SOME INTEGRALS INVOLVING E-FUNCTIONS

by R. K. SAXENA

(Received 26 October, 1959)

1. In this paper we evaluate some integrals involving E-functions by the methods of the Operational Calculus. The results obtained are quite general and many of them include, as particular cases, some known results.

A function $\psi(p)$ is operationally related with another function $f(t)$, if they satisfy the integral equation

$$\psi(p) = p \int_0^\infty e^{-pt} f(t)\, dt. \quad \ldots\ldots(1)$$

As usual, we shall denote (1) by the symbolic expression

$$\psi(p) \doteqdot f(t).$$

2. Theorem. *If*

$$\psi(p) \doteqdot f(t)$$

and

$$\phi(p) \doteqdot t^{n\alpha-1} f(t^n),$$

then

$$\phi(p) = (2\pi)^{-\frac{1}{2}(n+1)} n^{n\alpha-\frac{1}{2}} p^{1-n\alpha} \int_0^\infty \frac{1}{t} \sum_{i,-i} \frac{1}{i}\, E\left(\alpha, \alpha+\frac{1}{n}, \ldots, \alpha+\frac{n-1}{n} :: \frac{p^n e^{i\pi}}{n^n t}\right) \psi(t)\, dt, \quad \ldots\ldots(2)$$

provided that the integral is convergent. Here $R(\alpha) > 0$, $R(p) > 0$, $n = 2, 3, 4, \ldots$, *and* $\sum_{i,-i}$ *means that in the expression following it, i is to be replaced by $-i$ and the two expressions are to be added.*

Proof. We have

$$\psi(p) \doteqdot f(t) \quad \ldots\ldots(3)$$

and, by Ragab [7, p. 119],

$$\sum_{i,-i} \frac{1}{i}\, E\left(\alpha, \alpha+\frac{1}{n}, \ldots, \alpha+\frac{n-1}{n} :: \frac{ae^{i\pi}}{n^n t}\right) \doteqdot (2\pi)^{\frac{1}{2}(n+1)} n^{-n\alpha-\frac{1}{2}} a^\alpha p^\alpha e^{-a^{1/n} p^{1/n}}, \quad \ldots\ldots(4)$$

where $R(\alpha) > 0$, $R(a) > 0$ and $n = 2, 3, 4, \ldots$.

Applying the Parseval-Goldstein theorem [2] that if

$$\phi_1(p) \doteqdot g_1(x) \quad \text{and} \quad \phi_2(p) \doteqdot g_2(x),$$

then

$$\int_0^\infty \phi_1(x) g_2(x) x^{-1}\, dx = \int_0^\infty g_1(x) \phi_2(x) x^{-1}\, dx, \quad \ldots\ldots(5)$$

to the relations (3) and (4), we obtain

$$\int_0^\infty t^{\alpha-1} e^{-a^{1/n} t^{1/n}} f(t)\, dt$$

$$= (2\pi)^{-\frac{1}{2}(n+1)} n^{\frac{1}{2}+n\alpha} a^{-\alpha} \int_0^\infty \frac{1}{t} \sum_{i,-i} \frac{1}{i}\, E\left(\alpha, \alpha+\frac{1}{n}, \ldots, \alpha+\frac{n-1}{n} :: \frac{ae^{i\pi}}{n^n t}\right) \psi(t)\, dt.$$

Now put $t = u^n$ in the integral on the left and replace a by a^n, so that

$$\int_0^\infty u^{n\alpha-1}e^{-au}f(u^n)\,du$$
$$= (2\pi)^{-\frac{1}{2}(n+1)}a^{-n\alpha}n^{n\alpha-\frac{1}{2}}\int_0^\infty \frac{1}{t}\sum_{i,-i}\frac{1}{i}E\left(\alpha, \alpha+\frac{1}{n}, \ldots, \alpha+\frac{n-1}{n} :: \frac{a^n e^{i\pi}}{n^n t}\right)\psi(t)\,dt.$$

The theorem follows immediately on multiplying both sides by a and then replacing a by p.

