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MINIMAL AND MAXIMAL OPERATOR THEORY WITH APPLICATIONS

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ABSTRACT. Let X be a complex Banach space and A a linear operator from X into X with dense domain. We construct the minimal and maximal operators of the operator A and prove that they are equal under reasonable hypotheses on the space X and operator A. As an application, we obtain the existence and regularity of weak solutions of linear equations on the space X. As another application we obtain a criterion for a symmetric operator on a complex Hilbert space to be essentially self-adjoint. An application to pseudo-differential operators of the Weyl type is given.

1. **Introduction.** Let X be a complex Banach space and $\mathcal{S}$ a dense subspace of X. Let A be a linear operator from X into X with domain $\mathcal{S}$. In applications, we are usually interested in the following question:

QUESTION 1.1. Does the operator A have a closed extension B with the property that the space $\mathcal{S}$ is contained in the domain of the adjoint of B?

If the answer to Question 1.1 is yes, then we can proceed to ask another question.

QUESTION 1.2. Is the extension unique?

In this paper, we show that under suitable conditions on the space X and operator A, the answers to Questions 1.1 and 1.2 are yes. This is achieved by constructing the minimal and maximal extensions of the operator A and then proving that the extensions are equal.

In Section 2, we briefly recall the well-known notions of adjoints and minimal operators. The minimal operator is the smallest closed extension of the operator A. These are explained clearly in the book [5] by Schechter. We include these topics in order to fix notation and make the paper more self-contained. In Section 3, we introduce the notions of formal adjoints and maximal operators. We prove in Theorem 3.8 that the maximal operator of A is the largest closed extension of A satisfying the requirements of Question 1.1. Hence, by proving in Section 4 that the minimal and maximal operators of A are equal, we can conclude that the answers to Questions 1.1 and 1.2 are yes under reasonable hypotheses on the space X and linear operator A. The detailed hypotheses are given in Section 4. In Section 5, we introduce the notion of weak solutions of linear equations on Banach spaces. Then we use the equality of minimal and maximal operators to obtain the existence and regularity of weak solutions of linear equations on Banach spaces.

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For a brief discussion of the meaning of regularity, see Remark 5.4. In Section 6, we give a criterion for a symmetric operator on a complex Hilbert space to be essentially self-adjoint. This is also based on the equality of minimal and maximal operators proved in Section 4. As an application to pseudo-differential operators, we prove that an elliptic pseudo-differential operator of the Weyl type with real-valued symbol is essentially self-adjoint.

Finally, it should be mentioned that the theory of minimal and maximal partial and pseudo-differential operators have been studied by many authors. See, for instance, Hörmander [2], Schechter [8], Wong [9], Wong [10] and Wong [11].

2. **Adjoints and minimal operators.** Let X, $\mathcal{S}$ and A be as in Section 1. Of fundamental importance in linear operator theory is the notion of the adjoint of a linear operator. We begin with a brief recall of the definition and some properties of the adjoint A^t of the linear operator A.

Let X' be the Banach space of all bounded linear functionals on X. We usually call X' the dual space of X. The value of a functional x' in X' at an element x in X is denoted by (x', x). We define a linear operator from X' into X' as follows: we let $\mathcal{D}(A^t)$ be the set of all functionals y' in X' for which there is a functional x' in X' such that

$$(y', Ax) = (x', x), \quad x \in \mathcal{S}. \tag{2.1}$$

For any y' in X', we can prove that there is at most one x' in X' for which (2.1) holds. Hence we can define $A^t y'$ to be equal to x' for all y' in $\mathcal{D}(A^t)$. We call A^t the *adjoint* of A. It can be proved easily that A^t is a closed linear operator from X' into X' with domain $\mathcal{D}(A^t)$. Finally, we observe that if B is a linear extension of A, then A^t is a linear extension of B^t.

