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LOCALIZATIONS OF INJECTIVE MODULES

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The question of whether an injective module E over a noncommutative noetherian ring R remains injective after localization with respect to a denominator set $X \subseteq R$ is addressed. (For a commutative noetherian ring, the answer is well-known to be positive.) Injectivity of the localization $E[X^{-1}]$ is obtained provided either R is fully bounded (a result of K. A. Brown) or X consists of regular normalizing elements. In general, $E[X^{-1}]$ need not be injective, and examples are constructed. For each positive integer n, there exists a simple noetherian domain R with Krull and global dimension $n+1$, a left and right denominator set X in R, and an injective right R-module E such that $E[X^{-1}]$ has injective dimension n; moreover, E is the injective hull of a simple module.

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1. Preservation of injectivity

Given a right or left denominator set X in a ring R, we write $t_X(E)$ and $E[X^{-1}]$ for the X-torsion submodule of an R-module E and the X-localization of E. Assuming that E is an injective R-module, we consider the problem of deciding whether $E[X^{-1}]$ must be an injective $R[X^{-1}]$-module. Recall that $E[X^{-1}]$ is injective as an $R[X^{-1}]$-module if and only if it is injective as an R-module [**11**, Exercise 12, p. 62].

In case R is commutative noetherian, or, more generally, if R is noetherian and X is central, $E[X^{-1}]$ must be injective [**2**, Lemma 1.2]. However, for R commutative but not noetherian, $E[X^{-1}]$ need not be injective [**5**, Theorems 25, 28]. In the noncommutative fully bounded noetherian case, K. A. Brown has proved the following positive result, and we thank him for communicating it for presentation here.

Theorem 1.1. (Brown) *Let R be a right and left noetherian right fully bounded ring, let X be a right and left denominator set in R, and let E be an injective right R-module. Then $E[X^{-1}]$ is an injective right $R[X^{-1}]$-module.*

Proof. It suffices to consider the case that E is indecomposable. If E is X-torsion, then $E[X^{-1}]=0$, while if E is X-torsion-free, then $E[X^{-1}]=E$; in either case, $E[X^{-1}]$ is injective. Hence, it is enough to show that E is either X-torsion or X-torsion-free.

Let $T=t_X(E)$ and suppose that $T\neq 0$ and $T\neq E$. Since $T\neq E$ we may choose a finitely generated submodule B of E such that $B\nsubseteq T$ and $\operatorname{ann}(B)$ is maximal among the annihilators of such submodules. Set $C=B\cap T$ and $Q=\operatorname{ann}(B/C)$; thus $BQ\subseteq T$. By the H-condition [3, Theorem 7.8], the annihilator of BQ equals the annihilator of some finite subset of BQ, and hence there exists $x\in X$ such that $BQx=0$. It follows that Q is prime, for if I and J are ideals of R such that $I\nsubseteq Q$ and $IJ\subseteq Q$, then $BIJx=0$ and Jx is contained in $\operatorname{ann}(BI)$, which, by maximality, equals $\operatorname{ann}(B)$; then $BJx=0$ and so $BJ\subseteq \mathrm{T}$, whence $J\subseteq Q$.

The image $\bar{X}$ of X in $R/\operatorname{ann}(B)$ is a left Ore set, and, because $R/\operatorname{ann}(B)$ is left noetherian, $\bar{X}$ must be a left denominator set [11, Proposition II.1.5]. Hence, since $Qx\subseteq\operatorname{ann}(B)$, there exists $y\in X$ such that $yQ\subseteq\operatorname{ann}(B)$, and $ByQ=0$. But $ByR\nsubseteq T$ because B is not X-torsion, and so, by maximality of $\operatorname{ann}(B)$, we obtain $\operatorname{ann}(ByR)=\operatorname{ann}(B)$. Thus $Q\subseteq\operatorname{ann}(B)$, and so $Q=\operatorname{ann}(B)$.

By the H-condition, R/Q embeds in a finite direct sum of copies of B. Let U be a uniform right ideal of R/Q. There is a finite set of homomorphisms $U\to B$ whose kernels intersect to zero, and hence one of these maps is injective, so that B has a submodule D isomorphic to U. Since E is indecomposable, $D\cap T\neq 0$. It follows that $t_X(U)\neq 0$ and hence that $t_X(R/Q)\neq 0$.

