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BOUNDARY BLOW-UP SOLUTIONS TO EQUATIONS INVOLVING THE INFINITY LAPLACIAN

CUICUI LI, FANG LIU and PEIBIAO ZHAO

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Abstract

In this paper, we study the boundary blow-up problem related to the infinity Laplacian

$$\begin{cases} \Delta_\infty^h u = u^q & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_\infty^h u = |Du|^{h-3}\langle D^2uDu, Du\rangle$ is the highly degenerate and h-homogeneous operator associated with the infinity Laplacian arising from the stochastic game named Tug-of-War. When $q > h > 1$, we establish the existence of the boundary blow-up viscosity solution. Moreover, when the domain satisfies some regular condition, we establish the asymptotic estimate of the blow-up solution near the boundary. As an application of the asymptotic estimate and the comparison principle, we obtain the uniqueness result of the large solution. We also give the nonexistence of the large solution for the case $q \leq h$.

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1. Introduction

In this paper, we consider the following family of degenerate elliptic equations with a parameter $h > 1$:

$$\begin{cases} \Delta_\infty^h u = u^q & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where q is a given constant and

$$\Delta_\infty^h u := |Du|^{h-3}\langle D^2uDu, Du\rangle = |Du|^{h-3}\sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j}$$

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denotes the h-homogeneous nonlinear operator. Due to the high degeneracy of the operator $\Delta_\infty^h u$, the associated problems do not have smooth solutions in general. Therefore, solutions are understood in the viscosity sense (see Section 2 for the precise definition). Notice that the h stands for the homogeneous degree of the operator. Throughout this paper, we denote Ω to be a bounded domain in the Euclidean space $\mathbb{R}^n (n \geq 2)$ and $\partial\Omega$ to be its boundary.

The boundary condition in (1-1) is understood in the following sense:

$$\lim_{x\to z} u(x) = \infty, \quad z \in \partial\Omega,$$

and the solution of (1-1) is called the 'boundary blow-up solution', 'large solution', or 'explosive solution' due to the explosive boundary condition. The motivation for the name is as follows: if U is a boundary blow-up solution, the comparison principle (Theorem 2.3) implies that any solution V to $\Delta_\infty^h u = u^q$ in Ω with bounded boundary data satisfies $V(x) \leq U(x), x \in \Omega$. Hence, the boundary blow-up solution provides local uniform bounds for all other solutions in the domain Ω, regardless of the boundary data.

The boundary blow-up problem (1-1) has been studied in two special cases, $h = 1$ [19] and $h = 3$ [29]. For $h = 1$, $\Delta_\infty^h u$ is the 1-homogeneous normalized infinity Laplacian operator,

$$\Delta_\infty^N u := |Du|^{-2}\langle D^2uDu, Du\rangle.$$

For $h = 3$, $\Delta_\infty^h u$ is the 3-homogeneous infinity Laplacian operator,

$$\Delta_\infty u := |Du|^2 \Delta_\infty^N u.$$

And for other h, we have

$$\Delta_\infty^h u = |Du|^{h-3}\Delta_\infty u = |Du|^{h-1}\Delta_\infty^N u.$$

The infinity Laplacian Δ_∞ was first introduced by Aronsson [2] in the 1960s in connection with the geometric problem of finding the so-called absolutely minimizing Lipschitz extension. For more properties of the infinity harmonic functions (the viscosity solution to $\Delta_\infty u = 0$), one can see the works of Crandall [10], Crandall *et al.* [11], Aronsson *et al.* [3], and the references therein.

For the inhomogeneous equation

$$\Delta_\infty u = f(x),$$

Lu and Wang [26] proved the existence and uniqueness of a viscosity solution of the Dirichlet problem when the inhomogeneous term f does not change its sign. In [4, 5], Bhattacharya and Mohammed studied the existence or nonexistence of viscosity solutions to the Dirichlet problem

$$\begin{cases} \Delta_\infty u = f(x, u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

under some conditions for f and g.

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The normalized version $\Delta_\infty^N u$ was first introduced by Peres, Schramm, Sheffield, and Wilson from the point of randomized theory named Tug-of-War [33]. Now let us briefly recall the random-turn, ε-tug-of-war game. This is a zero sum game with two players in which the earnings of one of them are the losses of the other. Given a step size $\varepsilon > 0$, let $f \in C(\Omega)$ be a running payoff function and $g \in C(\partial\Omega)$ be a final payoff function. The starting position is $x_0 \in \Omega$. At the kth step, a fair coin is tossed, and the player who wins the toss may move the token from x_{k-1} to any x_k with $|x_k - x_{k-1}| < \varepsilon$. The game ends when $x_m \in \partial\Omega$, and player II pays to player I the amount

$$\text{Payoff} = g(x_m) + \sum_{i=1}^{m} f(x_{i-1}).$$

If the token never reaches $\partial\Omega$ and the game thus fails to terminate, each of the players must pay a fine of $+\infty$. The value function $V_I(x_0)$ for Player I is, roughly, the minimum that Player I can expect to win by playing optimally, while the value function $V_{II}(x_0)$ is the maximum that Player II can expect to be required to pay, by playing optimally.

According to the dynamic programming principle, one gets $V_I(x) = V_{II}(x)$ and V_I satisfies

$$\begin{cases} \frac{1}{2}(u(x) - \min\limits_{B(x,\varepsilon)\cap\Omega} u(y)) - \frac{1}{2}(\max\limits_{B(x,\varepsilon)\cap\Omega} u(y) - u(x)) = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Furthermore, the corresponding expectation value function u^ε of this game exists and converges (as $\varepsilon \to 0$) to a function u which is the unique solution of the problem

$$\begin{cases} \Delta_\infty^N u = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

For more stochastic games related to the normalized infinity Laplacian, one can consult the papers [21, 23, 24, 32, 35], and so on. The normalized infinity Laplacian equation was also well studied by Lu and Wang based on partial differential equation methods in [27]. Notice that $\Delta_\infty^N u$ is not only degenerate but also singular when the gradient of u vanishes. In the last three decades, the infinity Laplacian has received much attention because it is not only highly degenerate but also has many applications in image processing [1, 6, 13] and optimal mass transportation problems [14].

In [22], Liu and Yang established the existence of the viscosity solutions for the Dirichlet problem of the inhomogeneous equation

$$\Delta_\infty^h u(x) = f(x).$$

In [34], Portilheiro and Vázquez studied the parabolic version of the operator Δ_∞^h. They proved the existence and uniqueness of viscosity solutions for the

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initial-Dirichlet boundary problem

$$\begin{cases} u_t - \Delta_\infty^h u = 0 & \text{in } Q =: \Omega \times (0, T), \\ u = g & \text{on } \partial_p Q. \end{cases}$$

They also established the asymptotic behavior of the viscosity solution for the problem posed in the whole space.

The boundary blow-up solutions to the elliptic equations have many applications in stochastic differential processes [20], population dynamics [15, 31], and the equilibrium state of charged gas in a container [16], and so on. Specifically, Lasry and Lions [20] considered a stochastic control problem

$$dX_t = a(X_t)\, dt + dB_t, \quad X_0 = x \in \Omega, \quad t > 0,$$

where the state of the controlled system is a diffusion process, B_t is a standard Brownian motion, a is the control process, X_t is the state process, and $P(X_t \in \partial\Omega) > 0$. To get the state constraints (that is, Brownian motion in a bounded domain), one needs to use the unbounded drifts a. In other words, one will have to choose feedback controls that push back the state process inside Ω when it gets near $\partial\Omega$. And the intensity of the state blows up at the boundary of Ω. Therefore, they defined the class $\mathcal{A}$ of such feedback controls. And for each $a \in \mathcal{A}$, they considered the cost function

$$J(x, a) = E\Big\{ \int_0^\infty \Big[f(X_t) + \frac{1}{q}|a(X_t)|^q \Big] e^{-\lambda t} dt \Big\} \quad \text{for all } x \in \Omega,$$

where E denotes the expectation, $q > 1$, f is a given function on Ω which is bounded from below and the positive parameter λ denotes the so-called discount factor. Then, by the dynamic programming principle and probability analysis, they showed that the minimum u (called the value function) of the function J

$$u(x) = \inf_{a \in \mathcal{A}} J(x, a) \quad \text{for all } x \in \Omega,$$

satisfies the following boundary blow-up problem

$$\begin{cases} -\dfrac{1}{2}\Delta u + \dfrac{1}{p}|Du|^p + \lambda u = f(x), & x \in \Omega, \\ u(x) = \infty, & x \in \partial\Omega, \end{cases}$$

where $p = q/(q - 1)$ and the optimal feedback control $a(\cdot) = -|Du|^{p-2}Du(\cdot)$. Note that for $h = 1$, the operator Δ_∞^h shares the same structure with the standard Laplacian for 1-dimension. Hence, it is meaningful to consider the boundary blow-up problem of Δ_∞^h.