We say that A is *closable* if and only if

$$\varphi_k \in \mathcal{S},\ \varphi_k \longrightarrow 0 \text{ in } X,\ A\varphi_k \longrightarrow x \text{ in } X \Rightarrow x = 0.$$

If A is a closable operator, then we can construct a closed linear extension A_0 of A as follows: we let $\mathcal{D}(A_0)$ be the set of all x in X for which there exists a sequence $\{\varphi_k\}$ of elements in $\mathcal{S}$ such that $\varphi_k \longrightarrow x$ in X and $A\varphi_k \longrightarrow y$ in X for some y in X as $k \longrightarrow \infty$. For any x in $\mathcal{D}(A_0)$, we define $A_0 x$ to be equal to y. It can be proved that the definition of A_0 does not depend on the particular choice of the sequence $\{\varphi_k\}$. It can also be proved that A_0 is the smallest closed linear extension of A. This means that if B is any closed linear extension of A, then B is also a linear extension of A_0. For this reason, we call A_0 the minimal operator of A.

To determine when A is closable, we use a criterion based on the adjoint A^t of A.

PROPOSITION 2.1. *The operator A is closable if and only if the domain $\mathcal{D}(A^t)$ of A^t is total in X'.*

Let us recall that a subset T of X' is total in X' if and only if the zero vector is the only element x in X with the property that $(x', x) = 0$ for all x' in T. Proposition 2.1 is a well-known result in functional analysis and hence its proof is omitted. See Problem 5 in Chapter 12 of the book [5] by Schechter.

REMARK 2.2. Let us recall that if X is a reflexive complex Banach space, then a subset of X' is total in X' if and only if it is dense in X'. See Lemma 4.2 in Chapter 7 of Schechter [5].

From now on, we assume that X is a reflexive complex Banach space.

3. **Formal adjoints and maximal operators.** The aim of this section is to define another closed linear extension of the operator A and study some of its properties. To do this, we first introduce the notion of the formal adjoint of the operator A. This requires more restrictions on the space X and the operator A. To wit, we make the following assumptions:

(i) The space X and its dual space X' can be continuously embedded in some topological space Y. Henceforth, the spaces X and X' will be identified as subspaces of Y.

(ii) There exists a subspace $\mathcal{S}$ of Y such that $\mathcal{S}$ is a dense subspace of X and X'.

Let A be a linear operator from X into X with domain $\mathcal{S}$.

DEFINITION 3.1. The formal adjoint A^* of the operator A, if it exists, is defined to be the restriction of the true adjoint A^t to the space $\mathcal{S}$.

REMARK 3.2. It is clear from Definition 3.1 that the formal adjoint A^* exists if and only if $\mathcal{S}$ is contained in the domain of A^t.

We are now ready to define another closed linear extension of the operator A.

DEFINITION 3.3. We define the linear operator A_1 from X into X by $A_1 = (A^*)^t$.

PROPOSITION 3.4. *A_1 is a closed linear operator from X into X with domain $\mathcal{D}(A_1)$ containing the space $\mathcal{S}$.*

PROOF. It is clear that A_1 is a closed linear operator. Let $\psi \in \mathcal{S}$. Then, by Definition 3.1,

$$(A^*\varphi,\psi) = (\varphi, A\psi), \quad \varphi,\psi \in \mathcal{S}.$$

So, by Definition 3.3, $\psi \in \mathcal{D}(A_1)$ and $A_1\psi = A\psi$.

PROPOSITION 3.5. *The domain $\mathcal{D}(A_1^t)$ of the adjoint of A_1 contains the space $\mathcal{S}$.*

PROOF. By Proposition 3.4, A_1 is a closed linear operator from X into X with domain $\mathcal{D}(A_1)$ containing the space $\mathcal{S}$. It follows that A_1^t is a closed linear operator from X' into X'. Let $\psi \in \mathcal{S}$. Then, for all x in $\mathcal{D}(A_1)$, we have, by Definition 3.3,

$$(\psi, A_1x) = (A^*\psi, x).$$

Hence, by the definition of the adjoint, $\psi \in \mathcal{D}(A_1^t)$ and $A_1^t\psi = A^*\psi$.

