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A Note on p-Harmonic 1-Forms on Complete Manifolds

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Abstract. In this paper we prove that there is no nontrivial L^q-integrably p-harmonic 1-form on a complete manifold with nonnegatively Ricci curvature ($0 < q < \infty$).

1 Introduction

Let (M, g) be a Riemannian manifold, and let u be a real C^∞ function on M. Fix $p \in R$, $p > 1$ and consider a compact domain $\Omega \subset M$. The p-energy of u on Ω, is defined to be

$$E_p(\Omega, u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dv_g. \tag{1}$$

The function u is said to be p-harmonic on M if u is a critical point of $E_p(\Omega, *)$ for every compact domain $\Omega \subset M$. Equivalently, u satisfies the Euler-Lagrange equation.

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0. \tag{2}$$

Thus, the concept of p-harmonic function is a natural generalization of that of harmonic function, that is, of a critical point of the 2-energy functional.

Definition 1.1 A p-harmonic 1-form is a differentiable 1-form on M satisfying the following properties:

$$\begin{cases} d\omega = 0 \\ d^*(|\omega|^{p-2}\omega) = 0 \end{cases} \tag{3}$$

where d^* is the codifferential operator. It is easy to see that the differential of a p-harmonic function is a p-harmonic 1-form.

In [1], R. E. Greene and H. Wu showed that there is no nonzero L^q ($1 < q < \infty$) harmonic 1-form on a complete noncompact manifold with nonnegative Ricci curvature. The purpose of this paper is to prove a nonexistence theorem of L^q p-harmonic 1-form.

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In [2], L. Saloff-Coste showed that the condition of nonnegative Ricci curvature implies a Sobolev inequality of the form:

$$\left(\int_{B_x(r)} f^{\frac{2\mu}{\mu-2}}\right)^{\frac{\mu-2}{\mu}} \leq C_0 V_x(r)^{\frac{-2}{\mu}} \left(r^2 \int_{B_x(r)} |\nabla f|^2\right) \tag{4}$$

for any compactly supported function $f \in H^c_{1,2}\big(B_x(r)\big)$, where $C_0 > 0$ and $\mu > 2$ are some fixed constants, and $x \in M$, $r > 0$ are arbitrary. By the Bochner formula and running the Moser iteration argument in [3] or [4], using (4), we obtain:

Main Theorem *If M is a complete noncompact manifold with nonnegative Ricci curvature, then no nonzero p-harmonic 1-form is in $L^q(M)$, $0 < q < \infty$ ($p > 1$).*

As the application of the above theorem, we obtain a Liouville-type theorem of p-harmonic maps which can be viewed as a generalization of the result due to Schoen and Yau [5].

2 Proof of Main Theorem

Lemma 2.1 **([2])** *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature; if $n > 2$, then there exists C_0, depending on n, such that, for any ball $B_x(r)$, we have:*

$$\left(\int_{B_x(r)} f^{\frac{2\mu}{\mu-2}}\right)^{\frac{\mu-2}{\mu}} \leq C_0 V_x(r)^{\frac{-2}{\mu}} \left(r^2 \int_{B_x(r)} |\nabla f|^2\right) \tag{5}$$

where $f \in C_0^\infty(B)$, $\mu = n$.

If $n \leq 2$, then the above inequality holds for any $\mu > 2$.

Let ω be a smooth 1-form on M, and X be a vector field on M. Then we know $\nabla_X|\omega|^2 = 2 \cdot \langle \nabla_X \omega, \omega\rangle$. Then, by the Schwarz inequality, we have:

Lemma 2.2 **(First Kato inequality)** *Let ω be a differentiable 1-form on M. Then*

$$|\omega| \cdot |\nabla\omega| \geq \frac{1}{2}\big|\nabla|\omega|^2\big|. \tag{6}$$