In the present work, we are interested in the boundary blow-up problem (1-1). The h-degree homogeneous infinity Laplacian equation (1-1) is of intrinsic interest, because it is not only strongly degenerate, but also has no variational structure and divergence form.

Our main results are summarized as follows.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $q > h$, then there exists a positive viscosity solution to the boundary blow-up problem (1-1).*

When the domain possesses C^1 regularity, we can establish the following asymptotic estimate near the boundary and uniqueness of large solutions to (1-1).

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that there exists a neighborhood $\mathcal{N}$ of $\partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) \in C^1(\mathcal{N} \cap \Omega)$ and $q > h$, then the viscosity solution u to (1-1) satisfies the precise boundary behavior*

$$u(x) \sim \bigg(\Big(\frac{h+1}{q-h}\Big)^h \frac{q+1}{q-h}\bigg)^{1/(q-h)} \operatorname{dist}(x, \partial\Omega)^{-(h+1)/(q-h)} \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0. \tag{1-2}$$

Furthermore, the viscosity solution of (1-1) is unique.

THEOREM 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $q \le h$, then the boundary blow-up problem (1-1) has no positive solution.*

In [19], Juutinen and Rossi established the existence, uniqueness, and asymptotic behavior for the large solutions of the normalized infinity Laplacian

$$\begin{cases} \Delta_\infty^N u = u^q & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

for $q > 1$. In [29, 30], Mohammed and Mohammed studied the boundary blow-up solutions of the infinity Laplacian

$$\begin{cases} \Delta_\infty u = b(x) f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1-3}$$

where $b \in C(\overline{\Omega})$ is nonnegative, $f \in C[0, \infty) \cap C^1(0, \infty)$, $f(0) = 0$, $f(s) > 0$, $s > 0$, and $f(s)$ is nondecreasing on $(0, \infty]$. They proved that the boundary blow-up problem of (1-3) has a nonnegative viscosity solution if the following Keller–Osserman-type condition holds:

$$\Psi(r) := \int_r^\infty \frac{d\tau}{4F(\tau)^{1/4}} < \infty \quad \text{for all } r > 0,$$

where $F(\tau) = \int_0^\tau f(\nu)\, d\nu$.

Most recently, Wang *et al.* [36] studied the second-order asymptotic behavior of boundary blow-up viscosity solutions (1-3) based on Karamata regular variation theory which was first introduced by Cîrstea and Rădulescu in a stochastic process to study the boundary behavior and uniqueness of solutions to boundary blow-up elliptic problems. A series of rich and significant information about the boundary behavior of solutions was obtained based on such theory [7–9]. In [37], under appropriate structure conditions on the nonlinear term f, Zhang established the following the boundary

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estimate of large solutions to problem (1-3):

$$\lim_{d(x)\to 0}\frac{u(x)}{\psi(K(d(x)))}=1,$$

where $k\in C^1$ is positive and nondecreasing, $K(t)=\int_0^t k(s)\,ds$ satisfies

$$\lim_{t\to 0^+}\frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big)=C_k,$$

and ψ satisfies

$$\int_{\psi(t)}^{\infty}\frac{ds}{(4F(s))^{1/4}}=t.$$

In [28], the boundary behavior of the boundary blow-up viscosity solutions to problem (1-3) was studied under different conditions on the weight function $b(x)$ and the nonlinear term f.

To obtain the existence of boundary blow-up viscosity solutions of (1-1), we first establish the comparison principle and then combine Perron's method with compactness arguments. Due to the strong degeneracy of the operator Δ_∞^h and the boundary blow-up condition, it is difficult to study the comparison principle for viscosity solutions of (1-1). To overcome this difficulty, we employ the perturbation method and the logarithmic transformation of functions so that the double variables method can be carried out in the usual way. Then we consider the approximate problems with boundary condition $u=M$, where $M\ge 1$ is a constant. To establish the existence of large solutions, a difficulty with respect to the degenerate operators is the lack of the existence of barriers. Thanks to the particular structure of Δ_∞^h, we can construct 'good' barriers and use the standard Perron method to get the existence of approximate solutions. Finally, based on compactness analysis, we establish that the limit function of the approximate solutions is the desired boundary blow-up solution. To conclude that the limit is finite, we use again the comparison principle with a radial large solution obtained by analyzing the corresponding ordinary differential equation.

One should notice that if the regularity assumption of Theorem 1.2 holds, then the distance function is a solution of $\Delta_\infty^h v=0$ near the boundary. Therefore, we can perturb the distance function to analyze the asymptotic behavior near the boundary. Based on the asymptotic estimates and the comparison principle, the uniqueness result of the viscosity solution follows immediately. Let us point out that, unlike the case $h=1$, the operator Δ_∞^h is quasi-linear even in dimension 1. Therefore, we must make a subtle analysis.

Due to the high degeneracy of Δ_∞^h, we employ the logarithmic transformation of functions and comparison principle to obtain the nonexistence of the large solution of (1-1) for the case $q\le h$.

The outline of this paper is as follows. In Section 2, we give the definition of the viscosity solution to (1-1) and prove the comparison principle for the equation $\Delta_\infty^h u=u^q$ based on the perturbation method of viscosity solutions. In Section 3, based

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on the comparison principle, we establish the existence of viscosity solutions to (1-1) by Perron's method and compactness analysis for $q > h$. In Section 4, under some regular assumption of the domain, we give the characteristic of the boundary blow-up solution near the boundary, and then by the comparison principle, we obtain the uniqueness of solutions to (1-1). Finally, in Section 5, we give the nonexistence of viscosity solutions to (1-1) for $q \le h$.

2. Comparison principles

In this section, we first give the definition of the viscosity solutions to the equation

$$\Delta_\infty^h u = u^q \quad \text{in } \Omega, \tag{2-1}$$

where $q > 0$, and then establish the comparison results by the double variables method based on the viscosity solutions theory. Notice that the operator Δ_∞^h is singular for $1 < h < 3$ at points where the gradient of the function vanishes. To give a reasonable explanation when the gradient vanishes, we use the definition of viscosity solutions based on semi-continuous extension and we refer the reader to [12, 22, 27], and so on. In fact, the singularity is removable when $h > 1$ since we are not considering here $h = 1$. Therefore, we can write (2-1) as

$$F_h(D^2u, Du) = u^q \quad \text{in } \Omega,$$

where $F_h : \mathbb{S} \times (\mathbb{R}^n \backslash \{0\}) \to \mathbb{R}$ and $F_h(M, p) := |p|^{h-3}(Mp) \cdot p$. Here $\mathbb{S}$ denotes the set of $n \times n$ real symmetric matrices. Since $h > 1$, we have $\lim_{p \to 0} F_h(M, p) = 0$ for arbitrary $M \in \mathbb{S}$. Hence, we can define the continuous extension of F_h as follows:

$$\overline{F}_h(M, p) := \begin{cases} F_h(M, p) & \text{if } p \neq 0, \\ 0 & \text{if } p = 0. \end{cases}$$

We remark that due to the strong degeneracy of (2-1), it is not clear that the functions that one would like to call solutions are actually differentiable even once.

Definition 2.1. Suppose that $u : \Omega \to \mathbb{R}$ is an upper semi-continuous and nonnegative function. If, for every $x_0 \in \Omega$ and test function $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \le \varphi(x)$ for all $x \in \Omega$ near x_0, there holds

$$\overline{F}_h(D^2\varphi(x_0), D\varphi(x_0)) \ge \varphi(x_0)^q,$$

then we say u is a viscosity subsolution of (2-1).