PROPOSITION 3.6. *A_1 is a linear extension of A_0.*

PROOF. From the proof of Proposition 3.4, we know that A_1 is a closed linear extension of A. Hence the conclusion of Proposition 3.6 follows immediately.

REMARK 3.7. As a corollary of Proposition 3.6, we see that $(A_0)^t$ is a linear extension of $(A_1)^t$. Since, by Proposition 3.5, the domain of $(A_1)^t$ contains the space $\mathcal{S}$, it follows that the domain of $(A_0)^t$ contains the space $\mathcal{S}$ as well.

We can now give the main result of this section.

THEOREM 3.8. *A_1 is the largest closed linear extension of A with the property that the space $\mathcal{S}$ is contained in the domain of its adjoint. In other words, if B is any closed linear extension of A such that $\mathcal{S} \subseteq \mathcal{D}(B^t)$, then A_1 is a linear extension of B.*

PROOF. Let $x \in \mathcal{D}(B)$. Then, for all ψ in $\mathcal{S}$, we have $\psi \in \mathcal{D}(B^t)$ by hypothesis. Hence, by the definition of B^t,

$$(\psi, Bx) = (B^t\psi, x). \tag{3.1}$$

Since B is a linear extension of A, it follows that A^t is a linear extension of B^t. Hence, by (3.1) and the definition of A^*,

$$(\psi, Bx) = (A^t\psi, x) = (A^*\psi, x).$$

So, by the definition of A_1, $x \in \mathcal{D}(A_1)$ and $A_1x = Bx$.

REMARK 3.9. Because of Theorem 3.8, we call A_1 the maximal operator of A.

4. **The equality of minimal and maximal operators.** In this section, we impose conditions on the space X and the operator A to ensure that the minimal operator A_0 and maximal operator A_1 of A are equal.

Let X be a reflexive complex Banach space. Let $\mathcal{S}$ be a dense subspace of X. We assume that $\mathcal{S}$ is a topological vector space of which the topology is defined by a countable family of semi-norms $\{ |\ \ |_j : j = 1, 2, \ldots \}$. A sequence $\{ \varphi_k \}$ in $\mathcal{S}$ is said to converge to an element φ in $\mathcal{S}$ if and only if $|\varphi_k - \varphi|_j \longrightarrow 0$ as $k \longrightarrow \infty$ for all $j = 1, 2, \ldots$. We let $\mathcal{S}'$ be the space of all continuous linear functionals on the space $\mathcal{S}$. We denote the value of a functional u in $\mathcal{S}'$ at an element φ in $\mathcal{S}$ by (u, φ). We say that a functional u is continuous if and only if $(u, \varphi_k) \longrightarrow 0$ as $k \longrightarrow \infty$ for all sequences $\{ \varphi_k \}$ converging to zero in $\mathcal{S}$ as $k \longrightarrow \infty$. A sequence $\{ u_k \}$ in $\mathcal{S}'$ is said to converge to an element u in $\mathcal{S}'$ if and only if $(u_k, \varphi) \longrightarrow (u, \varphi)$ as $k \longrightarrow \infty$ for all φ in $\mathcal{S}$.

We assume that the spaces X and X' are continuously embedded in $\mathcal{S}'$. Furthermore, let us suppose that there exist a one-parameter family of reflexive complex Banach spaces X_s with norms denoted by $\|\ \ \|_s$, $-\infty < s < \infty$, and a one-parameter group of continuous linear mappings $J_s : \mathcal{S}' \longrightarrow \mathcal{S}'$, $-\infty < s < \infty$, satisfying the following conditions:

(i) J_s maps $\mathcal{S}$ into $\mathcal{S}$, $-\infty < s < \infty$.

(ii) $X_s = \{ u \in \mathcal{S}' : J_{-s}u \in X \}$, $-\infty < s < \infty$.

