EXTENSION OF COVERINGS, OF PSEUDOMETRICS, AND OF LINEAR-SPACE-VALUED MAPPINGS

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1. Introduction. Let A be a closed subset of a topological space. We show that the following three conditions are equivalent.

1.1 *Any countable neighbourhood-finite open covering of* A (the topological terms referring to the relative topology of A) *has a refinement which can be extended to be a countable neighbourhood-finite open covering of* X.

1.2 *Any separable pseudometric on* A *can be extended to a separable pseudometric on* X.

1.3 *Any mapping of* A *into a separable closed convex subset* S *of a Banach space* B *can be extended to* X, *keeping the values still in* S.

The three conditions obtained by omitting "separable" and "countable" throughout are also shown to be equivalent.

Thereupon we show that 1.1 to 1.3 are always true if X is normal. This is a true generalization of Tietze's extension theorem.

Without the words "countable," "separable," the conditions hold when X is paracompact (and thus normal). We do not know if normality suffices in the general case.

We then take a closer look at the fact that S in 1.3 is restricted to be *closed*. We find that this restriction may be relaxed (that is, the word "closed" omitted) when X is normal and B is one-dimensional. On the other hand, we refer to a paracompact space X and a mapping f on A with values in the *plane* whose extension to X always requires enlargement of the convex hull of the values.

2. Some properties equivalent to extendability. A *pseudometric* r defined on a topological space X is a real-valued function of two variables in X such that $r(x, x) = 0$, $r(x, y) \geqslant 0$, $r(x, z) \leqslant r(x, y) + r(y, z)$ and such that for each y the set of all x such that $r(x, y) < c$ is open. The latter set is an *r-sphere of radius c about y*. We shall call r *γ-separable* if γ is an infinite cardinal number and if there is a subset G of power (cardinal number) not greater than γ which has a non-void intersection with every r-sphere of positive radius.

We shall abbreviate "neighbourhood" by "nbd." When we talk about a subset A of a space it is always implied that this set A is closed. Now consider the following statements about such a set A.

Received November 12, 1951.

https://doi.org/10.4153/CJM-1953-023-1 Published online by Cambridge University Press

2.1 *Let C be a nbd-finite* [cf. **7**] *open covering of A, where the power of C does not exceed γ. Then there is a refinement C' of C which can be extended to a nbd-finite open covering C'' of X, where the power of C'' also does not exceed γ.*

2.2 *Any γ-separable pseudometric r defined on A can be extended to a γ-separable pseudometric r' of X.*

2.3 *Let f be a continuous mapping of A into a metrizable γ-separable subset S of a convex topological linear space L. Unless there is a real-valued continuous function g which vanishes precisely on A, assume that S is complete in some one of its metrics. Then f can be extended to X so as to map X into S.*

Now the relation between these is the following.

2.4 THEOREM. *The statements* 2.1, 2.2, 2.3 *are equivalent.*

We shall prove first that each of 2.1 and 2.3 implies 2.2, and then we shall prove that 2.2 implies the others. It is to be noted that in the proof that 2.3 implies 2.2 (and thus 2.1), 2.3 is utilized only for the case in which L is a Banach space.

Let q be a γ-separable pseudometric on A. For each positive integer n cover A by q-spheres of radius $2^{-(n+2)}$. Now the theorem in [**7**] clearly holds for pseudometrics as well as for metrics, so that there is a nbd-finite refinement C, and C can be supposed to have not more than γ members. Let C_1 be the covering of X provided by 2.1. By repeated application of a corollary in [**6**, p. 17] we can obtain a normal sequence [**8**, p. 46] of coverings U_n ($n = 1, 2, \ldots$) of which U is the first covering U_1. By [**8**, p. 51] there is an "associated pseudometric" r_m such that [**8**, p. 51, 7.5] $r_m(x, y) \leqslant 2^{-(n+2)}$ implies x and y are both in some common element of U_n, and thus that $q(x, y) < 2^{-(m+2)}$. Moreover, $r_m(x, y)$ is bounded by 1. Let $r(x, y) = \Sigma_m 2^{-m} r_m(x, y)$. Suppose $r(x, y) < 2^{-k}$ for some positive integer k. Then $r_m(x, y) < 2^{m-k}$ for all m. Let $m = k + 3$ and $n = 1$. Then $r_m(x, y) < 2^{-3}$ and so $q(x, y) < 2^{-(k+5)}$. Let X/r be the class of sets of r-diameter zero, metrized with r. The set A corresponds to a set B in X/r, and from the dominance of q by r shows that q can be defined continuously on B and even to the closure of B. By [**1**, 3.5] q can be extended to X/r and thus to X, as desired. Since for every positive e there is a covering of X by not more than γ q-spheres of radius less than e, the pseudometric q is γ-separable on X.