Similarly, suppose that $u : \Omega \to \mathbb{R}$ is a lower semi-continuous and nonnegative function. If, for every $x_0 \in \Omega$ and test function $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \ge \varphi(x)$ for all $x \in \Omega$ near x_0, there holds

$$\overline{F}_h(D^2\varphi(x_0), D\varphi(x_0)) \le \varphi(x_0)^q,$$

then we say u is a viscosity supersolution of (2-1).

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If a continuous function u is both a viscosity subsolution and a viscosity supersolution of (2-1), then we say u is a viscosity solution of (2-1).

REMARK 2.2. Let $u, v \in C(\Omega)$. The two statements below follow easily from the definition of a viscosity sub (super)-solution of (2-1).

(i) If u and v are both viscosity subsolutions of (2-1), then $\max\{u, v\}$ is also a viscosity subsolution of (2-1).
(ii) If u and v are both viscosity supersolutions of (2-1), then $\min\{u, v\}$ is also a viscosity supersolution of (2-1).

Because of the degeneracy at the points where the gradient vanishes and the explosive boundary condition, the general comparison result stated in [12] does not apply to (2-1). Now we present a comparison principle which is proved via a perturbation argument.

THEOREM 2.3. *Let $q \geq h$. Suppose that u and v are nonnegative continuous functions defined in a bounded domain Ω and satisfy*

$$\Delta_\infty^h u(x) \geq u(x)^q \quad \textit{for all } x \in \Omega$$

and

$$\Delta_\infty^h v(x) \leq v(x)^q \quad \textit{for all } x \in \Omega$$

in the viscosity sense. If

$$\limsup \frac{u(x)}{v(x)} \leq 1 \qquad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0, \tag{2-2}$$

then we have $u \leq v$ in Ω.

One should notice that assumption (2-2) implies that $v > 0$ near $\partial\Omega$ so as to make the ratio u/v well defined.

PROOF. We argue it by contradiction. Suppose that

$$\sup_\Omega (u - v) > 0.$$

Then, due to (2-2), there exist $\varepsilon > 0$ small enough and $x_0 \in \Omega$ such that

$$M := u(x_0) - v_\varepsilon(x_0) = \sup_{x\in\Omega}(u(x) - v_\varepsilon(x)) > 0, \tag{2-3}$$

where $v_\varepsilon := (1 + \varepsilon)v$. Notice that assumptions (2-2) and (2-3) also imply that there exists an open set $\Omega_0 \subset\subset \Omega$ such that $x_0 \in \Omega_0$ and

$$M = \sup_{x\in\Omega_0}(u(x) - v_\varepsilon(x)) > \sup_{x\in\partial\Omega_0}(u(x) - v_\varepsilon(x)).$$

Since $q \geq h$ and v is a viscosity supersolution of (2-1), one can verify that

$$\Delta_\infty^h v_\varepsilon = (1 + \varepsilon)^h \Delta_\infty^h v \leq (1 + \varepsilon)^h v^q = (1 + \varepsilon)^{h-q} v_\varepsilon^q \leq v_\varepsilon^q$$

in the viscosity sense, that is, v_ε is also a viscosity supersolution of (2-1).

Based on the ideas in [12], we double the variables

$$w_j(x,y) := u(x) - v_\varepsilon(y) - \frac{j}{4}|x-y|^4, \quad (x,y) \in \Omega_0 \times \Omega_0,\ j = 1,2,\ldots.$$

We denote the maximum point of w_j over $\overline{\Omega}_0 \times \overline{\Omega}_0$ by (x_j, y_j) and $M_j := w_j(x_j, y_j)$. According to Proposition 3.7 in [12], we have

$$\lim_{j\to\infty} M_j = \lim_{j\to\infty}(u(x_j) - v_\varepsilon(y_j) - j|x_j - y_j|^4/4) = M$$

and

$$\lim_{j\to\infty} j|x_j - y_j|^4/4 = 0.$$

Furthermore, $x_j \to x_0$, $y_j \to x_0$ as $j \to \infty$. It is obvious that $M = u(x_0) - v_\varepsilon(x_0)$. Due to $M > \sup_{\partial\Omega_0}(u - v_\varepsilon)$, we have x_j, y_j are interior points of Ω_0 for j large enough.

Set

$$\psi(x) = j|x - y_j|^4/4, \quad \phi(y) = -j|x_j - y|^4/4.$$

It is clear that the functions $u - \psi$ and $v_\varepsilon - \phi$ have a local maximum at x_j and a local minimum at y_j. We consider the two cases: either $x_j \neq y_j$ or $x_j = y_j$ for j large enough.

Case 1: If $x_j = y_j$, we have $D\psi(x_j) = 0$ and $D^2\psi(x_j) = 0$. Since u is a viscosity subsolution, we have

$$0 \geq u^q(x_j) = \psi^q(x_j). \tag{2-4}$$

By (2-3), we have

$$u(x_j) \geq u(x_j) - v_\varepsilon(y_j) \geq u(x_j) - v_\varepsilon(y_j) - \frac{j}{4}|x_j - y_j|^4 = w_j(x_j, y_j) \geq u(x_0) - v_\varepsilon(x_0) > 0. \tag{2-5}$$

Obviously, (2-5) contradicts (2-4).

Case 2: If $x_j \neq y_j$, we use jets and the maximum principle for semi-continuous functions, see [12]. Now we recall the definitions of super-jets and sub-jets. The second-order super-jet of an upper semi-continuous function γ at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,+}\gamma(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(\Omega) \text{ and } \gamma - \varphi \text{ has a local maximum at } x_0\},$$

and its closure is

$$\overline{\mathcal{J}}^{2,+}\gamma(x_0) := \{(p, M) \in \mathbb{R}^n \times \mathbb{S} : \text{there exists } (x_i, p_i, M_i) \in \Omega \times \mathbb{R}^n \times \mathbb{S} \text{ such that } (p_i, M_i) \in \mathcal{J}^{2,+}\gamma(x_i) \text{ and } (x_i, \gamma(x_i), p_i, M_i) \to (x_0, \gamma(x_0), p, M)\}.$$

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Here $\mathbb{S}$ denotes the set of $n \times n$ real symmetric matrices. Similarly, the second-order sub-jet of a lower semi-continuous function γ at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,-}\gamma(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(\Omega) \text{ and } \gamma - \varphi \text{ has a local minimum at } x_0\},$$

and its closure is

$$\begin{aligned}\overline{\mathcal{J}}^{2,-}\gamma(x_0) :=\{(p, M) \in \mathbb{R}^n \times \mathbb{S} :\ & \text{there exists } (x_i, p_i, M_i) \in \Omega \times \mathbb{R}^n \times \mathbb{S} \\ & \text{such that } (p_i, M_i) \in \mathcal{J}^{2,-}\gamma(x_i) \text{ and } (x_i, \gamma(x_i), p_i, M_i) \to (x_0, \gamma(x_0), p, M)\}.\end{aligned}$$

By the maximum principle for semi-continuous functions in [12], there exist symmetric matrices X_j, $Y_j \in \mathbb{S}$ such that $Y_j - X_j \geq 0$ and

$$\begin{aligned}(j|x_j - y_j|^2(x_j - y_j), X_j) &\in \overline{\mathcal{J}}^{2,+}u(x_j),\\ (j|x_j - y_j|^2(x_j - y_j), Y_j) &\in \overline{\mathcal{J}}^{2,-}v_\varepsilon(y_j).\end{aligned}$$

For simplicity, we denote $p_j = j|x_j - y_j|^2(x_j - y_j)$. Again since u and v_ε are a viscosity subsolution and supersolution, respectively, we have

$$\begin{aligned}0 &\leq |p_j|^{h-3}\langle X_j p_j, p_j\rangle - u^q(x_j)\\ &\leq |p_j|^{h-3}\langle Y_j p_j, p_j\rangle - v_\varepsilon^q(y_j) + v_\varepsilon^q(y_j) - u^q(x_j)\\ &\leq v_\varepsilon^q(y_j) - u^q(x_j),\end{aligned}$$

where we use $Y_j - X_j \geq 0$. Passing to the limit and noting $q \geq h > 1$, we obtain $v_\varepsilon(x_0) \geq u(x_0)$, which is a contradiction to (2-3). □

REMARK 2.4. In the above proof, the assumption $q \geq h$ was used to guarantee that $(1 + \varepsilon)v$ is also a supersolution if v is, and the rest of the argument does still work for all $q > 0$. Therefore, we can also establish the following comparison result which will be needed to establish the existence of the large solution in Section 3. We leave its proof to the reader.