Alternative form of the theorem. If $\psi(p) \doteqdot f(t^{1/n})$

and $$\phi(p) \doteqdot t^{n\alpha-1}f(t),$$

then

$$\phi(p) = (2\pi)^{-\frac{1}{2}(n+1)}n^{n\alpha-\frac{1}{2}}p^{1-n\alpha}\int_0^\infty \frac{1}{t}\sum_{i,-i}E\left(\alpha, \alpha+\frac{1}{n}, \ldots, \alpha+\frac{n-1}{n} :: \frac{p^n e^{i\pi}}{n^n t}\right)\psi(t)\,dt,$$

provided that the integral is convergent, where $R(\alpha) > 0$, $R(p) > 0$, $n = 2, 3, 4, \ldots$.

When $n = 2$ the theorem reduces to a known theorem [**1**, p. 132] by virtue of the relation [**7**, p. 122]

$$\frac{1}{i}E\left(\alpha, \alpha+\tfrac{1}{2} :: \frac{ae^{i\pi}}{4t}\right) - \frac{1}{i}E\left(\alpha, \alpha+\tfrac{1}{2} :: \frac{ae^{-i\pi}}{4t}\right) = 2^{\frac{3}{2}-3\alpha}\pi a^\alpha t^{-\alpha}e^{-a/8t}\,D_{2\alpha-1}\left(\frac{a^{\frac{1}{2}}}{2^{\frac{1}{2}}t^{\frac{1}{2}}}\right)$$

Example. If we take [**9**, p. 133]

$$f(t) = t^{\beta-1}E\left(l\,;\ \alpha_r : \rho_1, \rho_2, \ldots, \rho_m, \beta : \frac{1}{t}\right) \doteqdot p^{1-\beta}E(l\,;\ \alpha_r : m\,;\ \rho_s : p) = \psi(p),$$

where $R(\beta) > 0$, $R(p) > 0$, then [**10**, p. 172]

$$t^{n\alpha-1}f(t^n) = t^{n\alpha+n\beta-n-1}E\left(l\,;\ \alpha_r : \rho_1, \rho_2, \ldots, \rho_m, \beta : \frac{1}{t^n}\right)$$
$$\doteqdot (2\pi)^{\frac{1}{2}-\frac{1}{2}n}n^{n\beta+n\alpha-n-\frac{1}{2}}p^{n+1-n\alpha-n\beta}E\{l+n\,;\ \alpha_r : \rho_1, \rho_2, \ldots, \rho_m, \beta : (p/n)^n\}$$
$$= \phi(p),$$

where $R(\alpha+\beta-1) > 0$, $R(p) > 0$, and $\alpha_{l+k+1} = (n\alpha+n\beta-n+k)/n$ for $k = 0, 1, \ldots, (n-1)$.

Applying (2) and replacing $(p/n)^n$ by a, we find that

$$\int_0^\infty t^{-\beta}E(l\,;\ \alpha_r : m\,;\ \rho_s : t)\sum_{i,-i}\frac{1}{i}E\left(\alpha, \alpha+\frac{1}{n}, \ldots, \alpha+\frac{n-1}{n} :: \frac{ae^{i\pi}}{t}\right)dt$$
$$= 2\pi\, a^{1-\beta}E(l+n\,;\ \alpha_r : \rho_1, \rho_2, \ldots, \rho_m, \beta : a), \quad \ldots\ldots\ldots(6)$$

where $R(\alpha+\beta-1) > 0$, $R(\alpha) > 0$, $n = 2, 3, 4, \ldots$, $R(a) > 0$, and α_{l+k+1} is defined as before.