Now assume 2.3 and let q be given as before. Let $C(A, q)$ be that class of bounded real-valued functions f on A, such that for every positive e there is a positive d such that if $q(x, y) < d$ then $|f(x) - f(y)| < e$. It is not hard to see that $C(A, q)$ is a γ-Lindelöf Banach space with the norm $||f|| = \sup_{x \epsilon A} |f(x)|$. The mapping ϕ, where

$$\phi(a)\,(x) = q(a, x) - q(a_0, x), \qquad a_0 \text{ fixed in } A,$$

sends A continuously into $C(A, q)$. This mapping can presumably be extended

https://doi.org/10.4153/CJM-1953-023-1 Published online by Cambridge University Press

to X. Then $q(x, y) = || \phi(x) - \phi(y) ||$ extends q to all of X, as desired. The γ-separable property is easily seen to hold for the extension.

For both parts of the following second half of the proof of 2.4, let 2.2 hold. First 2.1 will be established. Let U be a nbd-finite open covering for A, with not more than γ sets. There exists a Δ-refinement U_2 of U [**6**], and it can be seen from Morita's proof that U_2 need have no more than γ sets. Iteration, and taking every other covering so constructed gives a normal sequence beginning with U. Then the associated [**7**] pseudometric r being γ-separable, can be extended to all of X, so as to be γ-separable. Its relation to U is such that the spheres of r-radius less than $\frac{1}{4}$ are (on A) a refinement of U. By the obvious extension of A. H. Stone's theorem to the case of a pseudometric space, the covering by the spheres of r-radius $\frac{1}{4}$ has a nbd-finite refinement which may be assumed to have not more than γ sets. Thus 2.1 is derived from 2.2.

If S is complete, then the deduction of 2.3 from 2.2 is given by the proof of 4.1 in [**1**] except for the detail about γ, which can be supplied by the reader. In the proof in question, the completeness of S is used in extending f from A to A [**1**]. Now if there is a continuous real-valued g which vanishes precisely on A (in a normal space, this is equivalent to A's being the intersection of a countable family of open sets), then q in the proof of 4.1 in the paper just referred to can be replaced by q', $q'(x, y) = q(x, y) + | g(x) - g(y) |$, whereupon $A_0 = A$ and no completeness is needed. Thus 2.4 is proved.

We shall see later that the requirement that either some g vanish precisely on A or that S be complete cannot be ignored.

If A is a closed subset of a topological space such that any and hence all of 2.1, 2.2, and 2.3 hold, we shall say that A is *γ-normally embedded* in X. An earlier result [**1**, 3.5 and 4.1] can thus be formulated as follows [**8**; **7**]:

2.5 THEOREM. *In a fully normal space X every closed subset A is γ-normally embedded in X for every infinite cardinal number γ.*

3. Countable normal imbedding. In this section we shall show that a (closed) set A in a normal space is always $\aleph_0$-normally imbedded. If it were known that every normal space was "countably paracompact" [**3**] the proof would be particularly easy, via 2.1 as follows: extend each V in C to X, adjoin the complement, and obtain a nbd-finite refinement of this countable covering as promised by the axiom of countable paracompactness. However, it is possible to circumvent this assumption.

3.1 THEOREM. *If A lies in a normal space, then A is countably, normally imbedded, that is,* 2.1 *to* 2.4 *hold with* $\gamma = \aleph_0$.