THEOREM 2.5. *Let $q > 0$. Suppose that u and v are nonnegative continuous functions defined in the closure of a bounded domain Ω, and satisfy*

$$\Delta_\infty^h u(x) \geq u(x)^q \quad \text{for all } x \in \Omega$$

and

$$\Delta_\infty^h v(x) \leq v(x)^q \quad \text{for all } x \in \Omega$$

in the viscosity sense. Then $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in Ω.

3. Existence of boundary blow-up solutions for $q > h$

In this section, we focus on the existence of boundary blow-up solutions to (1-1) for the case $q > h$. First, we show certain radial solutions to (1-1) on a ball. Then we establish the existence of large solutions of (1-1) based on Perron's method and the compactness argument.

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First, we consider the following boundary blow-up problem in the ball $B_R(x_0)$:

$$\begin{cases} \Delta_\infty^h u = u^q & \text{in } B_R(x_0), \\ u = \infty & \text{on } \partial B_R(x_0), \end{cases} \tag{3-1}$$

where $q > h$. We look for solutions of the form $u(x) = g(r)$, where $r = |x - x_0|$. By direct calculation, we get

$$Du(x) = g'(r)\frac{x - x_0}{|x - x_0|}$$

and

$$D^2u(x) = \Big[g''(r)\frac{(x - x_0) \otimes (x - x_0)}{|x - x_0|^2} + g'(r)\frac{1}{|x - x_0|}I - g'(r)\frac{(x - x_0) \otimes (x - x_0)}{|x - x_0|^3}\Big]$$

for $x \neq x_0$, where $\otimes$ denotes the tensor product. Then u satisfies

$$\Delta_\infty^h u = u^q, \quad x \in B_R(x_0)\backslash\{x_0\}, \tag{3-2}$$

if and only if

$$|g'(r)|^{h-1}g''(r) = g^q(r), \quad r \in (0, R), \tag{3-3}$$

which is a 1-dimensional case for (3-2).

Now we consider the nonnegative solution of the following initial value problem:

$$\begin{cases} |g'(r)|^{h-1}g''(r) = g^q(r), \quad r \in (0, R), \\ g(0) = \tau, \\ g'(0) = 0, \end{cases} \tag{3-4}$$

where $\tau > 0$ and $0 < R < \infty$. Our aim is to find a solution satisfying (3-4) and

$$\lim_{r \to R} g(r) = +\infty. \tag{3-5}$$

LEMMA 3.1. *Let $0 < R < \infty$ and $q > h > 1$ be given. Then, there exists $\tau > 0$ (depending on R, q, and h) such that problem (3-4) admits a nonnegative solution $g \in C^2((0, R)) \cap C^1([0, R))$ satisfying (3-5).*

PROOF. Notice that the above discussion implies $g \in C^2((0, R)) \cap C^1([0, R))$. Since g^q is nonnegative and $g'(0) = 0$, we integrate (3-3) on $(0, r)$, and then we get g' is nonnegative in $(0, R)$. Next we multiply both sides of the differential equation (3-3) by g' and integrate on $(0, r)$. We get

$$g'(r) = \Big[\frac{h + 1}{q + 1}(g(r)^{q+1} - g(0)^{q+1})\Big]^{1/(h+1)}.$$

Therefore, an implicit solution of the initial value problem (3-4) is

$$\int_\tau^{g(r)} \frac{ds}{(s^{q+1} - \tau^{q+1})^{1/(h+1)}} = \Big(\frac{h + 1}{q + 1}\Big)^{1/(h+1)} r \quad \text{for every } r \in (0, R). \tag{3-6}$$

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Using the change of variables $k^{q+1} = s^{q+1} - \tau^{q+1}$, we get

$$ds = k^q \cdot (k^{q+1} + \tau^{q+1})^{-q/(q+1)}\, dk$$

and

$$\int_\tau^g \frac{ds}{(s^{q+1} - \tau^{q+1})^{1/(h+1)}} = \int_0^v \frac{k^{(qh-1)/(h+1)}}{(k^{q+1} + \tau^{q+1})^{q/(q+1)}}\, dk,$$

where $v = (g^{q+1} - \tau^{q+1})^{1/(q+1)}$. Since $q > h$, this integral is finite for any $v > 0$. Thus for any $g > \tau$, we have

$$\begin{aligned}\lim_{v\to\infty} \int_0^v \frac{k^{(qh-1)/(h+1)}}{(k^{q+1} + \tau^{q+1})^{q/(q+1)}}\, dk &\le \int_0^\tau \tau^{-q} k^{(qh-1)/(h+1)}\, dk + \lim_{v\to\infty} \int_\tau^v k^{-(q+1)/(h+1)}\, dk \\ &= \Big(\frac{h+1}{h(q+1)} + \frac{h+1}{q-h}\Big)\tau^{-(q-h)/(h+1)}\end{aligned}$$

and

$$\begin{aligned}\lim_{v\to\infty} \int_0^v \frac{k^{(qh-1)/(h+1)}}{(k^{q+1} + \tau^{q+1})^{q/(q+1)}}\, dk &\ge \int_0^\tau \frac{k^{(qh-1)/(h+1)}}{2^{q/(q+1)}\tau^q}\, dk \\ &= 2^{-q/(q+1)} \frac{h+1}{h(q+1)} \tau^{-(q-h)/(h+1)}.\end{aligned}$$

By continuity, this means that for any fixed $\tau > 0$, the function

$$g \longmapsto \int_\tau^{g(r)} \frac{ds}{(s^{q+1} - \tau^{q+1})^{1/(h+1)}}$$

is a bijection from $[\tau, \infty)$ to $[0, \ell_\tau)$ for some constant $\ell_\tau > 0$. Moreover, $\ell_\tau \to 0$ as $\tau \to \infty$ and $\ell_\tau \to \infty$ as $\tau \to 0$. Choosing τ such that

$$\ell_\tau = \Big(\frac{h+1}{q+1}\Big)^{1/(h+1)} R,$$

we get the implicit function defined in (3-6) satisfying (3-4) and (3-5). □

REMARK 3.2. During the above procedure, we have obtained

$$C_1 \tau^{-(q-h)/(h+1)} \le \int_\tau^\infty \frac{ds}{(s^{q+1} - \tau^{q+1})^{1/(h+1)}} \le C_2 \tau^{-(q-h)/(h+1)},$$

where $C_1 = 2^{-q/(q+1)}(h+1)/(h(q+1))$ and $C_2 = (h+1)/(h(q+1)) + (h+1)/(q-h)$. This implies

$$C_1 \tau^{-(q-h)/(h+1)} \le \ell_\tau \le C_2 \tau^{-(q-h)/(h+1)}.$$

Since $g(0) = \tau$ and

$$\ell_\tau = \Big(\frac{h+1}{q+1}\Big)^{1/(h+1)} R,$$

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we get

$$\widetilde{C_1}R^{-(h+1)/(q-h)} \le g(0) \le \widetilde{C_2}R^{-(h+1)/(q-h)}, \tag{3-7}$$

where $\widetilde{C_1}$ and $\widetilde{C_2}$ are positive constants, which depend only on q. Moreover, the solution g obtained as above satisfies

$$\widetilde{C_1}(R-r)^{-(h+1)/(q-h)} \le g(r) \le \widetilde{C_2}(R-r)^{-(h+1)/(q-h)}, \quad r \in (0, R).$$

Indeed, for $0 < r < R$, $\varepsilon > 0$, let $\widetilde{g}(s)$ be the solution of (3-4) in the interval $(0, r + R_\varepsilon)$, where $R_\varepsilon = R - r - \varepsilon$. Since $\widetilde{g}(s)$ blows up on the boundary of the ball $B_{r+R_\varepsilon}(x_0)$, that is,

$$\lim_{s\to r+R_\varepsilon} \widetilde{g}(s) = +\infty,$$

the comparison principle implies $g \le \widetilde{g}$ in the interval $(0, r + R_\varepsilon)$. Especially, we have

$$g(r) \le \widetilde{g}(r) \le \widetilde{C_2}R_\varepsilon^{-(h+1)/(q-h)},$$

where we use (3-7). Letting $\varepsilon \to 0$, we have $g(r) \le \widetilde{C_2}(R-r)^{-(h+1)/(q-h)}$. For the lower bound estimate, see Remark 3.4.