3. We now prove the formula

$$\int_0^\infty e^{-zt}t^{\lambda-1}(1+t)^{\alpha+\beta-\delta}E\{\gamma-\lambda, \delta-\alpha-\beta :: (1+t)z\}\,{}_3F_2(\alpha, \beta, \gamma\,;\ \delta, \lambda\,;\ -t)\,dt$$
$$= \frac{\Gamma(\lambda)\Gamma(\delta)\Gamma(\delta-\alpha-\beta)\Gamma(\gamma-\lambda)}{\Gamma(\gamma)\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}z^{-\lambda}E(\delta-\alpha, \delta-\beta, \gamma : \delta : z), \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where $R(\lambda) > 0$, $R(z) > 0$.

In the proof of (7) we require the integral [**10**, p. 171]

https://doi.org/10.1017/S2040618500034122 Published online by Cambridge University Press

$$\int_0^\infty e^{-u}u^{\delta-\alpha-\beta-1}(u+v)^{-\gamma}E(\alpha,\beta,\gamma:\delta:u+v)\,du$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}\,v^{-\gamma}E(\delta-\alpha,\delta-\beta,\gamma:\delta:v), \quad \text{.........(8)}$$

where $R(\delta-\alpha-\beta) > 0$, $|\arg v| < \pi$.

Take [8, p. 169]

$$g_1(t) = e^{-zt}t^{\lambda-1}{}_3F_2(\alpha,\beta,\gamma;\ \delta,\lambda;\ -t) \doteqdot \frac{\Gamma(\lambda)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\,p(p+z)^{-\lambda}E(\alpha,\beta,\gamma:\delta:p+z) = \phi_1(p),$$

where $R(\lambda) > 0$, $R(p) > 0$, $R(z) > 0$, and [1, p. 294]

$$g_2(t) = e^{-t}t^{\delta-\alpha-\beta-1}(t+z)^{\lambda-\gamma} \doteqdot \frac{p(1+p)^{\alpha+\beta-\delta}z^{\lambda-\gamma}}{\Gamma(\gamma-\lambda)}E\{\gamma-\lambda,\ \delta-\alpha-\beta::(1+p)z\} = \phi_2(p),$$

where $R(\delta-\alpha-\beta) > 0$, $R(p) > 0$.

Using these relations in (5) and evaluating the integral on the left with the help of (8), we arrive at the result.

In particular, when $\lambda=\alpha$, (7) reduces to

$$\int_0^\infty e^{-zt}t^{\alpha-1}(1+t)^{\alpha+\beta-\delta}E\{\gamma-\alpha,\ \delta-\alpha-\beta::(1+t)z\}\,{}_2F_1(\beta,\gamma;\ \delta;\ -t)\,dt$$
$$= \frac{\Gamma(\alpha)\Gamma(\delta)\Gamma(\gamma-\alpha)\Gamma(\delta-\alpha-\beta)}{\Gamma(\gamma)\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}\,z^{-\alpha}E(\delta-\alpha,\delta-\beta,\gamma:\delta:z), \quad \text{..................(9)}$$

where $R(\alpha) > 0$, $R(z) > 0$.

4. Next we establish the result

$$\int_0^1 t^{\gamma-1}(1-t)^{\delta-1}\{1+\lambda t+\mu(1-t)\}^{-\gamma-\delta}\,E\left\{l;\ \alpha_r:m;\ \rho_s:z\left(\frac{1+\lambda t+\mu(1-t)}{t}\right)^n\right\}dt$$
$$= \Gamma(\delta)n^{-\delta}(1+\lambda)^{-\gamma}(1+\mu)^{-\delta}E\{l+n;\ \alpha_r:m+n;\ \rho_s:(1+\lambda)^n z\}, \quad \text{......(10)}$$

where $R(\gamma) > 0$, $R(\delta) > 0$, $R(z) > 0$, n is a positive integer, $\lambda \geqslant 0$, $\mu \geqslant 0$, and

$$\alpha_{l+k+1} = \frac{\gamma+k}{n},\quad \rho_{m+k+1} = \frac{\gamma+\delta+k}{n}\quad \text{for } k = 0, 1, \ldots, (n-1).$$

In the proof we shall require the following theorem [3, p. 44].

