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AN ELEMENTARY APPROACH TO THE MATRICIAL NEVANLINNA–PICK INTERPOLATION CRITERION

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The matricial Nevanlinna–Pick interpolation criterion determines when there is an analytic matrix contraction valued function on the complex unit disc which assumes preassigned $n\times n$ matrix values $w_1,\dots,w_m$ at preassigned interpolation points $z_1,\dots,z_m$. Taking $\|w_i\|<1$, for $i=1,\dots,m$, the necessary and sufficient condition is the positivity of the $nm\times nm$ matricial Pick matrix,

$$C=\left(\frac{I-w_iw_j^*}{1-z_i\bar{z}_j}\right)_{i,j=1}^m.$$

Much has been written about such interpolation problems and the nature of the interpolating functions. Matricial Pick-type conditions appear to originate with Sz-Nagy and A. Koranyi [**12**]. The matricial criterion was obtained in the form above by Fedcina [**5**], using ideas of Adamjan, Arov and Krein [1]. Delsarte, Genin and Kamp [4] give a comprehensive account and a proof through the connection with classical moment problems and the Riesz Herglotz theorem. Modern operator theoretic approaches are based on the ideas of the commutant lifting theorem [7, **8**, **9**], which go back to Sarason's paper on H^∞ interpolation [**10**], or on the elegant theory of spaces with an indefinite inner product [**2**, **3**, **11**]. For related matter and further literature see [**13**, **14**].

In this note we give an elementary constructive approach that is an adaptation of Marshall's elementary proof for the scalar case [**6**]. The method relies on basic symplectic formulae for Potapov factors and is immediately applicable to functions taking values in a von Neumann algebra (cf. Parrott [7]) and other mild generalizations. The method may also be helpful in the more intriguing study of symplectic interpolation for functions defined on the matricial disc.

We first give a positivity criterion for 2×2 block matrices which will allow us ultimately to relate the Pick matrix C for the data $\{z_1,\dots,z_m,w_1,\dots,w_m\}$ of order m, to the Pick matrix of some transformed data of order $m-1$. We then present some fundamental formulae for Potapov and symplectic factors which will allow us to transform data of order m to data of order $m-1$. To make the paper entirely complete, proofs of these lemmas are given at the end of the paper.

Lemma 1. *Let*

$$C=\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

be an invertible self-adjoint matrix with $k \times k$ matrix entries. Then C is positive if and only if a is invertible and $c \geqq b^* a^{-1} b$.

Write $\mathbb{D}$ for the open unit disc in $\mathbb{C}$ and Δ for the open unit ball of $n \times n$ complex matrices, centred at the origin. Usually we denote matrices by x, y, w and complex scalars by ξ, z, λ. The following terminology is convenient.

Definition 2 (i). For $x \in \Delta$ the rational matrix function $b_x(\xi)$, $\xi \in \mathbb{D}$, given by

$$b_x(\xi)=(1-xx^*)^{-1/2}(x-\xi)(1-x^*\xi)^{-1}(1-x^*x)^{1/2}$$

is called a *Potapov factor.*

(ii) The matrix valued function $b_x(y)$, defined for $y \in \Delta$ by the same formula, is called a *symplectic factor.*

If x is a normal matrix then we obtain the simplification $b_x(\xi)=(x-\xi)(1-x^*\xi)^{-1}$, a function analogous to the usual scalar Blaschke factor.

Lemma 3 *The symplectic factor $b_x(y)$ satisfies the following for all x, y, y_1, y_2 in Δ.*

(i) $b_x(y)=(1-xx^*)^{1/2}(1-yx^*)^{-1}(x-y)(1-x^*x)^{-1/2}$

(ii) $b_x(y)=x-(1-xx^*)^{1/2}y(1-x^*y)^{-1}(1-x^*x)^{1/2}$

(iii) $b_x(y) \in \Delta$

(iv) $b_x(0)=x, b_x(x)=0$

(v) $1-b_x(y_1)b_x(y_2)^*=(1-xx^*)^{1/2}(1-y_1x^*)^{-1}(1-y_1y_2^*)(1-xy_2^*)^{-1}(1-xx^*)^{1/2}$

The key equation (v), which is the symplectic version of a familiar fact about Blaschke factors, shows that $1-b_x(y)b_x(y)^*$ is positive and invertible for y in Δ, and so (iii) follows. On the other hand if y is a unitary matrix then $1-b_x(y)b_x(y)^*=0$, and so, when extended to the unit sphere, $b_x(y)$ takes unitary values on unitary matrices. It follows that a Potapov factor is a rational analytic function on $\mathbb{D}$ taking unitary values on $|\xi|=1$. That is, $b_x(\xi)$ is a rational matrix valued inner function.