But $t_X(R/Q)$ is an ideal of R/Q and so, since Q is prime, $t_X(R/Q)$ must contain a regular element c. As $cy=0$ for some $y\in X$, we conclude that $y\in Q$ and $By=0$, so that B is X-torsion. This contradicts the choice of B.

Therefore E is either X-torsion or X-torsion-free, as desired. □

Another case in which localizations of injective modules are injective is that of a denominator set consisting of regular normalizing elements. (Recall that a *normalizing element* in a ring R is any element $c\in R$ such that $cR=Rc$.)

Lemma 1.2. *Let x be a regular normalizing element in a ring R, let E be an injective right R-module, and set $A=\{a\in E\,|\,ax=0\}$. Then E/A is an injective right R-module.*

Proof. That A is a submodule of E follows because x is a normalizing element. Since x is regular, there exist homomorphisms $xR\to E$ sending x to any element of E; by injectivity, $Ex=E$.

For any $r\in R$, there is a unique element $\varphi(r)\in R$ such that $xr=\varphi(r)x$. Observe that φ is a ring automorphism of R.

Now right multiplication by x defines an abelian group epimorphism $E\to E$ with kernel A. This induces an abelian group isomorphism $f:E/A\to E$ such that $fq(b)=bx$ for all $b\in E$, where $q:E\to E/A$ is the quotient map. For all $b\in E$ and $r\in R$, we compute that

$$f(q(b)\varphi(r))=fq(b\varphi(r))=b\varphi(r)x=bxr=fq(b)r.$$

Hence, $f(c\varphi(r))=f(c)r$ for all $c\in E/A$ and all $r\in R$.

Let J be a right ideal of R and $g:J\to E/A$ an R-module homomorphism. Then $\varphi^{-1}(J)$ is a right ideal of R and $fg\varphi$ is a group homomorphism from $\varphi^{-1}(J)$ to E. For all $t\in J$ and $r\in R$, we check that

$$fg\varphi(\varphi^{-1}(t)r)=fg(t\varphi(r))=f(g(t)\varphi(r))=fg(t)r=fg\varphi(\varphi^{-1}(t))r,$$

so that $fg\varphi$ is an R-module homomorphism. Hence, there exists $b\in E$ such that $fg\varphi(s)=bs$ for all $s\in\varphi^{-1}(J)$. Also, $b=f(c)$ for some $c\in E/A$. For all $t\in J$, we have

$$fg(t)=fg\varphi(\varphi^{-1}(t))=b\varphi^{-1}(t)=f(c)\varphi^{-1}(t)=f(ct),$$

whence $g(t)=ct$.

Therefore E/A is injective. □

Theorem 1.3. *Let R be a right noetherian ring, let X be a right denominator set of regular normalizing elements of R, and let E be an injective right R-module. Then $E[X^{-1}]$ is an injective right $R[X^{-1}]$-module.*

Proof. As X consists of regular elements, E is X-divisible. Hence, $E[X^{-1}]=E/A$, where $A=t_X(E)$. Set $A_x=\{a\in E \mid ax=0\}$ for all $x\in X$, and note that A is the union of the submodules A_x. Given any $x,y\in X$, there exist $r\in R$ and $z\in X$ such that $xr=yz$, whence $yz\in X$ and $A_x\cup A_y\subseteq A_{yz}$. Thus A is a directed union of the A_x.

Now $E[X^{-1}]$ is isomorphic to a direct limit of the modules E/A_x, each of which is injective by Lemma 1.2. Since R is right noetherian, $E[X^{-1}]$ is injective as a right R-module, and therefore also as a right $R[X^{-1}]$-module. □

2. Loss of injectivity

An example of an injective module which has a localization that is not injective is constructed in this section. As the coefficient ring for this example is a differential operator ring, we recall some of the terminology associated with such rings.