Lemma 2.3 *Let ω be a p-harmonic 1-form on M. Let η be a compactly supported nonnegative smooth function on M, and $\phi = \eta \cdot |\omega|^{\tilde{q}-1}$, $\tilde{q} \geq p + 5$. Then*

$$\begin{aligned}\int_M \phi^2 \cdot \langle \Delta\omega, \omega\rangle = {} & \frac{(p-2)(2\tilde{q}-p-4)}{4} \cdot \int_M \eta^2 \cdot |\omega|^{2\tilde{q}-6} \cdot (\langle d|\omega|^2, \omega\rangle)^2 \\ & + (p-2) \cdot \int_M \eta \cdot |\omega|^{2\tilde{q}-4} \cdot \langle d|\omega|^2, \omega\rangle \cdot \langle d\eta, \omega\rangle.\end{aligned} \tag{7}$$

Proof By a straightforward computation, we have

$$
\begin{aligned}
-\int_M &\langle d(\eta^2\cdot|\omega|^{\bar q-1}\cdot d^*\omega), |\omega|^{\bar q-1}\cdot\omega\rangle\\
&=\int_M \bigl(d^*(\eta^2\cdot|\omega|^{\bar q-1}\cdot\omega)+\langle d(\eta^2\cdot|\omega|^{\bar q-1}),\omega\rangle\bigr)\cdot\langle d|\omega|^{\bar q-p+1}, |\omega|^{p-2}\cdot\omega\rangle\\
&=\int_M \bigl(\langle d(\eta^2\cdot|\omega|^{\bar q-1}),\omega\rangle-\langle d(\eta^2\cdot|\omega|^{\bar q-p+1}), |\omega|^{p-2}\omega\rangle\bigr)\\
&\qquad\cdot\langle d(|\omega|^{\bar q-p+1}), |\omega|^{p-2}\omega\rangle\\
&=\int_M \bigl(\eta^2\langle d(|\omega|^{\bar q-1}),\omega\rangle-\eta^2\langle d(|\omega|^{\bar q-p+1}), |\omega|^{p-2}\omega\rangle\bigr)\cdot\langle d(|\omega|^{\bar q-p+1}), |\omega|^{p-2}\omega\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\int_M &\langle d^*\omega\cdot d(\eta^2|\omega|^{\bar q-1}), |\omega|^{\bar q-1}\omega\rangle\\
&=\int_M \langle d(\eta^2\cdot|\omega|^{\bar q-1}),\omega\rangle\cdot\bigl(d^*(|\omega|^{\bar q-1}\omega)+\langle d|\omega|^{\bar q-1},\omega\rangle\bigr)\\
&=\int_M \bigl(\langle d|\omega|^{\bar q-1},\omega\rangle-\langle d(|\omega|^{\bar q-p+1}), |\omega|^{p-2}\omega\rangle\bigr)\cdot\langle d[\eta^2\cdot|\omega|^{\bar q-1}],\omega\rangle\\
&=\int_M \langle d|\omega|^{\bar q-1},\omega\rangle\cdot\{\eta^2\langle d[|\omega|^{\bar q-1}],\omega\rangle+|\omega|^{\bar q-1}\cdot\langle d\eta^2,\omega\rangle\}\\
&\qquad-\int_M \langle d[|\omega|^{\bar q-p+1}], |\omega|^{p-2}\omega\rangle\cdot\{\eta^2\langle d(|\omega|^{\bar q-1}),\omega\rangle+|\omega|^{\bar q-1}\cdot\langle d\eta^2,\omega\rangle\}
\end{aligned}
$$

where we have used $d^*(|\omega|^{p-2}\omega)=0$. By the last two inequalities, we have:

$$
\begin{aligned}
\int_M &\phi^2\cdot\langle\Delta\omega,\omega\rangle\\
&=\int_M \eta^2\cdot|\omega|^{2\bar q-2}\langle\Delta\omega,\omega\rangle\\
&=-\int_M \langle\eta^2\cdot|\omega|^{\bar q-1}\cdot dd^*\omega, |\omega|^{\bar q-1}\cdot\omega\rangle\\
&=-\int_M \langle d(\eta^2\cdot|\omega|^{\bar q-1}\cdot d^*\omega), |\omega|^{\bar q-1}\cdot\omega\rangle+\int_M \langle d^*\omega\cdot d(\eta^2|\omega|^{\bar q-1}), |\omega|^{\bar q-1}\omega\rangle\\
&=\int_M \eta^2\bigl(\langle d(|\omega|^{\bar q-1}),\omega\rangle^2-\langle d(|\omega|^{\bar q-p+1}), |\omega|^{p-2}\omega\rangle^2\bigr)\\
&\qquad+\int_M \bigl(\langle d|\omega|^{\bar q-1},\omega\rangle-\langle d(|\omega|^{\bar q-p+1}), |\omega|^{p-2}\omega\rangle\bigr)\cdot|\omega|^{\bar q-1}\cdot\langle d\eta^2,\omega\rangle
\end{aligned}
$$