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(iii)

$$\|u\|_s = \|J_{-s}u\|,\ u \in X_s,\ -\infty < s < \infty. \tag{4.1}$$

(iv) Let $s \leq t$. Then $X_t \subseteq X_s$ and

$$\|u\|_s \leq \|u\|_t, \quad u \in X_t. \tag{4.2}$$

(v) X_s can be continuously embedded in $\mathcal{S}'$, $-\infty < s < \infty$.

(vi) $\mathcal{S}$ can be continuously embedded in X'_s and X'_s can be continuously embedded in $\mathcal{S}'$, $-\infty < s < \infty$.

(vii)

$$(u, \varphi) = \overline{(\varphi, u)},\ u \in X_s,\ \varphi \in \mathcal{S}. \tag{4.3}$$

PROPOSITION 4.1. *Let $s, t \in (-\infty, \infty)$. Then*

(i) $J_t: X_s \longrightarrow X_{s+t}$ is a unitary operator.

(ii) $\mathcal{S}$ is dense in X_s.

PROOF. (i) Let $u \in X_s$. Then, by (4.1),

$$\|J_t u\|_{s+t} = \|J_{-s}u\| = \|u\|_s.$$

This proves that $J_t: X_s \longrightarrow X_{s+t}$ is an isometry. It remains to prove that $J_t: X_s \longrightarrow X_{s+t}$ is onto. To do this, let $y \in X_{s+t}$. Then $J_{-t}y \in X_s$ and $J_t(J_{-t}y) = y$. This proves that $J_t: X_s \longrightarrow X_{s+t}$ is onto.

(ii) Let $u \in X_s$. Then $J_{-s}u \in X$. Since $\mathcal{S}$ is dense in X, it follows that there is a sequence $\{\varphi_k\}$ of elements in $\mathcal{S}$ such that $\varphi_k \longrightarrow J_{-s}u$ in X as $k \longrightarrow \infty$. Let $\psi_k = J_s\varphi_k$, $k = 1, 2, \ldots$. Since J_s maps $\mathcal{S}$ into $\mathcal{S}$, it follows that $\psi_k \in \mathcal{S}$, $k = 1, 2, \ldots$. Also, by the definition of X_s again,

$$\|\psi_k - u\|_s = \|J_{-s}\psi_k - J_{-s}u\| = \|\varphi_k - J_{-s}u\|$$

for all $k = 1, 2, \ldots$. Hence $\psi_k \longrightarrow u$ in X_s as $k \longrightarrow \infty$. This proves that $\mathcal{S}$ is dense in X_s.

Let A be a linear operator from X into X with domain $\mathcal{S}$. We assume that A maps $\mathcal{S}$ into $\mathcal{S}$ and its formal adjoint A^* maps $\mathcal{S}$ into $\mathcal{S}$ continuously. In other words, if $\{\varphi_k\}$ is any sequence in $\mathcal{S}$ such that $\varphi_k \longrightarrow 0$ in $\mathcal{S}$ as $k \longrightarrow \infty$, then $A\varphi_k \longrightarrow 0$ and $A^*\varphi_k \longrightarrow 0$ in $\mathcal{S}$ as $k \longrightarrow \infty$. We can now extend the linear operator A to the space $\mathcal{S}'$ as follows: for any u in $\mathcal{S}'$, we define Au to be the element in $\mathcal{S}'$ given by

$$(Au, \varphi) = (u, A^*\varphi), \quad \varphi \in \mathcal{S}.$$

It is easy to prove that $A: \mathcal{S}' \longrightarrow \mathcal{S}'$ is a continuous linear mapping.

Before we formulate the main theorem of this section, we introduce some new notions.