The term *differential ring* is used to denote a ring (associative, with unit) equipped with a specified derivation. For ease of notation, all derivations in this paper will be denoted δ. A differential ring which is also a domain, or a field, is called a *differential domain*, or a *differential field*. The *subring of constants* of a differential ring R is the set $\{r\in R \mid \delta(r)=0\}$, which, as is readily seen, is a subring of R. In case R is a differential field, its subring of constants is a subfield of R, and so is called the *subfield* of constants.

The *formal linear differential operator ring* associated with a differential ring R, denoted $R[\theta;\delta]$, is a ring which additively is the abelian group of all polynomials over R in an indeterminate θ, and in which multiplication is induced from the multiplication in R via the rule $\theta r=r\theta+\delta(r)$, for all $r\in R$. If $T=R[\theta;\delta]$, then R can be viewed as a left T-module by extending the left R-module multiplication of ${}_RR$ to a left T-module multiplication $\circ$ under which $\theta\circ r=\delta(r)$ for all $r\in R$. (The module ${}_TR$ constructed in this fashion is isomorphic to $T/T\theta$.) Similarly, if S is a differential ring extension of R, then S can be viewed as a left $S[\theta;\delta]$-module and hence as a left T-module.

If R is a commutative differential domain, it will be convenient to obtain the injective hull $E({}_TR)$ as a T-submodule of a suitable differential field extension of the quotient field of R. This differential field extension is constructed so as to contain solutions for all linear differential equations, as follows.

Proposition 2.1. *Given a differential field F, there exists a differential field extension $\bar{F}$ of F such that every nonhomogeneous linear differential equation over $\bar{F}$ has a solution in $\bar{F}$.*

Proof. It is enough to show that any differential field F_1 has a differential field extension $\sigma(F_1)$ such that every nonhomogeneous linear differential equation over F_1 has a solution in $\sigma(F_1)$, since then the union of the differential fields

$$F \subseteq \sigma(F) \subseteq \sigma^2(F) \subseteq \dots$$

can be taken for $\bar{F}$. Such a differential field extension $\sigma(F_1)$ can be constructed by transfinite induction, provided it can be shown that any nonhomogeneous linear differential equation over a differential field F_2 has a solution in a differential field extension of F_2.

Hence, consider a differential equation

$$\alpha_n\delta^n(x) + \alpha_{n-1}\delta^{n-1}(x) + \cdots + \alpha_1\delta(x) + \alpha_0 x = \beta \qquad (*)$$

where $\alpha_i, \beta \in F_2$ and $\alpha_n \neq 0$. There is no loss of generality in assuming that $\alpha_n = 1$, for otherwise we can multiply throughout $(*)$ by α_n^{-1}. Set F_3 equal to a rational function field $F_2(y_0, y_1, \dots, y_{n-1})$ where the y_j are algebraically independent over F_2, and extend δ to a derivation of F_3 by the rules $\delta(y_j) = y_{j+1}$ for $j = 0, \dots, n-2$, while

$$\delta(y_{n-1}) = \beta - \alpha_{n-1}y_{n-1} - \dots - \alpha_1 y_1 - \alpha_0 y_0.$$

Then y_0 is a solution of $(*)$ in the differential field extension F_3 of F_2. □

The solvability of nonhomogeneous linear differential equations obtained in Proposition 2.1 says precisely that $\bar{F}$ is divisible as a left $\bar{F}[\theta;\delta]$-module. We shall use this in the situation where F is the quotient field of a commutative differential domain R. (The derivation on R extends uniquely to a derivation on F by the quotient rule.) Of course $\bar{F}$ is also divisible when viewed as an $F[\theta;\delta]$-module or as an $R[\theta;\delta]$-module.

Proposition 2.2. *Let R be a commutative differential domain, F its quotient field, and $\bar{F}$ as in Proposition* 2.1. *Set $T = R[\theta;\delta]$.*

(i) *The T-module ${}_T\bar{F}$ is injective, and the submodule ${}_TF$ is an essential extension of ${}_TR$. Consequently, the injective hull of ${}_TR$ can be identified with a T-submodule of $\bar{F}$ containing F.*

(ii) *For any left denominator set X in T, the localization $\bar{F}[X^{-1}]$ of ${}_T\bar{F}$ equals $\bar{F}/t_X(\bar{F})$.*

Proof. (i) Let U be the ring $F[\theta;\delta]$. Since U is a left principal ideal domain and $\bar{F}$ is a divisible U-module, $\bar{F}$ is an injective U-module [**10**, Theorem 2.8]. Since U is flat as a right T-module (e.g., [**6**, Lemma 7]), it follows that $\bar{F}$ is an injective left T-module [**8**, Theorem IV.12.5]. That ${}_TR$ is essential in ${}_TF$ is clear.