Theorem. *Let $z_1, \ldots, z_m$ be complex numbers in $\mathbb{D}$, let $w_1, \ldots, w_m$ be matrices in Δ, and* *let the Pick matrix* $((1-z_i\bar{z}_j)^{-1}(1-w_iw_j^*))$, $1\leqq i,j\leqq m$, *be positive definite. Then there exists a rational inner function* $p(z)$ *such that* $p(z_i)=w_i$, $1\leqq i\leqq m$.

Proof. If $m=1$ the theorem is true, so assume the theorem has been established for $m-1$ interpolation points, and that the data $\{z_1,\dots,z_m,w_1,\dots,w_m\}$ has a positive definite Pick matrix M. Let $b_m(z)=(z_m-z)/(1-\bar{z}_mz)$ be the unnormalized Blaschke factor, and let $B_m(x)$ be the symplectic factor $b_{w_m}(x)$, $x\in\Delta$. Consider the new interpolation data of order $m-1$, $\{\beta_1,\dots,\beta_{m-1},\beta_1^{-1}B_m(w_1),\dots,\ \beta_{m-1}^{-1}B_m(w_{m-1})\}$, where $\beta_i=b_m(z_i)$. Suppose that we can establish the positivity of the Pick matrix N for this data. By the induction hypothesis we obtain a rational inner function function, $p_{m-1}(z)$ say, that interpolates;

$$p_{m-1}(\beta_i)=\beta_i^{-1}B_m(w_i),\quad 1\leqq i\leqq m-1.$$

We can now construct an interpolant for the original data. Let B_m^{-1} be the symplectic factor such that $B_m^{-1}(B_m(x))=x, x\in\Delta$, (in fact $B_m^{-1}=B_m$) and consider the matrix function

$$p_m(z)=B_m^{-1}(b_m(z)p_{m-1}(b_m(z))),\quad z\in\mathbb{D},$$

which is simply the composition $B_m^{-1}\circ zp_{m-1}\circ b_m$. Then for $1\leqq i\leqq m-1$, $p_m(z_i)=B_m^{-1}(\beta_i(\beta_i^{-1}B_m(w_i)))=w_i$, and $p_m(z_m)=B_m^{-1}(0)=w_m$. In view of our remarks earlier, $p_m(z)$ takes unitary values on the unit circle, and provides the required interpolating function.

We complete the proof by using Lemma 1 to show that if M is positive definite then so too is N.

Partition the positive Pick matrix M into the 2×2 matrix

$$M=\begin{bmatrix} M_1 & b^* \\ b & a\end{bmatrix}$$

relative to the orthogonal decomposition $(\mathbb{C}^{m-1}\otimes\mathbb{C}^n)\oplus\mathbb{C}^n$ of $\mathbb{C}^{mn}=\mathbb{C}^m\otimes\mathbb{C}^n$. By Lemma 1 $\tilde{M}=M_1-b^*a^{-1}b$ is positive definite. For $i,j=1,\dots,m-1$

$$\tilde{M}_{ij}=\frac{1-w_iw_j^*}{1-z_i\bar{z}_j}-\left(\frac{1-w_iw_m^*}{1-z_i\bar{z}_m}\right)(1-|z_n|^2)(1-w_mw_m^*)^{-1}\left(\frac{1-w_mw_j^*}{1-z_m\bar{z}_j}\right)$$

$$=\frac{(1-w_iw_m^*)(1-w_mw_m^*)^{-1/2}(1-|z_m|^2)^{1/2}}{(1-z_i\bar{z}_m)}$$

$$\times\left[\frac{(1-w_mw_m^*)^{1/2}(1-w_iw_m^*)^{-1}(1-w_iw_j^*)(1-w_mw_j^*)^{-1}(1-w_mw_m^*)^{1/2}}{(1-z_i\bar{z}_m)^{-1}(1-|z_m|^2)(1-z_i\bar{z}_j)(1-z_m\bar{z}_j)^{-1}}-1\right]$$

$$\times\frac{(1-|z_m|^2)^{1/2}(1-w_mw_m^*)^{-1/2}(1-w_mw_j^*)}{1-z_m\bar{z}_j}$$

We have thus rewritten $\tilde{M}$ in the form $\tilde{M}=DM_2D^*$ where M_2 is the matrix defined by the expression in the square brackets. Since the diagonal matrix D is invertible, M_2 is positive definite. But for the Pick matrix $N=(N_{ij})$, $1\leqq i, j\leqq m-1$, we have

$$N_{ij}=\frac{1-\beta_i^{-1}\bar{\beta}_j^{-1}B_m(w_i)B_m(w_j)^*}{1-\beta_i\bar{\beta}_j}$$

$$=\frac{1}{\beta_i}\left(\frac{1-B_m(w_i)B_m(w_j)^*}{1-\beta_i\bar{\beta}_j}-1\right)\frac{1}{\bar{\beta}_j}$$

and so, by Lemma 3 (v), $N=D_2M_2D_2^*$ where D_2 is the diagonal matrix determined by $\beta_1^{-1},\dots,\beta_{m-1}^{-1}$, and hence N is positive definite. □

Remark 1. The proof remains valid for operator valued rational function interpolation, and it is clear that the interpolating function takes values in the *-algebra generated by $w_1,\dots,w_m$.