Let $T: \mathcal{S}' \longrightarrow \mathcal{S}'$ be a continuous linear mapping. Suppose that there exists a real number m such that $T: X_s \longrightarrow X_{s-m}$ is a bounded linear operator for all $s \in (-\infty, \infty)$. Then we call T an operator of order m if m is the least number for which $T: X_s \longrightarrow X_{s-m}$ is a bounded linear operator. If the least number m is equal to $-\infty$, then we call T an *infinitely smoothing operator.* If A is a linear operator from X into X with domain $\mathcal{S}$ such that A maps $\mathcal{S}$ into $\mathcal{S}$ and its formal adjoint A^* maps $\mathcal{S}$ into $\mathcal{S}$ continuously, then we call A an *operator of order m* if the extended operator $A: \mathcal{S}' \longrightarrow \mathcal{S}'$ is of order m.

We can now formulate the main theorem in this section.

THEOREM 4.2. *Let A be a linear operator from X into X with domain $\mathcal{S}$ such that A maps $\mathcal{S}$ into $\mathcal{S}$ and its formal adjoint A^* maps $\mathcal{S}$ into $\mathcal{S}$ continuously. Suppose that A is of positive order m and there exists a linear operator B of order $-m$ from X into X with domain $\mathcal{S}$ such that*

$$BA = I + R \tag{4.4}$$

where I is the identity operator, and R is an infinitely smoothing operator. Then the minimal and maximal operators of A are equal, i.e., $A_0 = A_1$.

To prove Theorem 4.2, we need some lemmas.

LEMMA 4.3. *Let A be as in Theorem 4.2. Then $\mathcal{D}(A_0) = X_m$.*

Lemma 4.3 follows from the following lemma:

LEMMA 4.4. *Let A be as in Theorem 4.1. Then there exist positive constants C_1 and C_2 such that*

$$C_1\|x\|_m \le \|Ax\| + \|x\| \le C_2\|x\|_m, \quad x \in X_m.$$

PROOF. By using the fact that A is of order m and (4.2), we can find a positive constant C such that

$$\|Ax\| + \|x\| \le C\|x\|_m, \quad x \in X_m.$$

Next, by (4.4),

$$x = BAx - Rx, \quad x \in X_m,$$

where B is an operator of order $-m$ and R an infinitely smoothing operator. Hence there exists another positive constant C' such that

$$\|x\|_m \le C'(\|Ax\| + \|x\|), \quad x \in X_m.$$

This proves Lemma 4.4.

We can now prove Lemma 4.3.

PROOF OF LEMMA 4.3. Let $x \in X_m$. Since $\mathcal{S}$ is dense in X_m, we can find a sequence $\{\varphi_k\}$ of elements in $\mathcal{S}$ such that $\varphi_k \longrightarrow x$ in X_m as $k \longrightarrow \infty$. By Lemma 4.4, $\{A\varphi_k\}$ and $\{\varphi_k\}$ are Cauchy sequences in X. Hence $\varphi_k \longrightarrow u$ and $A\varphi_k \longrightarrow f$ in X for some u and f in X as $k \longrightarrow \infty$. Hence, by the definition of A_0, $u \in \mathcal{D}(A_0)$ and $A_0u = f$. By (4.2), $\varphi_k \longrightarrow x$ in X as $k \longrightarrow \infty$. Hence $x = u$. This proves that $x \in \mathcal{D}(A_0)$. On the other hand, if $x \in \mathcal{D}(A_0)$, then by the definition of A_0 again, we can find a sequence $\{\varphi_k\}$ of elements in $\mathcal{S}$ for which $\varphi_k \longrightarrow x$ in X and $A\varphi_k \longrightarrow f$ in X for some f in X as $k \longrightarrow \infty$. Hence $\{\varphi_k\}$ and $\{A\varphi_k\}$ are Cauchy sequences in X_m. Since X_m is complete, it follows that $\varphi_k \longrightarrow u$ in X_m for some u in X_m as $k \longrightarrow \infty$. Hence, by (4.2), $\varphi_k \longrightarrow u$ in X as $k \longrightarrow \infty$. Therefore $x = u$. This proves that $x \in X_m$.