(ii) This is immediate from the divisibility of $\bar{F}$ as a left T-module. □

Continuing with the notation of Proposition 2.2, let X be a left denominator set in T such that $t_X(F) = R$. Our first aim is to establish sufficient conditions for the natural map from F/R to $\bar{F}/t_X(\bar{F}) = \bar{F}[X^{-1}]$ to be a split R-module monomorphism and hence for F/R to be isomorphic to an R-module direct summand of $E[X^{-1}]$, where E is the injective hull of ${}_TR$. (See Proposition 2.7.)

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Recall that a differential ring R is *δ-simple* provided R is nonzero and the only δ-ideals (that is, δ-invariant ideals) of R are 0 and R.

Proposition 2.3. *Let R be a δ-simple differential ring such that the subring K of constants of R is a field. Let S be a simple K-algebra, and extend δ to a derivation on $R \bigotimes_K S$ so that $\delta(r \otimes s) = \delta(r) \otimes s$ for all $r \in R$ and $s \in S$. Then $R \bigotimes_K S$ is δ-simple.*

Proof. Let I be a nonzero δ-ideal of $R \bigotimes_K S$, and choose a nonzero element

$$x = (r_1 \otimes s_1) + \cdots + (r_n \otimes s_n) \in I$$

with n minimal. By the minimality of n, the s_i are linearly independent over K and each $r_i \neq 0$.

As R is δ-simple, $\sum_{j \geqq 0} R\delta^j(r_1)R = R$ and hence

$$\sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}\delta^j(r_1)b_{jk} = 1$$

for some elements $a_{jk}, b_{jk} \in R$. Set

$$y = \sum_{j=0}^{p} \sum_{k=0}^{q} (a_{jk} \otimes 1)\delta^j(x)(b_{jk} \otimes 1).$$

Then $y \in I$ and

$$y = (1 \otimes s_1) + (r'_2 \otimes s_2) + \cdots + (r'_n \otimes s_n)$$

for some elements $r'_j \in R$. Note that $y \neq 0$ (because the s_i are linearly independent). Now

$$(\delta(r'_2) \otimes s_2) + \cdots + (\delta(r'_n) \otimes s_n) = \delta(y) \in I.$$

By the minimality of n and the linear independence of $s_2, \dots, s_n$, it follows that $\delta(r'_i) = 0$ and so $r'_i \in K$, for $i = 2, \dots, n$. But then

$$y = 1 \otimes (s_1 + r'_2 s_2 + \cdots + r'_n s_n).$$

Thus $n = 1$ and $y = 1 \otimes s_1$. By the simplicity of S, we have $Ss_1S = S$, from which we conclude that

$$1 \otimes 1 \in 1 \otimes (Ss_1S) \subseteq (R \bigotimes_K S)y(R \bigotimes_K S) \subseteq I$$

and hence that $I = R \bigotimes_K S$. □

Corollary 2.4. *Let $F \subseteq \bar{F}$ be differential fields with subfields K and $\bar{K}$ of constants. Then the multiplication map μ from $F \bigotimes_K \bar{K}$ to $\bar{F}$ is injective, and hence every K-linearly independent subset of F is also $\bar{K}$-linearly independent.*

Proof. Extend δ to a derivation on $F\bigotimes_K\bar{K}$ so that $\delta(x\otimes y)=\delta(x)\otimes y$ for all $x\in F$ and $y\in\bar{K}$. Since F is δ-simple, Proposition 2.3 shows that $F\bigotimes_K\bar{K}$ is δ-simple. It is easy to check that μ is a nonzero differential ring homomorphism, and hence that $\ker(\mu)$ is a proper δ-ideal of $F\bigotimes_K\bar{K}$. Thus $\ker(\mu)=0$ and μ is injective. The second conclusion follows immediately. □