Remark 2. It is tempting to apply the arguments above in other contexts where the Pick matrix makes sense. For example, let $Z_1,\dots,Z_m$ be matrices in Δ, and $w_1,\dots,w_m$ values in $\mathbb{D}$. When does there exist a symplectic Blaschke product P such that $P(Z_i)=w_iI$? The proof above breaks down since the revised data of order $m-1$ does not have the same character.

Proof of Lemma 1. Consider the matrices

$$A=\begin{bmatrix} a^{1/2} & a^{-1/2}b \\ 0 & I \end{bmatrix},\quad D=\begin{bmatrix} I & 0 \\ 0 & c-b^*a^{-1}b \end{bmatrix}$$

and verify that $C=A^*DA$. □

Proof of Lemma 2. The key to (i) and (ii) is the identity $x(1-x^*x)^{1/2}=(1-xx^*)^{1/2}x$ which holds for all $x\in\Delta$. We have

$$-x+(1-xx^*)^{1/2}y(1-x^*y)^{-1}(1-x^*x)^{1/2}$$

$$=(1-xx^*)^{-1/2}[-(1-x^*x)^{1/2}x+(1-xx^*)y(1-x^*y)^{-1}(1-x^*x)^{1/2}]$$

$$=(1-xx^*)^{-1/2}[-x(1-x^*x)^{1/2}+(1-xx^*)y(1-x^*y)^{-1}(1-x^*x)^{1/2}]$$

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$$=(1-xx^*)^{-1/2}[-x(1-x^*y)+(1-xx^*)y](1-x^*y)^{-1}(1-x^*x)^{1/2}]$$
$$=-b_x(y),$$

and so (ii) holds. Similarly, we could arrange a common factor $(1-x^*x)^{-1/2}$ to be taken out to the right to obtain the right hand expression in (i), and so (i) holds.

The crucial and important equality in (v) is obtained now with some calculations that are the symplectic versions of basic formulae for Blaschke factors. We have

$$\begin{aligned}
&1-b_x(y_1)b_x(y_2)^*\\
&\quad=1-(1-xx^*)^{1/2}(1-y_1x^*)^{-1}(x-y_1)(1-x^*x)^{-1}(x-y_2)^*(1-xy_2^*)^{-1}(1-xx^*)^{1/2}\\
&\quad=(1-xx^*)^{1/2}(1-y_1x^*)^{-1}\{(1-y_1x^*)(1-xx^*)^{-1}(1-xy_2^*)\\
&\qquad-(x-y_1)(1-x^*x)^{-1}(x-y_2)^*\}\\
&\qquad\times(1-xy_2^*)^{-1}(1-xx^*)^{1/2}.
\end{aligned}$$

Moreover the centrally bracketed term simplifies by making use of the identities $x^*(1-xx^*)^{-1}=(1-x^*x)^{-1}x^*$, $x(1-x^*x)^{-1}=(1-xx^*)^{-1}x$;

$$\begin{aligned}
&\{(1-y_1x^*)(1-xx^*)^{-1}(1-xy_2^*)-(x-y_1)(1-x^*x)^{-1}(x-y_2^*)\}\\
&\quad=(1-xx^*)^{-1}(1-xy_2^*)-y_1x^*(1-xx^*)^{-1}(1-xy_2^*)-x(1-x^*x)(x-y_2)^*\\
&\qquad+y_1(1-x^*x)^{-1}(x-y_2)^*\\
&\quad=(1-xx^*)^{-1}(1-xy_2^*)-y_1(1-x^*x)^{-1}x^*(1-xy_2^*)-(1-xx^*)^{-1}x(x^*-y_2^*)\\
&\qquad+y_1(1-x^*x)^{-1}(x^*-y_2^*)\\
&\quad=(1-xx^*)^{-1}(1-xx^*)-y_1(1-x^*x)^{-1}(1-x^*x)y_2^*\\
&\quad=1-y_1y_2^*.
\end{aligned}$$

We have now established (v). In particular when $y_1=y_2=y\in\Delta$ we see that the operator $1-b_x(y)b_x(y)^*$ is positive andf invertible. This means that $b_x(y)\in\Delta$, and so (v) holds and the proof is complete. □

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