If

$$f_1(p) \doteqdot h_1(t) \quad\text{and}\quad f_2(p) \doteqdot h_2(t),$$

then

$$\frac{f_1(p)f_2(p)}{p} = \int_0^{\frac{\pi}{2}} F(p,\cos^2\theta,\sin^2\theta)\sin 2\theta\,d\theta, \quad \text{.....................(11)}$$

where $t\,h_1(xt)h_2(yt) \doteqdot F(p,x,y)$.

https://doi.org/10.1017/S2040618500034122 Published online by Cambridge University Press

Taking [**10**, p. 172]

$$h_1(t) = e^{-at}t^{\gamma-1} E\left(l\,;\ \alpha_r : m\,;\ \rho_s : \frac{1}{t^n}\right)$$

$$\doteqdot (2\pi)^{\frac{1}{2}-\frac{1}{2}n} n^{\gamma-\frac{1}{2}} p\,(p+a)^{-\gamma} E\left\{l+n\,;\ \alpha_r : m\,;\ \rho_s : \left(\frac{p+a}{n}\right)^n\right\}$$

$$= f_1(p),$$

where $R(\gamma) > 0$, $R(a) \geqslant 0$, $R(p) > 0$, and

$$h_2(t) = t^{\delta-1}e^{-bt} \doteqdot p\Gamma(\delta)(p+b)^{-\delta} = f_2(p),$$

where $R(\delta) > 0$, $R(b) \geqslant 0$, $R(p) > 0$, we have from (11)

$$2\int_0^{\frac{\pi}{2}} \sin^{2\delta-1}\theta \cos^{2\gamma-1}\theta\,(p + a\cos^2\theta + b\sin^2\theta)^{-\gamma-\delta}$$
$$\times E\left\{l+n\,;\ \alpha_r{}^* : m\,;\ \rho_s : \left(\frac{p+a\cos^2\theta+b\sin^2\theta}{n\cos^2\theta}\right)^n\right\} d\theta$$
$$= n^{-\delta}(p+a)^{-\gamma}(p+b)^{-\delta}\Gamma(\delta)E\left\{l+n\,;\ \alpha_r : m\,;\ \rho_s : \left(\frac{p+a}{n}\right)^n\right\}, \quad\ldots\ldots\ldots\ldots\ldots(12)$$

where $R(\gamma) > 0$, $R(\delta) > 0$, $R(p) > 0$, $R(a) \geqslant 0$, $R(b) \geqslant 0$, and $\alpha^*_{l+k+1} = (\gamma+\delta+k)/n$ for $k = 0, 1, \ldots, (n-1)$.

On replacing a by λp, b by μp and $(p/n)^n$ by z, (12) can be put in a more compact form :

$$2\int_0^{\frac{\pi}{2}} \sin^{2\delta-1}\theta \cos^{2\gamma-1}\theta\,(1 + \lambda\cos^2\theta + \mu\sin^2\theta)^{-\gamma-\delta}$$
$$\times E\left\{l\,;\ \alpha_r : m\,;\ \rho_s : z\left(\frac{1+\lambda\cos^2\theta+\mu\sin^2\theta}{\cos^2\theta}\right)^n\right\} d\theta$$
$$= \Gamma(\delta)n^{-\delta}(1+\lambda)^{-\gamma}(1+\mu)^{-\delta}E\{l+n\,;\ \alpha_r : m+n\,;\ \rho_s : (1+\lambda)^n z\}, \quad\ldots\ldots\ldots\ldots\ldots(13)$$

where $R(\gamma) > 0$, $R(\delta) > 0$, $R(z) > 0$, $\lambda \geqslant 0$ and $\mu \geqslant 0$.