Proposition 2.5. *Let $F\subseteq\bar{F}$ be differential fields with subfields K and $\bar{K}$ of constants. Let R be a differential subring of F and set $\bar{R}=R\bar{K}$. Then the R-module homomorphism $j:F/R\to\bar{F}/\bar{R}$ induced by the inclusion map $F\to\bar{F}$ is a split monomorphism.*

Proof. The map j can be factorized as follows:

$$F/R\xrightarrow{f}(F/R)\bigotimes_K K\xrightarrow{1\otimes g}(F/R)\bigotimes_K\bar{K}\xrightarrow{h}F\bar{K}/\bar{R}\xrightarrow{i}\bar{F}/\bar{R}$$

where f is the isomorphism given by the rule $x\mapsto x\otimes 1$, the maps g and i are inclusion maps, and h is induced by the multiplication map $\mu:F\bigotimes_K\bar{K}\to\bar{F}$. Since g is a split K-module monomorphism, $1\otimes g$ is a split R-module monomorphism. It is enough, therefore, to show that h and i are split R-module monomorphisms.

Since $F\bar{K}$ is an F-subspace of $\bar{F}$, there is an F-subspace $V\subseteq\bar{F}$ such that $\bar{F}=F\bar{K}\oplus V$. As $\bar{R}\subseteq F\bar{K}$ it follows that $\bar{F}/\bar{R}$ is an R-module direct sum

$$\bar{F}/\bar{R}=(F\bar{K}/\bar{R})\oplus[(V+\bar{R})/\bar{R}]$$

and hence that i is a split R-module monomorphism.

Finally, consider h, which we claim is an isomorphism. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & R\bigotimes_K\bar{K} & \to & F\bigotimes_K\bar{K} & \to & (F/R)\bigotimes_K & \to & 0\\
 & & \mu'\downarrow & & \mu\downarrow & & h\downarrow & & \\
0 & \to & \bar{R} & \to & F\bar{K} & \to & F\bar{K}/\bar{R} & \to & 0
\end{array}$$

where μ' is the restriction of μ to $R\bigotimes_K\bar{K}$. Note that μ and μ' are surjective. By Corollary 2.4, μ is an isomorphism, and hence μ' is an isomorphism. Therefore h is an isomorphism, by the Five-Lemma. □

Lemma 2.6. *Let R be a differential ring and X a left denominator set in the ring $T=R[\theta;\delta]$. If $\theta\in X$, then $t_X({}_TR)=R$.*

Proof. Let $r\in R$. Since $\theta\in X$ there exists $x\in X$ such that $xr\in T\theta$. In addition, $xr-(x\circ r)$ lies in $T\theta$, whence

$$x\circ r\in T\theta\cap R=0$$

and so $r\in t_X({}_TR)$. □

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Proposition 2.7. *Let R be a commutative differential domain with quotient field F, and assume that the subfield K of constants of F is contained in R. Let X be a left denominator set in the ring $T=R[\theta;\delta]$ such that $\theta\in X$, and, for all $x\in X$, the K-dimension of the solution space*

$$\{r\in R \mid x\circ r=0\}$$

of x in R equals the order of x. Then F/R is isomorphic, as an R-module, to a direct summand of $E[X^{-1}]$ where E is the injective hull of ${}_TR$.

Proof. Let $\bar{F}$ be a differential field extension of F satisfying the conclusion of Proposition 2.1, and let $\bar{K}$ be the subfield of constants of $\bar{F}$.

We first show that $t_X({}_T\bar{F})=R\bar{K}$. That $t_X(\bar{F})\supseteq R\bar{K}$ follows from Lemma 2.6. To prove the reverse inclusion, it is enough to show that for any $x\in X$, the solution space of x in $\bar{F}$ is contained in $R\bar{K}$. Let the order of x be n. By hypothesis, we can choose a K-basis $\{r_1,\dots,r_n\}$ for the solution space of x in R. By Corollary 2.4, these r_i are $\bar{K}$-linearly independent. On the other hand, the solution space of x in $\bar{F}$ has $\bar{K}$-dimension at most n, by [**1**, Theorem 1]. Thus

$$\{\alpha\in\bar{F} \mid x\circ\alpha=0\}=\bar{K}r_1+\cdots+\bar{K}r_n\subseteq R\bar{K},$$

as desired.