PROOF OF THEOREM 4.2. Since A_1 is a linear extension of A_0 and $\mathcal{D}(A_0) = X_m$, it is sufficient to prove that $\mathcal{D}(A_1) \subseteq X_m$. To this end, let $x \in \mathcal{D}(A_1)$. Then, by (4.4),

$$x = BAx - Rx,$$

where B is of order $-m$ and R infinitely smoothing. Now, we claim that

$$A_1 u = Au, \quad u \in \mathcal{D}(A_1). \tag{4.5}$$

Let us assume (4.5) for a moment. Then $Ax \in X$. Since B is of order $-m$, it follows that $BAx \in X_m$. Since $x \in X$ and R is infinitely smoothing, we can conclude that $Rx \in X_m$. Hence $x \in X_m$.

It remains to prove (4.5). By the definition of A_1, we have

$$(\varphi, A_1 u) = (A^*\varphi, u), \quad \varphi \in \mathcal{S}. \tag{4.6}$$

Next, by the definition of A on $\mathcal{S}'$, we have

$$(Au, \varphi) = (u, A^*\varphi), \quad \varphi \in \mathcal{S}. \tag{4.7}$$

But, by (4.3) and (4.7),

$$(\varphi, Au) = (A^*\varphi, u), \quad \varphi \in \mathcal{S}. \tag{4.8}$$

Hence, by (4.3), (4.6) and (4.8),

$$(A_1 u, \varphi) = (Au, \varphi), \quad \varphi \in \mathcal{S}.$$

Therefore $A_1 u = Au$ for all $u \in \mathcal{D}(A_1)$.

5. **Linear equations on Banach spaces.** Let X be a reflexive complex Banach space and A a linear operator from X into X with domain $\mathcal{S}$. We assume that the formal adjoint A^* of A exists. For any element f in X, we call an element u in X a *weak solution* of the linear equation $Au = f$ if and only if

$$(A^*\varphi, u) = (\varphi, f), \quad \varphi \in \mathcal{S}.$$

From the definition of the maximal operator A_1 of A, it is obvious that the following statement is true.

PROPOSITION 5.1. *Let $f \in X$. Then an element u in X is a weak solution of the linear equation $Au = f$ if and only if $u \in \mathcal{D}(A_1)$ and $A_1 u = f$.*

The following theorem characterizes the elements f in X for which the linear equation $Au = f$ has a weak solution u in X. For a special case of it in the theory of linear partial differential equations, see Theorem 1.12 in Chapter 1 of Schechter's book [6].

THEOREM 5.2. *Let $f \in X$. Then the linear equation $Au = f$ has a weak solution u in X if and only if there exists a positive constant C such that*

$$|(\varphi, f)| \leq C\|A^*\varphi\|, \quad \varphi \in \mathcal{S}. \tag{5.1}$$

PROOF. Suppose that $Au = f$ has a weak solution u in X. Then

$$(\varphi, f) = (A^*\varphi, u), \quad \varphi \in \mathcal{S}.$$

Hence

$$|(\varphi, f)| \leq \|A^*\varphi\| \, \|u\|, \quad \varphi \in \mathcal{S}$$

and the inequality (5.1) holds with $C = \|u\|$. Conversely, suppose that the inequality (5.1) is true. Let W' be a nonempty subspace of X' defined by

$$W' = \{ w' \in X' : A^*\varphi = w' \text{ for some } \varphi \in \mathcal{S} \}.$$

We define a linear functional $F: W' \longrightarrow \mathbb{C}$ by

$$Fw' = (\varphi, f), \quad w' \in W',$$

where φ is any element in $\mathcal{S}$ with the property that $A^*\varphi = w'$. The definition of $F: W' \longrightarrow \mathbb{C}$ is independent of the choice of the element φ. Indeed, if φ_1 and φ_2 are elements in $\mathcal{S}$ such that $A^*\varphi_1 = w'$ and $A^*\varphi_2 = w'$, then, by (5.1),