LEMMA 3.3. *There exists a positive viscosity solution of problem (3-1).*

PROOF. Set

$$U(x) := g(|x - x_0|), \tag{3-8}$$

where g and τ are defined in Lemma 3.1. We want to show that the function U in (3-8) is the desired viscosity solution of the problem (3-1).

Noticing that $U \in C^2(B_R(x_0) \setminus \{x_0\}) \cap C^1(B_R(x_0))$, we have that U is a classical solution in $B_R(x_0) \setminus \{x_0\}$, which implies U is also a viscosity solution in the ball $B_R(x_0)$ except x_0.

Now we show that U is indeed a viscosity solution at x_0. Let $\varphi \in C^2(B_R(x_0))$ be such that φ touches U from below at $x_0 \in B_R(x_0)$. Noting that $D\varphi(x_0) = DU(x_0) = 0$, we have $\Delta_\infty^h \varphi(x_0) = 0 \le U(x_0)^q$. Therefore, U is a viscosity supersolution at x_0.

Next we show that U is a viscosity subsolution at x_0. Consider the test function $\varphi \in C^2(B_R(x_0))$ such that $U - \varphi$ has a local maximum at $x_0 \in B_R(x_0)$. Since $D\varphi(x_0) = DU(x_0) = 0$, we have

$$\begin{aligned} U(x) - U(x_0) &\le \varphi(x) - \varphi(x_0) \\ &= \tfrac{1}{2}\langle D^2\varphi(x_0)(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \quad \text{as } |x - x_0| \to 0. \end{aligned}$$

Taking $x - x_0 = t\vec{e}$, where $\vec{e}$ is a unit vector and $t > 0$, we get

$$U(x_0 + t\vec{e}) - U(x_0) \le \tfrac{1}{2}\langle D^2\varphi(x_0)\vec{e}, \vec{e}\rangle t^2 + o(t^2) \quad \text{as } t \to 0.$$

This is equivalent to

$$g(t) - g(0) \le \tfrac{1}{2}\langle D^2\varphi(x_0)\vec{e}, \vec{e}\rangle t^2 + o(t^2) \quad \text{as } t \to 0. \tag{3-9}$$

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Multiplying by $g'(r)$ in (3-3) and integrating on $(0, r)$, we get

$$\frac{1}{h+1}|g'(r)|^{h+1} = \int_{\tau}^{g(r)} u^q du \geq \tau^q(g(r) - \tau).$$

Since $g'(r)$ is nonnegative in $(0, R)$, one has

$$g'(r)[(h+1)(g(r) - \tau)]^{-1/(h+1)} \geq \tau^{q/(h+1)}.$$

Integrating this again on $(0, r)$, we have

$$g(r) - \tau \geq Cr^{(h+1)/h}, \quad r \in (0, R),$$

where C is a positive constant. Then we have

$$Ct^{(h+1)/h} \leq g(t) - \tau = g(t) - g(0),$$

which contradicts (3-9). Hence, U is a viscosity subsolution at x_0. □

REMARK 3.4. One can verify that

$$u_R(x) := \left(\left(\frac{h+1}{q-h}\right)^h \frac{q+1}{q-h}\right)^{1/(q-h)} (R - |x - x_0|)^{-(h+1)/(q-h)}$$

is a classical solution of (3-2) in $B_R(x_0) \setminus \{x_0\}$. Noticing $q > h$, we have that $\lim_{x \to \partial B_R(x_0)} u_R(x) = \infty$. Since u_R is not differentiable at $x = x_0$, we have u_R is a classical boundary blow-up solution of (3-1) in $B_R(x_0) \setminus \{x_0\}$. Additionally, u_R is a viscosity subsolution of (3-1) in $B_R(x_0)$ since there are no test functions touching u_R from above at x_0. However, it is not a viscosity supersolution at x_0 because one can test it from below by test functions with small but nonzero gradient and any Hessian.

REMARK 3.5. If u is a viscosity solution of (3-1) in $B_R(x_0)$, then $u_R \leq u$. In fact, for any $\varepsilon > 0$, we have $u_{R+\varepsilon} \leq u$ in $B_R(x_0)$ based on the comparison principle. Therefore, $u_{R+\varepsilon}$ converges locally uniformly to u as $\varepsilon \to 0$.

Next, we study the following problem:

$$\begin{cases} \Delta_\infty^h u = u^q & \text{in } \Omega, \\ u(x) = M & \text{on } \partial\Omega, \end{cases} \tag{3-10}$$

where $M > 1$ is a fixed constant. Since we want to study the boundary blow-up solution of the problem (1-1), in fact, we are interested in M sufficiently large.

THEOREM 3.6. *For any $M > 1$, there exists a unique, nonnegative viscosity solution $u_M \in C(\overline{\Omega})$ to (3-10).*

PROOF. The uniqueness follows immediately by the comparison principle, Theorem 2.5. The proof of the existence relies on Perron's method applied to viscosity solutions. It is obvious that $\overline{u} = M$ is a viscosity supersolution of (3-10).

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Next we want to construct a viscosity subsolution with appropriate boundary value. For $z \in \partial\Omega$, let

$$v_z(x) = M - C|x - z|^\alpha, \quad x \in \Omega,$$

where $\alpha \in (0, 1)$ and $C \geq 1$ is to be determined. One can verify that

$$\Delta_\infty^h v_z(x) = (1-\alpha)(\alpha C)^h |x - z|^{h(\alpha-1)-1} \geq (1-\alpha)\alpha^h |x - z|^{h(\alpha-1)-1}.$$

Choosing $0 < \delta = M^q/((1-\alpha)\alpha^h)^{1/(h(\alpha-1)-1)} < 1$ and $C = (\delta/2)^{-\alpha} M$, we get $v_z \leq 0$ outside $B_{\delta/2}(z)$ and

$$\Delta_\infty^h v_z(x) \geq M^q \geq v_z^q(x) \quad \text{for all } x \in B_\delta(z).$$

Then the function

$$\underline{u}(x) = \max\{0, \sup_{z\in\partial\Omega}(M - C|x - z|^\alpha)\}$$

is the desired viscosity subsolution. Hence, using the standard Perron method, we can obtain the existence of a nonnegative viscosity solution u_M to the problem (3-10). □

REMARK 3.7. From the above proof, we can see that Theorem 3.6 still holds for any $q > 0$, not just for $q > h$. We use this fact in the proof of the nonexistence of boundary blow-up solutions to (1-1) for $0 < q \leq h$ in Section 5.

With the aid of Theorem 3.6 and the compactness method, we can establish the existence of boundary blow-up viscosity solutions to (1-1). To establish the existence of large solutions, a difficulty with respect to the degenerate operators is the lack of existence of barriers. However, thanks to the special structure of the operator Δ_∞^h, we can construct appropriate barriers without assuming any regularity of the boundary of the domain.

Proof of Theorem 1.1. By the comparison principle, we have that the sequence $\{u_M\}$ obtained in Theorem 3.6 is increasing. Hence, the limit function $u_\infty : \overline{\Omega} \to [0, \infty]$

$$u_\infty(x) := \lim_{M\to\infty} u_M(x)$$

exists. Next we show that u_∞ is finite in Ω and gives the desired solution to (1-1) for $q > h$.