Now by Proposition 2.2, $\bar{F}[X^{-1}]=\bar{F}/t_X(\bar{F})=\bar{F}/R\bar{K}$, and E may be identified with a T-submodule of $\bar{F}$ containing F. Since E is an injective T-module, it is divisible, whence $E[X^{-1}]=E/t_X(E)$. By Lemma 2.6, $R\subseteq t_X(F)$. Now the inclusions $F\subseteq E\subseteq\bar{F}$ induce R-module homomorphisms

$$\rho:F/R\to E[X^{-1}] \quad\text{and}\quad \sigma:E[X^{-1}]\to\bar{F}/R\bar{K}$$

whose composition equals the split R-module monomorphism j of Proposition 2.5. It follows that ρ is a split R-module monomorphism, and so F/R is isomorphic to an R-module direct summand of $E[X^{-1}]$. □

Our next aim is to construct examples of R and X satisfying the hypotheses of Proposition 2.7. The method which we shall use to construct X is given by the following result.

Proposition 2.8. *Let T be a ring, c a regular element of T, and G a group of automorphisms of T. Let X be the multiplicatively closed subset of T generated by the set $\{g(c) \mid g\in G\}$. Then X is a left denominator set in T if and only if*

$$\textit{For each } t\in T \textit{ there exists } x\in X \textit{ such that } xt\in Tc. \qquad (*)$$

Proof. Note that X is closed under the action of G, and that X consists of regular elements. In particular, the reversibility condition is trivially satisfied, and so only the Ore condition is of concern.

That (*) is necessary is clear since $c\in X$. Conversely, suppose that (*) holds. We prove by induction that X satisfies the left Ore condition; namely, given $t \in T$ and $g_1, \ldots, g_n \in G$ there exist $x_n \in X$ and $t_n \in T$ such that

$$x_n t = t_n g_n(c) g_{n-1}(c) \ldots g_2(c) g_1(c).$$

For the case $n=1$, condition (*) provides us with elements $x \in X$ and $u \in T$ such that $x g_1^{-1}(t) = uc$. Thus $g_1(x)t = g_1(u) g_1(c)$ and hence $x_1 t = t_1 g_1(c)$ where $x_1 = g_1(x) \in X$ and $t_1 = g_1(u) \in T$. For the inductive step, let $n > 1$ and suppose that there exist $x_{n-1} \in X$ and $t_{n-1} \in T$ satisfying

$$x_{n-1} t = t_{n-1} g_{n-1}(c) g_{n-2}(c) \ldots g_2(c) g_1(c).$$

By the case $n=1$ there exist $y \in X$ and $t_n \in T$ such that $y t_{n-1} = t_n g_n(c)$. Therefore, setting $x_n = y x_{n-1} \in X$, we obtain

$$x_n t = y x_{n-1} t = t_n g_n(c) g_{n-1}(c) \ldots g_2(c) g_1(c),$$

completing the inductive step. □

We shall apply Proposition 2.8 in the case that T is a differential operator ring and $c = \theta$. In order to see that X is a right as well as left denominator set, we use an involution to reverse sides in T. Provided $T = R[\theta; \delta]$ for a commutative differential ring R, there is a natural involution * on T such that $\theta^* = -\theta$ and $r^* = r$ for all $r \in R$.

Proposition 2.9. *Let R be a commutative differential ring, A an additive group of constants of R, and T the ring $R[\theta; \delta]$. Let X be the multiplicatively closed subset of T generated by the set $\{\theta + \alpha \mid \alpha \in A\}$. Then the following conditions are equivalent:*

(i) *X is a left denominator set in T.*
(ii) *X is a right denominator set in T.*
(iii) *For each $r \in R$ there exists $x \in X$ such that $x \circ r = 0$.*

Proof. (i)⇔(ii): Let Y be the multiplicatively closed set $\{\pm x \mid x \in X\}$. As A is an additive subgroup of R, we see that

$$Y = Y^* = \{\pm z \mid z \in X^*\}.$$

Now X is a left denominator set if and only if $Y = Y^*$ is a left denominator set, if and only if X^* is a left denominator set, if and only if X is a right denominator set.