The substitution $\cos^2\theta = t$ in (13) yields (10).

Below we give some particular cases of (10) obtained by giving suitable values to its parameters.

(i) If we take $\lambda = 0$ and $\mu = 1$, then after slight changes in the variable we get

$$\int_0^1 t^{\delta-1}(1-t)^{\gamma-1}(1+t)^{-\gamma-\delta} E\left\{l\,;\ \alpha_r : m\,;\ \rho_s : z\left(\frac{1+t}{1-t}\right)^n\right\} dt$$
$$= \Gamma(\delta)(2n)^{-\delta} E\{l+n\,;\ \alpha_r : m+n\,;\ \rho_s : z\}, \quad\ldots\ldots\ldots\ldots\ldots(14)$$

where $R(\gamma) > 0$, $R(\delta) > 0$ and $R(z) > 0$.

(ii) Similarly when $\mu = 0$ and $\lambda = 1$, we find that

$$\int_0^1 t^{\gamma-1}(1-t)^{\delta-1}(1+t)^{-\gamma-\delta} E\left\{l\,;\ \alpha_r : m\,;\ \rho_s : z\left(1+\frac{1}{t}\right)^n\right\} dt$$
$$= \Gamma(\delta)2^{-\gamma}n^{-\delta} E\{l+n\,;\ \alpha_r : m+n\,;\ \rho_s : 2^n z\}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

where $R(\gamma) > 0$, $R(\delta) > 0$ and $R(z) > 0$.

Lastly when $\lambda = \mu$, then (10) reduces to a known result [**4**, p. 407].

5. The following results are to be established here :

$$\int_0^\infty \exp(-t^n)\, E(\lambda, \mu : : t^n) t^{n\beta-1} E\left(l ;\ \alpha_\sigma : m ;\ \rho_i : \frac{z}{t^s}\right) dt$$

$$= \Gamma(\lambda)\Gamma(\mu) n^{\Sigma\alpha_\sigma - \Sigma\rho_i - \frac{1}{2}l + \frac{1}{2}m - \frac{s}{2}} s^{\beta-\frac{1}{2}} (2\pi)^{(\frac{1}{2}-\frac{1}{2}n)(l-m)+\frac{1}{2}n-\frac{1}{2}s} \sum_{r=0}^{n-1} \left[\left(-s^{-s/n} n^{m-l+1} z \right)^{-r} \right.$$

$$\left. \times E \left\{ \begin{matrix} \frac{\alpha_1+r}{n}, \dots, \frac{\alpha_l+r+n-1}{n}, \gamma_1, \dots, \gamma_{2s} : s^{-s}(n^{m-l+1}z)^n \\ \frac{r+1}{n}, \dots * \dots, \frac{r+n}{n}, \frac{\rho_1+r}{n}, \dots, \frac{\rho_m+r+n-1}{n}, \delta_1, \dots, \delta_s \end{matrix} \right\} \right], \quad \dots\dots\dots(16)$$

where n and s are positive integers such that n is odd and $s < n$, $R(z) > 0$, $R(\lambda+\beta) > 0$, $R(\mu+\beta) > 0$,

$$\gamma_{k+1} = \frac{n\beta + sr + n\lambda + nk}{ns},$$

$$\gamma_{k+s+1} = \frac{n\beta + sr + n\mu + nk}{ns}$$

and

$$\delta_{k+1} = \frac{n\beta + sr + n\lambda + n\mu + nk}{ns}.$$

The asterisk indicates that the parameter n/n is omitted. When n is even, the argument of the E-function is to be multiplied by $e^{\pm i\pi}$.