Fix $x_0 \in \Omega$ and $r > 0$ so that $B_r(x_0) \subset \Omega$. By Lemma 3.3, there exists a radial function U as constructed in (3-8) satisfying

$$\begin{cases} \Delta_\infty^h U(x) = U^q(x), & x \in B_r(x_0), \\ \lim_{x\to z} U(x) = \infty, & z \in \partial B_r(x_0). \end{cases}$$

The comparison principle implies $u_M \leq U$ in $B_r(x_0)$ for every $M > 1$. Hence, $u_\infty \leq U$ in $B_r(x_0)$. This means that u_∞ is locally bounded in Ω.

Since $\Delta_\infty^h u_M = u_M^q \geq 0$ in the viscosity sense and $\{u_M\}$ is locally uniformly bounded, the Lipschitz continuity of u_M (see for example Lemma 2.9 in [3]) implies that $\{u_M\}$

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is also equicontinuous. Therefore, u_M converges to u_∞ locally uniformly as $M \to \infty$. Then we have u_∞ is a viscosity solution of $\Delta_\infty^h u = u^q$ in Ω by the stability theory of viscosity solutions [12].

Next we show that $\lim_{x\to z} u_\infty(x) = \infty$ for all $z \in \partial\Omega$. Consider the barrier function

$$w(x) = H_q(|x-z|+\varepsilon)^\alpha, \quad x \in \Omega,$$

where $z \in \partial\Omega, \varepsilon > 0$, $\alpha = -(h+1)/(q-h)$, and

$$H_q = \left(\left(\frac{h+1}{q-h}\right)^h \frac{q+1}{q-h}\right)^{1/(q-h)}.$$

Then, a direct calculation yields

$$Dw(x) = \alpha H_q(|x-z|+\varepsilon)^{\alpha-1}\frac{x-z}{|x-z|}$$

and

$$\begin{cases} D^2w(x) = \alpha(\alpha-1)H_q(|x-z|+\varepsilon)^{\alpha-2}\dfrac{x-z}{|x-z|}\otimes\dfrac{x-z}{|x-z|} \\ \qquad\qquad + \alpha H_q(|x-z|+\varepsilon)^{\alpha-1}\left(\dfrac{I}{|x-z|} - \dfrac{(x-z)\otimes(x-z)}{|x-z|^3}\right). \end{cases}$$

Hence, for any $x \in \Omega$, we have

$$\begin{cases} \Delta_\infty^h w(x) &= (|\alpha|H_q(|x-z|+\varepsilon)^{\alpha-1})^{h-1}\alpha(\alpha-1)H_q(|x-z|+\varepsilon)^{\alpha-2} \\ &= |\alpha|^{h-1}\alpha(\alpha-1)H_q^h(|x-z|+\varepsilon)^{(\alpha-1)h-1} \\ &= \left(\dfrac{h+1}{q-h}\right)^h\dfrac{q+1}{q-h}H_q^h(|x-z|+\varepsilon)^{\alpha q} \\ &= H_q^q(|x-z|+\varepsilon)^{\alpha q} \\ &= w^q(x). \end{cases} \tag{3-11}$$

By the comparison principle, we get $u_M \ge w$ in Ω for all $M \ge H_q\varepsilon^{-(h+1)/(q-h)}$. Letting first $M \to \infty$ and then $\varepsilon \to 0$, we have

$$u_\infty(x) \ge H_q|x-z|^{-(h+1)/(q-h)}, \quad x \in \Omega. \tag{3-12}$$

Noticing $q > h$, we obtain $\lim_{x\to z} u_\infty(x) = \infty$ for all $z \in \partial\Omega$. By (3-12), it is obvious that the boundary blow-up solution u_∞ is positive in Ω. This finishes the proof.

4. Boundary asymptotic estimate and uniqueness

In this section, we investigate the boundary asymptotic estimate of the large solutions to problem (1-1). The lower growth estimate can be easily obtained by the proof of the existence, Theorem 1.1. Hence, the key is to establish the upper growth estimate. To reach to this goal, we use the perturbation of the distance function and comparison principle. The idea comes from [19]. However, for $h > 1$, the operator Δ_∞^h is quasi-linear even in dimension 1. Therefore, we must make a subtle analysis.

LEMMA 4.1. *Let* $q > h$, $H_q = (((h+1)/(q-h))^h(q+1)/(q-h))^{1/(q-h)}$ *and* u *be a viscosity solution to (1-1) in a bounded domain* Ω*. Assume that there exists a neighborhood* $\mathcal{N}$ *of* $\partial\Omega$ *such that* $\operatorname{dist}(x,\partial\Omega) \in C^1(\mathcal{N}\cap\Omega)$*. Then there are constants* $\gamma > 0$ *and* $\mu > 0$ *depending only on the domain* Ω *such that*

$$u(x) \le H_q \operatorname{dist}(x,\partial\Omega)^{-(h+1)/(q-h)} + \gamma, \quad x \in \Omega_\mu, \tag{4-1}$$

where $\Omega_\mu := \{x | \ x \in \Omega \text{ and } \operatorname{dist}(x,\partial\Omega) < \mu\}$.

PROOF. Let $\mu > 0$ such that $\Omega_\mu \subset (\mathcal{N}\cap\Omega)$. Denote

$$D_\mu := \{x \mid x \in \Omega \text{ and } \operatorname{dist}(x,\partial\Omega) = \mu\}.$$

Then there exists some $x_0 \in D_\mu$ such that

$$u(x_0) = \sup_{x\in D_\mu} u(x).$$

For any $0 < r < \mu$, let v_r be the radial function satisfying

$$\begin{cases} \Delta_\infty^h v_r(x) = v_r^q(x), & x \in B_r(x_0), \\ \lim\limits_{x\to z} v_r(x) = \infty, & z \in \partial B_r(x_0), \end{cases}$$

in the viscosity sense. The comparison principle implies $u(x_0) \le v_r(x_0)$ for all $r > 0$. Therefore, we have

$$\tau := u(x_0) = \sup_{x\in D_\mu} u(x) \le v_\mu(x_0) := \gamma. \tag{4-2}$$

For simplicity, we denote $d(x) := \operatorname{dist}(x,\partial\Omega)$. Then we have $|Dd(x)| = 1$ for all $x \in \Omega_\mu$. Furthermore, $d(x)$ is a viscosity solution of $\Delta_\infty^h u = 0$ in Ω_μ (see for example [3]).

Now for $0 < \varepsilon < \mu$, set

$$w_\varepsilon(x) := H_q(d(x)-\varepsilon)^\alpha, \quad x \in \Omega_\mu \setminus \overline{\Omega}_\varepsilon,$$

where $\alpha = -(h+1)/(q-h) < 0$. By direct calculations, we have

$$\begin{cases} \Delta_\infty^h w_\varepsilon(x) & = (|\alpha|H_q(d(x)-\varepsilon)^{\alpha-1})^{h-1}\alpha(\alpha-1)H_q(d(x)-\varepsilon)^{\alpha-2} \\ & \quad +(|\alpha|H_q(d(x)-\varepsilon)^{\alpha-1})^{h-1}\alpha H_q(d(x)-\varepsilon)^{\alpha-1}\Delta_\infty^h d(x) \\ & = |\alpha|^{h-1}\alpha(\alpha-1)H_q^h(d(x)-\varepsilon)^{(\alpha-1)h-1} \\ & = \left(\dfrac{h+1}{q-h}\right)^h \dfrac{q+1}{q-h} H_q^h(d(x)-\varepsilon)^{\alpha q} \\ & = H_q^q(d(x)-\varepsilon)^{\alpha q} \\ & = w_\varepsilon^q(x) \end{cases}$$

in the viscosity sense in $\Omega_\mu \setminus \overline{\Omega}_\varepsilon$. Indeed, one can see that if $w_\varepsilon(x) - \varphi(x)$ has a local maximum (respectively minimum) at some point $\hat{x} \in \Omega_\mu \setminus \overline{\Omega}_\varepsilon$, then $d(x) - \psi(x)$ with $\psi(x) = (1/H_q\varphi(x))^{1/\alpha} + \varepsilon$ has a local maximum (respectively minimum) at $\hat{x}$. Since

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$d(x)$ is a viscosity solution of $\Delta_\infty^h u = 0$ in Ω_μ, we obtain that w_ε verifies $\Delta_\infty^h w_\varepsilon = w_\varepsilon^q$ in the viscosity sense in $\Omega_\mu \setminus \overline{\Omega}_\varepsilon$.