(i)⇔(iii): For each $\alpha \in A$ there is an automorphism g_α of T such that $g_\alpha(\theta) = \theta + \alpha$ and $g_\alpha(r) = r$ for all $r \in R$. The set X is the multiplicatively closed subset of T generated by $\{g(\theta) \mid g \in G\}$ where G is the group $\{g_\alpha \mid \alpha \in A\}$ of automorphisms of T. By Proposition 2.8, X is a left denominator set if and only if

$$\text{For each } t \in T \text{ there exists } x \in X \text{ such that } xt \in T\theta. \qquad (*)$$

Given $t \in T$ and $x \in X$, write $t = r_0 + r_1\theta + \cdots + r_n\theta^n$ for some $r_i \in R$ and observe that

$$xt - (x \circ r_0) \in T\theta.$$

Hence, $xt \in T\theta$ for some $x \in X$ if and only if $x \circ r_0 = 0$ for some $x \in X$. Therefore (*) and (iii) are equivalent. □

Lemma 2.10. *Let R be a commutative differential domain with quotient field F. If R is δ-simple then all nonzero constants of F are units of R.*

Proof. Let α be a nonzero constant of F, and let

$$I = \{r \in R \mid \alpha r \in R\}.$$

If $r \in I$, then $\alpha\delta(r) = \delta(\alpha r) \in R$ and so $\delta(r) \in I$. Hence, I is a δ-ideal of R, and we note that I is nonzero. It follows, by the δ-simplicity of R, that $I = R$, whence $\alpha \in R$. Now αR is a nonzero δ-deal of R, and so $\alpha R = R$. Thus α must be a unit of R. □

Proposition 2.11. *Let K be a field of characteristic zero, let n be a positive integer, and assume that K contains n elements $\alpha_1, \dots, \alpha_n$ that are $\mathbb{Q}$-linearly independent. Let A denote the additive group $\mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$. Let F be a rational function field $K(x_0, x_1, \dots, x_n)$ with the x_i algebraically independent over K, and let δ be the K-linear derivation on F such that $\delta(x_0) = 1$ and $\delta(x_i) = \alpha_i x_i$ for $i > 0$. Let R be the differential subring $K[x_0, x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of F, and let X be the multiplicatively closed subset of the ring $T = R[\theta; \delta]$ generated by the set $\{\theta + \alpha \mid \alpha \in A\}$.*

(i) *R is δ-simple.*
(ii) *The subfield of constants of F is K.*
(iii) *X is a left and right denominator set in T.*
(iv) *For all $x \in X$, the K-dimension of the solution space of x in R equals the order of x.*

Proof. (i) The differential subring $S = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of R is δ-simple by [**9**, Theorem 2.1]. Note that an element $s \in S$ cannot satisfy $\delta(s) \in K$ unless $s \in K$, in which case $\delta(s) = 0$.

Let I be a nonzero δ-ideal in $R = S[x_0]$, and let m be the minimal degree in x_0 of nonzero elements of I. The set

$$J = \{s_m \in S \mid s_m x_0^m + s_{m-1} x_0^{m-1} + \cdots + s_0 \in I \text{ for some } s_{m-1}, \dots, s_0 \in S\}$$

is a nonzero δ-ideal of S, and so $J = S$. Hence, there exist elements $s_{m-1}, \dots, s_0$ in S such that the element

$$y = x_0^m + s_{m-1} x_0^{m-1} + \cdots + s_1 x_0 + s_0$$

lies in I. But the element

$$\delta(y) = (m + \delta(s_{m-1})) x_0^{m-1} + [\text{terms of degree} \leqq m-2 \text{ in } x_0]$$

also lies in I, whence $\delta(y) = 0$, by the minimality of m. As a result, $\delta(s_{m-1}) = -m$, and so $m = 0$. Thus $I \cap S \neq 0$, and therefore, since S is δ-simple, $I \cap S = S$ and $I = R$.