$$|(\varphi_1 - \varphi_2, f)| \leq C\|A^*(\varphi_1 - \varphi_2)\| = 0.$$

Hence $(\varphi_1, f) = (\varphi_2, f)$. This proves that the choice of the element φ is irrelevant to the definition of $F: W' \longrightarrow \mathbb{C}$. Since, by (5.1),

$$|Fw'| = |(\varphi, f)| \leq C\|A^*\varphi\| = C\|w'\|$$

for all $w' \in W'$, it follows that $F: W' \longrightarrow \mathbb{C}$ is a bounded linear functional. Hence, by using the Hahn-Banach theorem and the reflexivity of X, we can find an element u in X such that

$$Fw' = (\varphi, f) = (w', u), \quad w' \in W', \tag{5.2}$$

where φ is any element in $\mathcal{S}$ satisfying $A^*\varphi = w'$. Since $\{A^*\varphi : \varphi \in \mathcal{S}\}$ is obviously a subspace of W', it follows from (5.2) that

$$(\varphi, f) = (A^*\varphi, u), \quad \varphi \in \mathcal{S}.$$

This proves that u is a weak solution of the linear equation $Au = f$ and the proof is now complete.

We can prove the following regularity theorem easily by using Theorem 4.2 and Lemma 4.3.

THEOREM 5.3. *Let $f \in X$. Then, under the hypotheses of Theorem 4.2, every weak solution u of the linear equation $Au = f$ is in X_m.*

REMARK 5.4. Theorem 5.3 asserts that every weak solution u of the linear equation $Au = f$ lies in a more *selective* or *regular* space X_m. If we recall, by (4.2), that $X_m \subset X$, then the weak solution u can be thought of being m units more *selective* or *regular* than the given source data f lying in X. For this reason, we call Theorem 5.3 a regularity theorem.

PROOF OF THEOREM 5.3. Let u be a weak solution of $Au = f$. Then, by Proposition 5.1, $u \in \mathcal{D}(A_1)$. By Theorem 4.2, $A_1 = A_0$. Therefore $u \in \mathcal{D}(A_0)$. By Lemma 4.3, $\mathcal{D}(A_0) = X_m$. Hence $u \in X_m$.

6. **Essential self-adjointness.** In this section, we assume that the family of spaces $X_s, -\infty < s < \infty$, introduced in Section 4 are complex Hilbert spaces. The inner product of the Hilbert space X is denoted by $(\, , \,)$. We say that a linear operator A from X into X with domain $\mathcal{S}$ is *symmetric* if and only if $(Ax, y) = (x, Ay)$ for all x and y in $\mathcal{S}$. We say that A is *essentially self-adjoint* if and only if it has a unique self-adjoint extension. For discussions of symmetric and self-adjoint operators, see Section 2 in Chapter 8 of the book [4] by Reed and Simon or page 61 in the book [7] by Schechter.

We can prove that the following criterion for a symmetric operator to be essentially self-adjoint is an easy consequence of Theorem 4.2.

THEOREM 6.1. *Under the hypotheses of Theorem 4.2 and the assumption that the spaces $X_s, -\infty < s < \infty$, are complex Hilbert spaces, the linear operator A from X into X with domain $\mathcal{S}$ is essentially self-adjoint if and only if it is symmetric.*

PROOF. By Theorem 4.2, A has a unique closed linear extension B with the property that the domain of B^t contains the space $\mathcal{S}$. Let B_1 and B_2 be self-adjoint extensions of A. The B_1 and B_2 are obviously closed linear extensions of A such that the domains of B_1^t and B_2^t both contain the space $\mathcal{S}$. Hence $B_1 = B_2$ and the proof of the theorem is complete.

7. **An application to pseudo-differential operators.** As a prelude to an application to pseudo-differential operators, it is important to realize that mathematical physicists are interested in determining when a symmetric operator is essentially self-adjoint. See Section 2 in Chapter 8 of the book [4] by Reed and Simon or page 61 in the book [7] by Schechter for discussions of this problem. It is an interesting fact that for pseudo-differential operators of the Weyl type (to be explained later), the problem of determining when symmetry implies essential self-adjointness has a very nice solution.