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$$= \frac{(p-2)(2\tilde{q}-p-4)}{4} \cdot \int_M \eta^2 \cdot |\omega|^{2\tilde{q}-6} \cdot (\langle d|\omega|^2, \omega\rangle)^2 + (p-2) \cdot \int_M \eta \cdot |\omega|^{2\tilde{q}-4} \cdot \langle d|\omega|^2, \omega\rangle \cdot \langle d\eta, \omega\rangle.$$

Theorem 2.4 *If M is a complete noncompact manifold with nonnegative Ricci curvature, then no nonzero p-harmonic 1-form is in $L^q(M)$, $0 < q < \infty$ ($p > 1$).*

Proof Let $e_1, e_2, \ldots, e_m$ be a local orthonormal frame on M. By a straightforward computation, we have

$$\frac{1}{2}\Delta(|\omega|)^2 = \langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + \sum_{i=1}^{m} \omega\big(\operatorname{Ric}_M(e_i)\big) \cdot \omega(e_i) \geq \langle \Delta\omega, \omega\rangle + |\nabla\omega|^2. \tag{8}$$

Let η be a compactly supported nonnegative smooth function on M and $\phi = \eta \cdot |\omega|^{\tilde{q}-1}$, $\tilde{q} \geq q_0$, $q_0 = p + \frac{1}{p-1} + 5$. Integrating by parts, (8) yields

$$\int_M \phi \langle \nabla\phi, \nabla(|\omega|)^2\rangle + \int_M \phi^2 \langle \Delta\omega, \omega\rangle + \int_M \phi^2 |\nabla\omega|^2 \leq 0. \tag{9}$$

(a) When $P \geq 2$, (7) implies:

$$\int_M \phi^2 \langle \Delta\omega, \omega\rangle \geq -(p-2) \int_M \eta \cdot |d\eta| \cdot \big|\nabla(|\omega|)^2\big| \cdot |\omega|^{2\tilde{q}-2}. \tag{10}$$

By (6), (8), (10), we have

$$\begin{aligned} 0 &\geq \int_M \eta \cdot |\omega|^{\tilde{q}-1} \cdot \langle \nabla(\eta \cdot |\omega|^{\tilde{q}-1}), \nabla(|\omega|)^2\rangle \\ &\quad - (p-2) \cdot \int_M \eta \cdot |d\eta| \cdot \big|\nabla(|\omega|)^2\big| \cdot |\omega|^{2\tilde{q}-2} \\ &\quad + \int_M \eta^2 \cdot |\nabla\omega|^2 \cdot |\omega|^{2\tilde{q}-2} \\ &\geq \left(\frac{1}{4} + \frac{\tilde{q}-1}{2}\right) \cdot \int_M \eta^2 \cdot \big|\nabla|\omega|^2\big|^2 \cdot |\omega|^{2\tilde{q}-4} \\ &\quad - (p-1) \cdot \int_M \eta \cdot |d\eta| \cdot \big|\nabla|\omega|^2\big| \cdot |\omega|^{2\tilde{q}-2}. \end{aligned} \tag{11}$$

(b) When $1 < p < 2$, (7) implies:

$$\int_M \phi^2 \cdot \langle \Delta\omega, \omega\rangle \geq \frac{(p-2)(2\tilde{q}-p-4)}{4} \cdot \int_M \eta^2 \cdot \big|\nabla|\omega|^2\big|^2 \cdot |\omega|^{2\tilde{q}-4} + (p-2) \cdot \int_M \eta \cdot |d\eta| \cdot \big|\nabla|\omega|^2\big| \cdot |\omega|^{2\tilde{q}-2}. \tag{12}$$