Noticing $q > h > 1$, we get

$$\Delta_\infty^h (w_\varepsilon + \gamma) \le (w_\varepsilon + \gamma)^q$$

in the viscosity sense in $\Omega_\mu \setminus \overline{\Omega}_\varepsilon$. By (4-2), we have

$$u(x) \le \tau \le \gamma \le w_\varepsilon(x) + \gamma, \quad x \in D_\mu.$$

Since

$$\lim_{x \to z \in D_\varepsilon} w_\varepsilon(x) = \infty,$$

the comparison principle implies

$$u(x) \le w_\varepsilon(x) + \gamma, \quad x \in \Omega_\mu \setminus \overline{\Omega}_\varepsilon.$$

Letting $\varepsilon \to 0$, we have

$$u(x) \le H_q \operatorname{dist}(x, \partial\Omega)^{-(h+1)/(q-h)} + \gamma, \quad x \in \Omega_\mu. \qquad \square$$

REMARK 4.2. By Remark 3.2 and the proof above, we actually have that $\gamma \approx \mu^{-(h+1)/(q-h)}$. In particular, there is a constant $C > 0$ depending only on q such that $\gamma \le C\mu^{-(h+1)/(q-h)}$.

With Lemma 4.1 in hand, we can immediately establish the asymptotic estimate.

Proof of Theorem 1.2. By Lemma 4.1, we have

$$u(x) \le H_q \operatorname{dist}(x, \partial\Omega)^{-(h+1)/(q-h)} + \gamma, \quad x \in \Omega_\mu. \tag{4-3}$$

Recall that in the proof of Theorem 1.1, we have (3-12). That is, if u is a viscosity solution to (1-1) in a bounded domain Ω, then there holds

$$u(x) \ge H_q |x - z|^{-(h+1)/(q-h)} \quad \text{for all } x \in \Omega,$$

where $z \in \partial\Omega$ and $H_q = (((h+1)/(q-h))^h (q+1)/(q-h))^{1/(q-h)}$. Hence, we can immediately get the lower growth estimate

$$u(x) \ge H_q \operatorname{dist}(x, \partial\Omega)^{-(h+1)/(q-h)} \quad \text{for all } x \in \Omega. \tag{4-4}$$

By the inequalities (4-3), (4-4), and Remark (4.2), we get the asymptotic behavior estimate (1-2).

Finally, we give the uniqueness of the boundary blow-up viscosity solution. We argue it by contradiction. Suppose that u and v are both viscosity solutions to (1-1). By the inequality (4-4) and Lemma 4.1, we have

$$\frac{u(x)}{v(x)} \le \frac{H_q \operatorname{dist}(x, \partial\Omega)^{-(h+1)/(q-h)} + \gamma}{H_q \operatorname{dist}(x, \partial\Omega)^{-(h+1)/(q-h)}}, \quad x \in \Omega_\mu.$$

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Therefore,

$$\limsup \frac{u(x)}{v(x)} \leq 1 \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

By the comparison principle, Theorem 3.6, we obtain $u(x) \leq v(x)$ in the whole Ω. Similarly, we can obtain $v(x) \leq u(x)$ in the whole Ω. Hence, the uniqueness is completed.

REMARK 4.3. One should notice that the assumption condition on the domain in Lemma 4.1 holds for all C^2-domains [17] and Lipschitz domains satisfying a uniform interior ball condition [18]. In fact, the condition $\operatorname{dist}(x, \partial\Omega) \in C^1(\mathcal{N} \cap \Omega)$ is equivalent to the geometric condition that for any $x \in \mathcal{N} \cap \Omega$, there exists a unique $z \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = |x - z|$.

5. Nonexistence for $q \leq h$

In this section, we show that problem (1-1) has no positive viscosity solution when $q \leq h$. We consider two cases, $q \leq 0$ and $0 < q \leq h$.

THEOREM 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $q \leq 0$. Then problem (1-1) has no positive viscosity solution.*

PROOF. Suppose that there is a positive solution u to (1-1). Since $q \leq 0$, there exists $C > 0$ such that

$$0 \leq u^q(x) \leq C, \quad x \in \Omega.$$

In particular, we have that u is a viscosity supersolution of $\Delta_\infty^h u = C$ in Ω. We introduce

$$w(x) := L + \frac{h}{h+1}(Ch)^{1/h}|x - x_0|^{(h+1)/h}, \quad x \in \Omega,$$

where $L \in \mathbb{R}$ is an arbitrary constant and $x_0 \in \mathbb{R}^n \backslash \Omega$. Then we can verify that $\Delta_\infty^h w = C$ in Ω. Since $\lim_{x \to z} u(x) = \infty$ for all $z \in \partial\Omega$, we have $w \leq u$ in Ω by the comparison principle (see [25]). Letting $L \to \infty$, we get $u \equiv \infty$ which is a contradiction. □

To obtain the nonexistence of a large solution for the case $0 < q \leq h$, we need the following maximum/minimum principle which can be derived from Harnack's inequality [3, 10].

LEMMA 5.2. *Let $u \in C(\overline{\Omega})$ satisfy $\Delta_\infty u \geq 0$ (≤ 0) in the viscosity sense. Then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \quad (\inf_{\Omega} u = \inf_{\partial\Omega} u).$$

Moreover, the supremum (infimum) occurs only on the set $\partial\Omega$ unless u is a constant.

THEOREM 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $0 < q \leq h$. Then problem (1-1) has no positive viscosity solution.*

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PROOF. We argue it by contradiction. Assume that there is a viscosity solution u to (1-1) in Ω. For $x_0 \in \partial\Omega$, there exist $r > 0$ and $c > 1$ such that $u > c$ in $\Omega_0 := B(x_0, r) \cap \Omega$.

Denote

$$U(x) = \log u(x), \quad x \in \Omega_0.$$

By direct calculation, we have

$$DU(x) = \frac{1}{u(x)} Du(x),$$
$$D^2U(x) = -\frac{1}{u^2(x)} Du(x) \otimes Du(x) + \frac{1}{u(x)} D^2u(x),$$

where $\otimes$ denotes the tensor product. Then one can verify that

$$\Delta_\infty^h U(x) = -\Big(\frac{1}{u(x)}\Big)^{h+1} |Du(x)|^{h+1} + \Big(\frac{1}{u(x)}\Big)^h \Delta_\infty^h u(x) \le \Big(\frac{1}{u(x)}\Big)^h (u(x))^q \le 1 \quad \text{in } \Omega_0$$

in the viscosity sense, where we use $0 < q \le h$. Since $u = \infty$ on $\partial\Omega$, we have that $U = \infty$ on $B(x_0, r) \cap \partial\Omega$.

Let g be a continuous function supported on $\partial\Omega_0$ and

$$g(x) = \begin{cases} 1, & x \in B\Big(x_0, \dfrac{r}{3}z\Big) \cap \partial\Omega, \\ 0, & x \in \partial\Omega_0 \setminus \Big(B\Big(x_0, \dfrac{r}{2}\Big) \cap \partial\Omega\Big). \end{cases}$$

The existence and uniqueness of a viscosity solution v of the following problem:

$$\begin{cases} \Delta_\infty^h v = 1 & \text{in } \Omega_0, \\ v = g & \text{on } \partial\Omega_0, \end{cases}$$

were obtained in [22, 25]. Due to the h-homogeneity of the operator Δ_∞^h, for any constant $k \ge 1$, the function $v_k(x) := kv(x)$ satisfies

$$\begin{cases} \Delta_\infty^h v_k = k^h \ge 1 & \text{in } \Omega_0, \\ v_k = kg & \text{on } \partial\Omega_0, \end{cases}$$

in the viscosity sense. By the comparison principle in [25], there holds $v_k \le U$ in Ω_0 for any $k \ge 1$. Since $\Delta_\infty^h v = |Dv|^{h-3}\Delta_\infty v = 1 > 0$, by Lemma 5.2, there exists $x_1 \in \Omega_0$ such that $v(x_1) > 0$. Then we have $v_k(x_1) = kv(x_1) \le U(x_1)$ for all $k \ge 1$ which leads to a contradiction. □

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References

[1] E. Abderrahim, D. Xavier, L. Zakaria and L. Olivier, 'Nonlocal infinity Laplacian equation on graphs with applications in image processing and machine learning', *Math. Comput. Simulation* **102** (2014), 153–163.