$$\int_0^\infty t^{n\beta-1} K_\mu(t^n)\, E\left(l ;\ \alpha_\sigma : m ;\ \rho_i : \frac{z}{t^{2s}}\right) dt$$

$$= \sqrt{\frac{\pi}{2}}\ n^{\Sigma\alpha_\sigma - \Sigma\rho_i - \frac{1}{2}l + \frac{1}{2}m - \frac{s}{2}} (2s)^{\beta-1} (2\pi)^{(\frac{1}{2}-\frac{1}{2}n)(l-m)+\frac{1}{2}n-s} \sum_{r=0}^{n-1} \left[\left\{ (-2s)^{-2s/n}\, n^{m-l+1} z \right\}^{-r} \right.$$

$$\left. \times E \left\{ \begin{matrix} \frac{\alpha_1+r}{n}, \dots, \frac{\alpha_l+r+n-1}{n}, \phi_1, \dots, \phi_{2s} : (2s)^{-2s}(n^{m-l+1}z)^n \\ \frac{r+1}{n}, \dots * \dots, \frac{r+n}{n}, \frac{\rho_1+r}{n}, \dots, \frac{\rho_m+r+n-1}{n} \end{matrix} \right\} \right], \quad \dots\dots(17)$$

where n and s are positive integers such that n is odd and $n > 2s$, $R(\beta \pm \mu) > 0$, $R(z) > 0$ and

$$\phi_{k+1} = \frac{n\beta + 2sr + n\mu + 2nk}{2ns}, \quad \phi_{s+k+1} = \frac{n\beta + 2sr - n\mu + 2nk}{2ns},$$

for $k = 0, 1, \dots, s-1$. For even n, the argument of the E-function is to be multiplied by $e^{\pm i\pi}$.

The following results are required in the proof [10, p. 172], [5, p. 92], [11, p. 116].

$$\int_0^\infty u^{\nu-1}(u+z)^{-\alpha}{}_2F_1\left(\lambda, \mu\,;\ \nu\,;\ -\frac{u}{z}\right) E\left\{l\,;\ \alpha_r : m\,;\ \rho_s : \left(\frac{u+z}{n}\right)^n\right\} du$$

$$= \Gamma(\nu) z^{\nu-\alpha} n^{-\nu} E\{l+2n\,;\ \alpha_r : m+2n\,;\ \rho_s : (z/n)^n\}, \qquad (18)$$

where $R(\nu) > 0$, $R(\alpha+\lambda-\nu) > 0$, $R(\alpha+\mu-\nu) > 0$, $|\arg z| < \pi$, $z \neq 0$, $n\alpha_{l+k+1} = \alpha+\lambda-\nu+k$, $n\alpha_{l+n+k+1} = \alpha+\mu-\nu+k$, $n\rho_{m+k+1} = \alpha+k$, $n\rho_{m+n+k+1} = \alpha+\lambda+\mu-\nu+k$, for $k = 0, 1, \ldots, (n-1)$.

$$\int_0^\infty \exp(-u^n)\, E(l\,;\ \alpha_\sigma : m\,;\ \rho_i : z/u^s) u^{\gamma-1}\, du$$

$$= n^{\Sigma\alpha_\sigma - \Sigma\rho_i - \frac{1}{2}l + \frac{1}{2}m - \frac{1}{2}s\gamma/n - \frac{1}{2}} (2\pi)^{(\frac{1}{2}-\frac{1}{2}n)(l-m)+\frac{1}{2}n-\frac{1}{2}s}$$

$$\times \sum_{t=0}^{n-1}\left[(-s^{-s/n} n^{m-l+1} z)^{-t} E\left\{\begin{matrix}\dfrac{\gamma+ts}{ns}, \ldots, \dfrac{\gamma+ts+(s-1)n}{ns}, \dfrac{\alpha_1+t}{n}, \ldots, \dfrac{\alpha_l+t+n-1}{n} \\ : s^{-s}(n^{m-l+1}z)^n \\ \dfrac{t+1}{n}, \ldots * \ldots \dfrac{t+n}{n}, \dfrac{\rho_1+t}{n}, \ldots, \dfrac{\rho_m+t+n-1}{n}\end{matrix}\right\}\right], \qquad (19)$$

n and s being positive integers such that n is odd, $s < n$, $R(\gamma) > 0$. If n is even, then the argument of the E-function should be multiplied by $e^{\pm i\pi}$.