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By (6), (8), (12), and $\tilde{q} \geq q_0$, $q_0 = p + \frac{1}{p-1} + 5$; we have

$$
\begin{aligned}
0 &\geq \left(\frac{1}{4} + \frac{\tilde{q}-1}{2} + \frac{(p-2)(2\tilde{q}-p-2)}{4}\right) \cdot \int_M \eta^2 \cdot \left|\nabla|\omega|^2\right|^2 \cdot |\omega|^{2\tilde{q}-4} \\
&\qquad + (p-3) \cdot \int_M \eta \cdot |d\eta| \cdot \left|\nabla|\omega|^2\right| \cdot |\omega|^{2\tilde{q}-2} \\
&\geq \left(\frac{1}{4}\right) \int_M \eta^2 \cdot \left|\nabla|\omega|^2\right|^2 \cdot |\omega|^{2\tilde{q}-4} \\
&\qquad + (p-3) \cdot \int_M \eta \cdot |d\eta| \cdot \left|\nabla|\omega|^2\right| \cdot |\omega|^{2\tilde{q}-2}.
\end{aligned} \tag{13}
$$

In any event, (11) and (13) imply that we have the inequality:

$$
\begin{aligned}
0 \geq \left(\frac{1}{4}\right) \int_M \eta^2 \cdot \left|\nabla|\omega|^2\right|^2 \cdot |\omega|^{2\tilde{q}-4} \\
- (p+1) \cdot \int_M \eta \cdot |d\eta| \cdot \left|\nabla|\omega|^2\right| \cdot |\omega|^{2\tilde{q}-2}.
\end{aligned} \tag{14}
$$

By Young's inequality, we have

$$
\begin{aligned}
(p+1) \cdot \eta \cdot |d\eta| \cdot \left|\nabla|\omega|^2\right| \cdot |\omega|^{2\tilde{q}-2} \leq \left(\frac{1}{8}\right) \eta^2 \cdot \left|\nabla|\omega|^2\right|^2 \cdot |\omega|^{2\tilde{q}-4} \\
+ 2(p+1)^2 \cdot |d\eta|^2 \cdot |\omega|^{2\tilde{q}}.
\end{aligned}
$$

(14) becomes:

$$
\int_M \eta^2 \cdot \left|\nabla|\omega|^{\tilde{q}}\right|^2 \leq 4 \cdot \tilde{q}^2 \cdot (p+1)^2 \cdot \int_M |d\eta|^2 \cdot |\omega|^{2\tilde{q}}. \tag{15}
$$

Next, we fix $0 < \rho < \gamma \leq R$, $o \in M$, and let $\eta \in C_0^\infty\big(B_0(\gamma)\big)$ be the cut-off function

$$
\eta(x) = \begin{cases} 1, & x \in B_o(\rho) \\ 0, & x \in M \setminus B_o(\gamma) \end{cases}
$$

$\eta(x) \in [0,1]$ on M, $|\nabla\eta| \leq \frac{2}{\gamma-\rho}$.

Let $k = \frac{\mu}{\mu-2}$, by the Sobolev inequality (5) and (15), we have:

$$
\begin{aligned}
\left\{\int_{B_o(\rho)} |\omega|^{2\tilde{q}\cdot k}\right\}^{\frac{1}{k}} &\leq \left\{\int_{B_o(R)} (\eta \cdot |\omega|^{\tilde{q}})^{2k}\right\}^{\frac{1}{k}} \\
&\leq C(n,p) \cdot \tilde{q}^2 \cdot \frac{R^2}{(\gamma-\rho)^2} \cdot V\big(B_o(R)\big)^{-\frac{2}{\mu}} \cdot \int_{B_o(\gamma)} |\omega|^{2\tilde{q}}
\end{aligned} \tag{16}
$$

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for some constant $C(n,p) > 0$.

Define:

$$\tilde{q}_i = q_0 \cdot k^i$$
$$R_i = \rho + 2^{-i} \cdot (\gamma - \rho)$$

for each $i = 0, 1, 2 \dots$.