Let $m > 0$. We define S^m to be the set of all C^∞ functions σ on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

We call any function in S^m a symbol of order m. Let $\sigma \in S^m$. Then the pseudo-differential operator T_σ is defined on the Schwartz space $\mathcal{S}$ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(\xi)\,d\xi, \quad x \in \mathbb{R}^n,$$

for all functions φ in $\mathcal{S}$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\,dx, \quad \xi \in \mathbb{R}^n. \tag{7.1}$$

It is an easy matter to prove that T_σ maps $\mathcal{S}$ into $\mathcal{S}$. Hence we can consider T_σ as a linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with dense domain $\mathcal{S}$. It is well-known that the formal adjoint T_σ^* of T_σ exists and is also a pseudo-differential operator with symbol in S^m. By using the formal adjoint, we can extend $T_\sigma : \mathcal{S} \longrightarrow \mathcal{S}$ to a continuous linear mapping from $\mathcal{S}'$ into $\mathcal{S}'$, where $\mathcal{S}'$ is the space of all tempered distributions. It is also well-known that $T_\sigma : H^{s,p} \longrightarrow H^{s-m,p}$ is a bounded linear operator for $1 < p < \infty$ and $-\infty < s < \infty$, where $H^{s,p}$ is the L^p Sobolev space of order s. For any fixed value of p in $(1,\infty)$, the one-parameter family of spaces $H^{s,p}$, indexed by $s, -\infty < s < \infty$, satisfies the conditions (i)–(vii) in Section 4 provided that J_s is chosen to be the pseudo-differential operator of which the symbol is given by

$$\sigma_s(\xi) = (1 + |\xi|^2)^{-s/2}, \quad \xi \in \mathbb{R}^n.$$

Let $\sigma \in S^m$, $m > 0$. Then we call σ an *elliptic symbol of order m* if there exist positive constants C and R such that

$$|\sigma(x,\xi)| \geq C(1 + |\xi|)^m, \quad |\xi| \geq R.$$

It is a well-known fact that for any elliptic symbol σ in S^m, $m > 0$, we can find a symbol τ in S^m such that

$$T_\sigma T_\tau = I + R$$

and

$$T_\tau T_\sigma = I + S,$$

where I is the identity operator, and R and S are pseudo-differential operators with symbols in $\bigcap_{k\in\mathbb{R}} S^k$. The above mentioned results concerning pseudo-differential operators can be found in the book [11] by Wong. The corresponding results for $p = 2$ can be found in the book [3] by Kumano-go.

For any symbol σ in S^m, $m > 0$, there is another way of associating a linear operator with σ. To wit, we define the linear operator W_σ on the Schwartz space $\mathcal{S}$ by

$$(W_\sigma \varphi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right)\varphi(y)\,dy\,d\xi, \quad x \in \mathbb{R}^n,$$

for all functions φ in $\mathcal{S}$, where the integral is an oscillatory integral. See Section 6 in Chapter 1 of the book [3] by Kumano-go for a discussion of oscillatory integrals. We call W_σ the *pseudo-differential operator of the Weyl type with symbol* σ. It can be proved that W_σ is a pseudo-differential operator T_κ, where κ is some other symbol in S^m. It can also be proved that if σ is elliptic, then κ is elliptic as well.

Let us now consider W_σ as a linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ with dense domain $\mathcal{S}$. Then an important property of pseudo-differential operators of the Weyl type is that W_σ is symmetric if and only if σ is real-valued. For a discussion of pseudo-differential operators of the Weyl type, see the book [1] by Folland.

By using Theorem 6.1 and the theory of pseudo-differential operators described in this section, we have proved the following theorem:

THEOREM 7.1. *Let σ be a symbol in S^m, $m > 0$, such that σ is real-valued and elliptic. Then the pseudo-differential operator W_σ of the Weyl type from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ with dense domain $\mathcal{S}$ is essentially self-adjoint.*

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