$$\int_0^\infty (p+u)^{-\gamma}(u^2+2pu)^{-\frac{1}{2}\lambda} E\left\{l\,;\ \alpha_r : m\,;\ \rho_s : \left(\frac{p+u}{2n}\right)^{2n}\right\} P_\mu^\lambda\left(1+\frac{u}{p}\right) du$$

$$= p^{1-\gamma-\lambda}(2n)^{\lambda-1} E\left\{l+2n\,;\ \alpha_r : m+2n\,;\ \rho_s : \left(\frac{p}{2n}\right)^{2n}\right\}, \qquad (20)$$

where $R(\lambda) < 1$, $R(\lambda+\gamma-\mu-1) > 0$, $R(\lambda+\gamma+\mu) > 0$, $p \neq 0$, $|\arg p| < \frac{1}{2}\pi(l-m+1)$, $2n\alpha_{l+k+1} = \lambda+\gamma+\mu+2k$, $2n\alpha_{l+n+k+1} = \lambda+\gamma-\mu-1+2k$, $2n\rho_{m+k+1} = \gamma+1+2k$, $2n\rho_{m+n+k+1} = \gamma+2k$, for $k = 0, 1, \ldots, (n-1)$.

From (19) we deduce that

$$g_1(t) = e^{-at} t^{\gamma-1} E\left(l\,;\ \alpha_\sigma : m\,;\ \rho_i : \frac{1}{t^{s/n}}\right)$$

$$\doteqdot p n^{\Sigma\alpha_\sigma - \Sigma\rho_i - \frac{1}{2}l + \frac{1}{2}m - \frac{1}{2}s\gamma - \frac{1}{2}} (2\pi)^{(\frac{1}{2}-\frac{1}{2}n)(l-m)+\frac{1}{2}n-\frac{1}{2}s} (p+a)^{-\gamma}$$

$$\times \sum_{r=0}^{n-1}\left[\{-s^{-s/n} n^{m-l+1}(p+a)^{s/n}\}^{-r} E\left\{\begin{matrix}\dfrac{n\gamma+sr}{ns}, \ldots, \dfrac{n\gamma+sr+(s-1)n}{ns}, \dfrac{\alpha_1+r}{n}, \ldots, \dfrac{\alpha_l+r+n-1}{n} \\ : s^{-s}(n^{m-l+1})^n (p+a)^s \\ \dfrac{r+1}{n}, \ldots * \ldots, \dfrac{r+n}{n}, \dfrac{\rho_1+r}{n}, \ldots, \dfrac{\rho_m+r+n-1}{n}\end{matrix}\right\}\right]$$

$$= \phi_1(p),$$

where $R(\gamma) > 0$, $R(p) > 0$, $R(a) > 0$, and [1, p. 212]

$$g_2(t) = t^{\nu-1}{}_2F_1(\lambda, \mu\,;\nu\,;\ -t/a)$$

$$\doteqdot \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)} p^{1-\nu} E(\lambda, \mu : : ap)$$

$$= \phi_2(p),$$

where $R(\nu) > 0$, $R(p) > 0$, $R(a) > 0$.

Applying (5) to the above relations and using (18), we obtain after a little simplification and changes in the parameters, the relation (16).