Observe that $\lim_{i\to\infty} R_i = \rho$. Applying (16) to $\tilde{q} = \tilde{q}_i$, $\rho = R_{i+1}$, and $\gamma = R_i$, and iterating the inequality. We conclude that:

$$\sup_{B_o(\rho)} |\omega|^{2q_0} \le C^1(n,p) \cdot \left(\frac{R}{\gamma - \rho}\right)^{\mu} \cdot V\big(B_o(R)\big)^{-1} \cdot \int_{B_o(\gamma)} |\omega|^{2q_0} \tag{17}$$

for some appropriate constant $C^1(n,p,\mu)$.

(a) When $q \ge 2q_0$, applying (17) to $\rho = \frac{R}{2}$, $\gamma = R$, we have:

$$\begin{aligned} \sup_{B_o(\frac{R}{2})} |\omega| &\le \left(2^{\mu} \cdot C^1(n,p,\mu)\right)^{\frac{1}{2q_0}} \cdot \left\{ \frac{\int_{B_o(R)} |\omega|^{2q_0}}{V\big(B_o(R)\big)} \right\}^{\frac{1}{2q_0}} \\ &\le \left(2^{\mu} \cdot C^1(n,p,\mu)\right)^{\frac{1}{2q_0}} \cdot \left\{ \frac{\int_{B_o(R)} |\omega|^{2q}}{V\big(B_o(R)\big)} \right\}^{\frac{1}{2q}}. \end{aligned} \tag{18}$$

(b) When $0 < q < 2q_0$. Let $h_i = \sum_{j=0}^{i} 2^{-j} \cdot \frac{R}{2}$, for each $i = 0, 1, 2, \dots$; applying (17) to $\rho = h_i$, $\gamma = h_{i+1}$, we have:

(19)

$$\begin{aligned} \sup_{B_o(h_i)} |\omega|^{2q_0} &\le C^1(n,p,\mu) \cdot \left(\frac{R}{h_{i+1} - h_i}\right)^{\mu} \cdot V\big(B_o(R)\big)^{-1} \cdot \int_{B_o(h_{i+1})} |\omega|^{2q_0} \\ &\le C^1(n,p,\mu) \cdot 2^{(i+2)\cdot\mu} \cdot V\big(B_o(R)\big)^{-1} \cdot \int_{B_o(h_{i+1})} |\omega|^{2q} \cdot \sup_{B_o(h_{i+1})} |\omega|^{2q_0 - q}. \end{aligned}$$

Denote $M(i) = \sup_{B_o(h_i)} |\omega|^{2q_0}$, (19) becomes:

$$M(i) \le C^1(n,p,\mu) \cdot 2^{(i+2)\cdot\mu} \cdot V\big(B_o(R)\big)^{-1} \cdot \int_{B_o(R)} |\omega|^{2q} \cdot M(i+1)^{1 - \frac{q}{2q_0}}. \tag{20}$$

Let $\lambda = 1 - \frac{q}{2q_0}$. Iterating the inequality, we conclude that:

$$M(0) \le \prod_{i=0}^{j-1} \left\{ C^1(n,p,\mu) \cdot 2^{(i+2)\cdot\mu} \cdot V\big(B_o(R)\big)^{-1} \cdot \int_{B_o(R)} |\omega|^{2q} \right\}^{\lambda^i} \cdot M(j)^{\lambda^j}. \tag{21}$$

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Let $j \to \infty$, we have:

$$\sup_{B_o(\frac{R}{2})} |\omega| \leq \left(2^{\mu} \cdot C^1(n,p,\mu)\right)^{\frac{1}{q}} \cdot 2^{\frac{2\mu \cdot q_0}{q^2}} \cdot \left\{ \frac{\int_{B_o(R)} |\omega|^q}{V\left(B_o(R)\right)} \right\}^{\frac{1}{q}}. \tag{22}$$

In any event, (18) and (22) imply that, for any $q > 0$, we have the inequality

$$\sup_{B_o(\frac{R}{2})} |\omega| \leq C^2(n,p,q,\mu) \cdot \left\{ \frac{\int_{B_o(R)} |\omega|^q}{V\left(B_o(R)\right)} \right\}^{\frac{1}{q}} \tag{23}$$

for some appropriate constant $C^2(n,p,q,\mu) > 0$ independent of $R > 0$.