[2] G. Aronsson, 'Extension of functions satisfying Lipschitz conditions', *Ark. Mat.* **6** (1967), 551–561.

[3] G. Aronsson, M. G. Crandall and P. Juutinen, 'A tour of the theory of absolutely minimizing functions', *Bull. Amer. Math. Soc. (N.S.)* **41** (2004), 439–505.

[4] T. Bhattacharya and A. Mohammed, 'On solutions to Dirichlet problems involving the infinity-Laplacian', *Adv. Calc. Var.* **4** (2011), 445–487.

[5] T. Bhattacharya and A. Mohammed, 'Inhomogeneous Dirichlet problems involving the infinity-Laplacian', *Adv. Differential Equations* **17**(3–4) (2012), 225–266.

[6] V. Caselles, J. M. Morel and C. Sbert, 'An axiomatic approach to image interpolation', *IEEE Trans. Image Process* **7** (1998), 376–386.

[7] F. Cîrstea and V. Rădulescu, 'Uniqueness of the blow-up boundary solution of logistic equations with absorption', *C. R. Math. Acad. Sci. Paris, Sér. I* **335** (2002), 447–452.

[8] F. Cîrstea and V. Rădulescu, 'Asymptotics for the blow-up boundary solution of the logistic equation with absorption', *C. R. Math. Acad. Sci. Paris, Sér. I* **336** (2003), 231–236.

[9] F. Cîrstea and V. Rădulescu, 'Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach', *Asymptot. Anal.* **46** (2006), 275–298.

[10] M. G. Crandall, 'A visit with the ∞-Laplacian', in: *Calculus of Variations and Non-linear Partial Differential Equations (CIME, Cetraro Italy, 2005)*, Lecture Notes in Mathematics, 1927 (Springer, Berlin, 2008), 75–122.

[11] M. G. Crandall, L. C. Evans and R. F. Gariepy, 'Optimal Lipschitz extensions and the infinity Laplacian,' *Calc. Var. Partial Differential Equations* **13** (2001), 123–139.

[12] M. G. Crandall, H. Ishii and P. L. Lions, 'User's guide to viscosity solutions of second order partial differential equations', *Bull. Amer. Math. Soc. (N. S.)* **27** (1992), 1–67.

[13] A. Elmoataz, M. Toutain and D. Tenbrinck, 'On the p-Laplacian and ∞-Laplacian on graphs with applications in image and data processing', *SIAM J. Imaging Sci.* **8**(4) (2015), 2412–2451.

[14] L. C. Evans and W. Gangbo, 'Differential equations methods for the Monge–Kantorovich mass transfer problem', *Mem. Amer. Math. Soc.* **137**(653) (1999), viii+66 pages.

[15] J. M. Fraile, P. Medina, J. López-Gómez and S. Merino, 'Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear equation', *J. Differential Equations* **127** (1996), 295–319.

[16] J. García-Melián, R. Gómez-Reñasco, J. López-Gómez and J. Sabinade, 'Pointwise growth and uniqueness of positive solutions for a class of sublinear elliptic problems where bifurcation from infinity occurs,' *Arch. Ration. Mech. Anal.* **145** (1998), 261–289.

[17] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Reprint of the 1998 edition, Classics in Mathematics, 224 (Springer-Verlag, Berlin, 2001).

[18] P. Juutinen, 'The boundary Harnack inequality for infinity harmonic functions in Lipschitz domains satisfying the interior ball condition', *Nonlinear Anal.* **69**(7) (2008), 1941–1944.

[19] P. Juutinen and J. D. Rossi, 'Large solutions for the infinity Laplacian', *Adv. Calc. Var.* **1**(3) (2008), 271–289.

[20] J. M. Lasry and P. L. Lions, 'Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem', *Math. Ann.* **283** (1989), 583–630.

[21] F. Liu, 'An inhomogeneous evolution equation involving the normalized infinity Laplacian with a transport term', *Commun. Pure Appl. Anal.* **17**(6) (2018), 2395–2421.

[22] F. Liu and X. Yang, 'Solutions to an inhomogeneous equation involving infinity-Laplacian', *Nonlinear Anal.* **75** (2012), 5693–5701.

[23] Q. Liu and A. Schikorra, 'General existence of solutions to dynamic programming equations', *Commun. Pure Appl. Anal.* **14** (2015), 167–184.

https://doi.org/10.1017/S1446788722000131 Published online by Cambridge University Press

[24] R. López-Soriano, J. C. Navarro-Climent and J. D. Rossi, 'The infinity Laplacian with a transport term', *J. Math. Anal. Appl.* **398** (2013), 752–765.
[25] G. Lu and P. Wang, 'A uniqueness theorem for degenerate elliptic equations', *Semin. Interdiscip. Mat.* **7** (2008), 207–222.
[26] G. Lu and P. Wang, 'Inhomogeneous infinity Laplace equation', *Adv. Math.* **217**(4) (2008), 1838–1868.
[27] G. Lu and P. Wang, 'A PDE perspective of the normalized infinity Laplacian,' *Comm. Partial Differential Equations* **33**(10) (2008), 1788–1817.
[28] L. Mi, 'Blow-up rates of large solutions for infinity Laplace equations', *Appl. Math. Comput.* **298** (2017), 36–44.
[29] A. Mohammed and S. Mohammed, 'On boundary blow-up solutions to equations involving the ∞-Laplacian', *Nonlinear Anal.* **74** (2011), 5238–5252.
[30] A. Mohammed and S. Mohammed, 'Boundary blow-up solutions to degenerate elliptic equations with non-monotone inhomogeneous terms', *Nonlinear Anal.* **75** (2012), 3249–3261.
[31] T. Ouyang, 'On the positive solutions of semilinear equation $\Delta u + \lambda u - hu^p = 0$ on the compact manifolds', *Trans. Amer. Math. Soc.* **331** (1992), 503–527.
[32] Y. Peres, G. Pete and S. Somersille, 'Biased tug-of-war, the biased infinity Laplacian, and comparison with exponential cones', *Calc. Var. Partial Differential Equations* **38**(3–4) (2010), 541–564.
[33] Y. Peres, O. Schramm, S. Sheffield and D. Wilson, 'Tug-of-war and the infinity Laplacian', *J. Amer. Math. Soc.* **22**(1) (2009), 167–210.
[34] M. Portilheiro and J. L. Vázquez, 'Degenerate homogeneous parabolic equations associated with the infinity-Laplacian', *Calc. Var. Partial Differential Equations* **46**(3–4) (2013), 705–724.
[35] J. D. Rossi, 'Tug-of-war games and PDEs', *Proc. Roy. Soc. Edinburgh Sect. A* **141(2)** (2011), 319–369.
[36] W. Wang, H. Gong and S. Zheng, 'Asymptotic estimates of boundary blow-up solutions to the infinity Laplace equations', *J. Differential Equations* **256** (2014), 3721–3742.
[37] Z. Zhang, 'Boundary behavior of large viscosity solutions to infinity Laplace equations', *Z. Angew. Math. Phys.* **66** (2015), 1453–1472.

CUICUI LI, Department of Mathematics,
School of Mathematics and Statistics, Nanjing University of Science and Technology,
Nanjing 210094, Jiangsu, PR China
e-mail: licui1121@njust.edu.cn

FANG LIU, Department of Mathematics,
School of Mathematics and Statistics, Nanjing University of Science and Technology,
Nanjing 210094, Jiangsu, PR China
e-mail: sdqdlf78@126.com, liufang78@njust.edu.cn

PEIBIAO ZHAO, Department of Mathematics,
School of Mathematics and Statistics, Nanjing University of Science and Technology,
Nanjing 210094, Jiangsu, PR China
e-mail: pbzhao@njust.edu.cn

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