak{a}$.

We may note also that although $\mathfrak{a}$ is not right primary it is left primary since $aRb \subseteq (x^2,xy)$ and $a \notin (x)$ together imply $b \in (x^2,xy)$.

THEOREM 18. *If* $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r$ *is a short representation of* $\mathfrak{a}$ *as the intersection of right primary ideals* $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r$ *then a prime ideal* $\mathfrak{p} \neq R$ *which divides* $\mathfrak{a}$ *is the radical of one of the ideals* $\mathfrak{q}_i$ *if and only if* $\mathfrak{p}$ *is* nrp *to* $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$. *The ring* R *is the radical of one of the* $\mathfrak{q}_i$ *if and only if* R *is* nrp *to* $\mathfrak{a}$.

Proof. (i) Let the radicals of $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r$ be $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$. If $\mathfrak{p} = \mathfrak{p}_i$ but $\mathfrak{p} \neq R$, then by Theorem 15,

(4) $$\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) = \mathfrak{q}_{1'} \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_k$$

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where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ are those primes among the $\mathfrak{p}_i$ which are contained in $\mathfrak{p}$. Now (4) is a short representation of $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$ and $\mathfrak{p}$ is the radical of one of the ideals $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_k$ and contains the radicals of the rest of these. Hence by Theorem 12, an element x is nrp to $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$ if and only if $x \in \mathfrak{p}$. Hence if $\mathfrak{p} = \mathfrak{p}_i$ then $\mathfrak{p}$ is nrp to $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$.

(ii) Now suppose $\mathfrak{p} \supseteq \mathfrak{a}$, $\mathfrak{p} \neq R$, and $\mathfrak{p}$ is nrp to $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$. Since, by Theorem 16, all minimal prime divisors of $\mathfrak{a}$ are among the primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$, $\mathfrak{p}$ must contain at least one of these. Suppose $\mathfrak{p}$ contains $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ but not $\mathfrak{p}_{k+1}, \ldots, \mathfrak{p}_r$. By Theorem 15,

$$\mathfrak{u}(\mathfrak{a}, \mathfrak{p}) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_k$$

is a short representation of $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$, and since $\mathfrak{p}$ is nrp to $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$ Theorem 12 gives

$$\mathfrak{p} \subseteq \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \ldots \oplus \mathfrak{p}_k,$$

where $\oplus$ denotes a set-theoretic sum. But since $\mathfrak{p} \supseteq \mathfrak{p}_i$ $(i = 1, 2, \ldots, k)$ it follows that

$$(5) \qquad \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \ldots \oplus \mathfrak{p}_k.$$

In the sum (5) any prime $\mathfrak{p}_i$ which is contained in the sum of the remaining primes may be omitted. We may assume therefore that

$$(6) \qquad \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \ldots \oplus \mathfrak{p}_l,$$

where $l \leqslant k$ and no one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ is contained in the set-theoretic sum of the remaining ones.

Now if $l > 1$ the product $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{l-1}$ cannot be contained in $\mathfrak{p}_l$, for if it were, since $\mathfrak{p}_l$ is prime, $\mathfrak{p}_l$ would contain one of the primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_{l-1}$, contrary to the assumption of the minimal length of the sum (6). Hence we can choose elements p_i from $\mathfrak{p}_i$ $(i = 1, 2, \ldots, l-1)$ such that $p_1 p_2 \ldots p_{l-1}$ does not belong to $\mathfrak{p}_l$. Moreover, we can choose an element p_l of $\mathfrak{p}_l$ which does not belong to $\mathfrak{p}_i$ for $i < l$. Form the element $x = p_1 p_2 \ldots p_{l-1} + p_l$. Being the sum of two elements of $\mathfrak{p}$, $x \in \mathfrak{p}$ and therefore $x \in \mathfrak{p}_j$ for some value of j such that $1 \leqslant j \leqslant l$. But this is impossible, for if $j < l$ then $p_1 p_2 \ldots p_{l-1} \in \mathfrak{p}_j$ but $p_l \notin \mathfrak{p}_j$, while if $j = l$, $p_l \in \mathfrak{p}_l$ but $p_1 p_2 \ldots p_{l-1} \notin \mathfrak{p}_l$. This contradiction leads to the conclusion that $l = 1$ and hence $\mathfrak{p} = \mathfrak{p}_i$ for some value of i.

(iii) Suppose R is the radical of one of the $\mathfrak{q}_i$, and let it be $\mathfrak{q}_1$. Since R is therefore the only minimal prime divisor of $\mathfrak{q}_1$, Theorem 10 gives $R^s \subseteq \mathfrak{q}_1$. Choose an element q which is contained in $\mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \ldots \cap \mathfrak{q}_r$ but not in $\mathfrak{q}_1$ so that $q \notin \mathfrak{a}$. Then $qR^s \subseteq \mathfrak{a}$. Assume s is the least exponent for which this holds, so that $s \geqslant 1$, and choose an element q' in qR^{s-1} such that q' is not contained in $\mathfrak{a}$. Then $q'Rr \subseteq \mathfrak{a}$ for all elements r of R and therefore R is nrp to $\mathfrak{a}$.

(iv) Conversely, suppose R is nrp to $\mathfrak{a}$ so that for every element r of R there is an element a_r not in $\mathfrak{a}$ such that $a_r Rr \subseteq \mathfrak{a}$. Hence for each i, $a_r Rr \subseteq \mathfrak{q}_i$ while for at least one j, $a_r \notin \mathfrak{q}_j$. Thus, since $\mathfrak{q}_j$ is right primary, $r \in \mathfrak{p}_j$ and

$$R = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \ldots \oplus \mathfrak{p}_r.$$

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Now let

$$R = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \ldots \oplus \mathfrak{p}_l \tag{7}$$

be the sum of minimal length which is equal to R. If $l > 1$ choose $p_1, p_2, \ldots, p_l$ as above and we find the element $p_1 p_2 \ldots p_{l-1} + p_l$ belongs to none of the primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_l$, in contradiction to (7). Hence $l = 1$ and $\mathfrak{p}_i = R$ for one value of i. This completes the proof of Theorem 18.

If an ideal $\mathfrak{a}$ can be represented as the intersection of right primary ideals $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r$, Theorem 18 shows that the radicals of these right primary components are uniquely determined since the criterion given to determine whether $\mathfrak{p}$ is one of these radicals or not depends only on $\mathfrak{p}$ and $\mathfrak{a}$. Similarly the number of right primary components in a short representation is also uniquely determined as the number of distinct primes among the radicals of $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r$. We may therefore summarize the results of this section as follows:

THEOREM 19. *Let R be a noncommutative ring in which the ascending chain condition holds for two sided ideals. If an ideal $\mathfrak{a}$ in R can be represented as the intersection of a finite number of right primary ideals then $\mathfrak{a}$ has a short representation as such. In any two short representations of $\mathfrak{a}$ the number of right primary components is the same and the radicals of the two sets of primary components coincide in some order. Moreover, the isolated primary components are the same for all short representations.*

Although Theorem 19 shows that the well-known results of E. Noether carry over to the noncommutative case for those ideals which can be represented as the intersection of a finite number of right primary ideals, a necessary and sufficient condition that such a representation exist is still unknown. The ascending chain condition is not sufficient to ensure this for all ideals as it is in a commutative ring. The necessary condition given by Theorem 17, Corollary 2, is not only not sufficient but is difficult to apply in a particular case owing to the difficulty of finding the ideals $\mathfrak{u}(\mathfrak{a}, \mathfrak{p})$. It is hoped to return to this problem in a later paper.

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