Proof. We shall establish 2.1. Let $C = \{U_1, U_2, \ldots\}$ be a countable nbd-finite open covering for A. Obtain closed sets $A_n \subset U_n$ which cover A [**5**]. For each n construct a continuous real-valued f_n such that $0 \leqslant f_n(a) \leqslant 1$, $f_n(a) = 0$ for a in A_n, $f_n(a) = 0$ for a not in U_n. Extend each f_n to all of X, without increase of bounds, and call the extension g_n. Let P be the topological product of countably many intervals; then P is a (compact) metric space. We can map X into P as follows:

$$x \rightarrow g(x) = \{g_1(x), g_2(x), \ldots\} \in P.$$

Let V_n in P be the set of all elements $\{t_1, t_2, \ldots\}$ in P for which t_n is positive. By the theorem in [**7**] or even the earlier result in [**2**] there is a nbd-finite system of open sets W_n in P whose union is the same as that of the V_n. That this system can be assumed countable is obvious since P is separable. The inverse images G_n under g of these W_n form nbd-finite system S which covers A. Adjoining the complement G_0 of A gives a covering of X. Now let $a \in G_n$, $a \in A$. Then $g(a) \in W_n \subset V_m$ for some m depending only on n. Therefore $g_m(a) > 0$ or $a \in U_m$. Hence S is a refinement of C. Thus 2.1 is proved for $\gamma = \aleph_0$, as desired.

This result generalizes an earlier result [**1**, 4.3]. Of course, it generalizes Tietze's extension theorem inasmuch as it says that a continuous function on A in a normal space X, with values in a separable Banach space (for example) can be extended without increasing the closed convex hull of the range of values.

In an abstract of this paper (Bull. Amer. Math. Soc., *57* (1951), 487) there was announced an example of a normal space in which a closed set A was not $\aleph_1$-normally imbedded, but the argument now appears to be fallacious and the question remains open.

4. Extension of mappings into non-closed convex sets. There is a refinement of Tietze's extension theorem which is not widely known. The theorem of Tietze says that if A is closed in a normal space X, and f maps A into a *closed* interval K, then f can be extended to X with values in K. But what if the interval K is open or half open? The following theorem shows that the extension is still possible without bringing in values outside K.

4.1 THEOREM. *Let A be a closed set in a normal space X. Let K be a convex set with non-void interior in a topological linear space L. Let T be*

4.11 *an F_σ-set*

contained in the frontier of K. Let f be a mapping of X into K but such that $f(A)$ avoids T. Then f can be so deformed at points not in A so as to have values in K but to avoid T altogether.

Proof. The inverse image S of T under f is an F_σ disjoint from A. We can therefore construct a real-valued mapping g vanishing on A with $0 \leqslant g \leqslant 1$ but $g(x) > 0$ for x in S. Select a point k in the interior of K. Define $h(t) = (1 - tg)f + tgk$ for real t, $0 \leqslant t \leqslant 1$. Then $h(t)$ agrees with f on A, $h(0) = f$, and $h(1)$ avoids T as desired.

The refinement of Tietze's theorem is obtained by letting T be the class of such end points of K as are not members of K.

A similar result can be obtained when A is a G_δ-set.

4.2 THEOREM. *Let* A, X, K, L, T, *and* f *be as in* 4.1, *ignoring* 4.11. *Suppose* A *is a* G_δ*-set. Then the conclusion of* 4.1 *holds.*

The proof is exactly as before, because now there exists a function g vanishing exactly on A.

It is of interest to point out that when the dimension of L is greater than 1, the omission of 4.11 would destroy the validity of 4.1. A corresponding statement holds for 4.2. It is possible to construct a normal (in fact, fully normal) space X with a closed subset A which can be mapped into a convex set K in the plane by a mapping which cannot be extended to X without exceeding K. For the set K take the "Example 5.1" of [**4**, p. 381]. Hanner tells how to imbed K as a closed set into a space which is normal and which can be used as the above X. For the sake of brevity, we omit the proof of the full normality of X.

5. Paracompactness not strictly necessary. When X is paracompact then each A is γ-normally imbedded for each γ (see 2.5). It is interesting to observe that the converse is not true. The space that shows this is the space T_{ω_1} of all countable ordinal numbers which is not paracompact [**2**] since it is not compact, but being Fréchet compact, permits no infinite family of sets to be nbd-finite. However, it satisfies 2.3 and in fact any mapping on a subset A can be extended. Any A is in fact a retract. We show how to construct, for A (closed in T_{ω_1}), a mapping f on T_{ω_1} to A such that $f(a) = a$ for a in A. For x in A define $f(x)$ to be the least element of A greater than x, but if there is no such element let $f(x)$ be the greatest element of A which must now exist because A is closed. This f is easily seen to be continuous.

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