(ii) Any unit u of R has the form

$$u=\beta x_1^{m(1)}x_2^{m(2)}\dots x_n^{m(n)}$$

for some nonzero $\beta\in K$ and some integers m(i), and

$$\delta(u)=[m(1)\alpha_1+m(2)\alpha_2+\cdots+m(n)\alpha_n]u.$$

If $u\notin K$, then at least one $m(i)\neq 0$ and $\delta(u)\neq 0$ by the $\mathbb{Q}$-linear independence of the α_i. Hence, (ii) follows from (i) and Lemma 2.10.

(iii) By Proposition 2.9 it is enough to show that for each $r\in R$ there exists $z\in X$ such that $z\circ r=0$. First suppose that

$$r=x_0^{m(0)}x_1^{m(1)}\dots x_n^{m(n)}$$

for some integers m(i) with $m(0)\geqq 0$. If

$$y=\theta-m(1)\alpha_1-\cdots-m(n)\alpha_n,$$

then $y^{m(0)+1}\in X$ and $y^{m(0)+1}\circ r=0$. Since R is spanned over K by monomials of the above form, and since X is commutative, the desired condition follows.

(iv) Let $x\in X$ be of order k. Then x can be written in the form

$$x=(\theta-\beta_1)^{m(1)}(\theta-\beta_2)^{m(2)}\dots(\theta-\beta_t)^{m(t)}$$

where the β_j are distinct elements of A, the $m(j)$ are positive integers, and $\sum m(j)=k$. There are integers $p(j,i)$, for $j=1,\dots,t$ and $i=1,\dots,n$, such that

$$\beta_j=\sum_{i=1}^{n}p(j,i)\alpha_i$$

for each j. Set $y_j=x_1^{p(j,1)}x_2^{p(j,2)}\cdots x_n^{p(j,n)}$ for $j=1,\dots,t$. It is a routine calculation to check that the set

$$\{x_0^m y_j\,|\,j=1,\dots,t \text{ and } m=0,\dots,m(j)-1\}$$

is a K-linearly independent set of k elements of the solution space of x in R. By [**1**, Theorem 1], this solution space has dimension exactly k. □

We are now in a position to construct our example.

Theorem 2.12. *Let n be a positive integer. There exist a simple noetherian domain T, a left and right denominator set X in T, and an injective left T-module E such that*

$$\text{K.dim.}(T)=\text{gl.dim.}(T)=n+1$$

and inj.dim.$(E[X^{-1}])=n$. *Moreover, E is the injective hull of a simple left T-module.*

Proof. Choose a field K of characteristic zero which has $\mathbb{Q}$-dimension at least n, and let R, T, X be as in Proposition 2.11. Then R is a δ-simple, commutative, noetherian, regular, differential domain of Krull and global dimension $n+1$. By [**2**, Theorem 6.3], ${}_R R$ has injective dimension $n+1$. If F is the quotient field of R, then since ${}_R F$ is injective, ${}_R(F/R)$ must have injective dimension n.

Note from the δ-simplicity of R that R is a simple left T-module. As R is a δ-simple noetherian $\mathbb{Q}$-algebra and $\delta \neq 0$, the domain T is a simple noetherian ring [**4**, Proposition 3.1, Theorem 3.2]. Consequently, T has Krull and global dimension $n+1$, by [**7**, Theorems 2.6 and 3.2].

Let E be the injective hull of ${}_T R$. By Propositions 2.11 and 2.7, F/R is isomorphic, as an R-module, to a direct summand of $E[X^{-1}]$. Hence,

$$\text{inj.dim.}({}_R E[X^{-1}]) \geqq \text{inj.dim.}({}_R(F/R)) = n.$$

Since T is flat as a right R-module, we conclude using [**8**, Theorem IV.12.5] that

$$\text{inj.dim.}({}_T E[X^{-1}]) \geqq \text{inj.dim.}({}_R E[X^{-1}]) \geqq n.$$

Recall that $E[X^{-1}] \cong E/t_X(E)$. If ${}_T E[X^{-1}]$ had injective dimension $n+1$, then $t_X(E)$ would have injective dimension $n+2$, which is impossible. Therefore $\text{inj.dim.}({}_T E[X^{-1}]) = n$. □

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