Next take

$$g_1(t) = e^{-at}t^{\gamma-1}E\left\{l\,;\ \alpha_\sigma : m\,;\ \rho_i : \frac{1}{t^{2s/n}}\right\}$$

$$\doteqdot n^{\Sigma\alpha_\sigma-\Sigma\rho_i-\frac{1}{2}l+\frac{1}{2}m-\frac{1}{2}}p\,(p+a)^{-\gamma}(2s)^{\gamma-\frac{1}{2}}$$

$$\times\ (2\pi)^{(\frac{1}{2}-\frac{1}{2}n)(l-m)+\frac{1}{2}n-s}\sum_{r=0}^{n-1}\{(-2s)^{-2s/n}n^{m-l+1}\}^{-r}(p+a)^{-2sr/n}$$

$$\times\ E\left\{\begin{matrix}\dfrac{n\gamma+2sr}{2ns},\ldots,\dfrac{n\gamma+2sr+(2s-1)n}{2ns},\dfrac{\alpha_1+r}{n},\ldots,\dfrac{\alpha_l+r+n-1}{n} \\ :(2s)^{-2s}(n^{m-l+1})^n(p+a)^{2s} \\ \dfrac{r+1}{n},\ldots*\ldots,\dfrac{r+n}{n},\dfrac{\rho_1+r}{n},\ldots,\dfrac{\rho_m+r+n-1}{n}\end{matrix}\right\}$$

$$= \phi_1(p),$$

where $R(\gamma) > 0$, and [**1**, p. 278]

$$g_2(t) = \left(\frac{\pi}{2a}\right)^{\frac{1}{2}}(t^2+2at)^{-\lambda/2}\,P_\mu^\lambda\left(1+\frac{t}{a}\right)$$

$$\doteqdot p^{\lambda+\frac{1}{2}}e^{ap}K_{\mu+\frac{1}{2}}(ap)$$

$$= \phi_2(p),$$

where $R(\lambda) < 1$, $|\arg a| < \pi$. Again using these relations in (5) and evaluating the integral on the left with the help of (20), we obtain (17) after a little simplification and changes in the parameters.

On taking $n = 2$, $s = 1$ in (16) we get a result given by Ragab [**6**, p. 418]. When λ or μ is zero, (16) reduces to (19).

Similarly, taking $n = 4$ and $s = 1$ in (17), we get an integral due to Sharma [**12**, p. 161]. When $\mu = \pm\frac{1}{2}$ then (17) reduces to (19).

I am highly grateful to Dr C. B. Rathie for his guidance in the preparation of this paper.

REFERENCES

1. A. Erdélyi, *Tables of integral transforms*, Vol. I (New York, 1954).

2. S. Goldstein, Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder functions, *Proc. London Math. Soc.*, (2) **34** (1932), 103–125.

3. Hari Shanker, Some definite integrals involving confluent hypergeometric functions, *J. London Math. Soc.*, **23** (1948), 44–49.

4. T. M. MacRobert, *Functions of a complex variable* (London, 1954).

5. T. M. MacRobert, Integrals allied to Airy's integrals, *Proc. Glasgow Math. Assoc.*, **3** (1957), 91–93.

6. F. M. Ragab, Integrals involving E-functions and Bessel functions, *Koninkl. Nederl. Akademie van Wetenschappen, Amsterdam*, **57** (1954), 414–423.

7. F. M. Ragab, The inverse Laplace transform of an exponential function, *Communications on Pure and Applied Mathematics* (New York University), **11** (1958), 115–127.

8. C. B. Rathie, Some infinite integrals involving E-functions, *J. Indian Math. Soc.*, (4), **17** (1953), 167–175.

9. C. B. Rathie, Some results involving hypergeometric and E-functions, *Proc. Glasgow Math. Assoc.*, **2** (1955), 132–138.

10. C. B. Rathie, A few infinite integrals involving E-functions, *Proc. Glasgow Math. Assoc.*, **2** (1956), 170–172.

11. K. C. Sharma, Infinite integrals involving products of Legendre functions, *Proc. Glasgow Math. Assoc.*, **3** (1957), 111–118.

12. K. C. Sharma, Infinite integrals involving E-functions, *Proc. Nat. Inst. Sci., India*, **25** (1959), 161–165.

Maharana Bhupal College
Udaipur