On the other hand, S. T. Yau in [6] and E. Calabi in [7] independently showed that when M is a complete noncompact manifold with nonnegative Ricci curvature, then the volume of M is infinite. Hence, if $\int_M |\omega|^q < \infty$, by taking $R \to \infty$, we conclude that:

$$\sup_M |\omega| \leq 0.$$

Therefore, ω must be identically 0.

By the definition, we know that the differential of a p-harmonic function is a p-harmonic 1-form; thus we have the following corollary.

Corollary 2.5 *Let M be a complete noncompact manifold with nonnegative Ricci curvature, and u be a p-harmonic function on M. If $\int_M \|\nabla u\|^q < \infty$, $0 < q < \infty$, then u must be constant ($p > 1$).*

If ω is a harmonic 1-form, *i.e.*, $\Delta\omega = 0$, from the proof of Theorem 2.4 one can conclude the following corollary that can be viewed as a generalization of the result due to R. E. Greene and H. Wu [1].

Corollary 2.6 *Let M be a complete noncompact manifold with nonnegative Ricci curvature, and ω be a harmonic 1-form on M. If $\int_M |\omega|^q < \infty$; $0 < q < \infty$, then $\omega \equiv 0$.*

3 Liouville Type Theorem of p-Harmonic Map

Let (M,g) be a complete Riemannian manifold (without boundary) of dimension m with metric g, and (N,h) be a complete one of dimension n with metric h. For a smooth map $U \colon M \longrightarrow N$, fix a number $1 < p < \infty$, and consider a compact domain $\Omega \subset M$, we defined the p-energy of u on Ω by

$$E_p(\Omega, u) = \frac{1}{p} \int_{\Omega} |du(x)|^p \, dv_g \tag{24}$$

where $|du(x)|$ is the norm of the differential $du(x)$ of u at $x \in \Omega$ and dv_g is the volume element of M. Let $u^{-1}TN$ be the induced vector bundle by u over M, then du can be viewed as a section of the bundle $\Lambda^1(u^{-1}TN) = T^*M \otimes u^{-1}TN$, and we denote by

$|du(x)|$ its norm at a point x of M, induced by the metrics g and h, *i.e.*, the Hilbert-Schmidt norm of the linear map $du(x)$.

We call u a p-harmonic map if it is a critical point of p-energy functional $E_p(\Omega, \cdot)$ for any compact domain $\Omega \subset M$. That is, u is a p-harmonic map if and only if

$$\frac{dE_p(u_t)}{dt}\Big|_{t=0} = 0 \tag{25}$$

for any one parameter family of maps $u_t \colon M \longrightarrow N$ with $u_0 = u$ and $u_t(x) = u(x)$ for $x \in M \setminus \Omega$. Note that 2-harmonic maps are harmonic maps by the definition. We define the p-tension field $\tau_p(u)$ of u by:

$$\tau_p(u) = -d^*(|du|^{p-2} du) \tag{26}$$

where $d^* \colon \Lambda^1(u^{-1}TN) \longrightarrow \Lambda^0(u^{-1}TN)$ is the codifferential operator. Equivalently, a smooth map $u \colon M \longrightarrow N$ is a p-harmonic map if and only if the p-tension field $\tau_p(u) = 0$.

Assuming that (M, g) is a complete noncompact Riemannian manifold with nonnegative Ricci curvature and (N, h) is a complete Riemannian manifold with nonpositive sectional curvature, denote the Ricci tensor of (M, g) by Ric_M, and the curvature tensor of (N, h) by ${}^N R$. Let $e_1, e_2, \ldots, e_m$ be a local orthonomal frame on M, by the Weitzenbock formula [8], we have:

$$\begin{aligned}\frac{1}{2}\Delta|du|^2 &= \langle \Delta du, du\rangle + |\nabla du|^2 + \sum_{i=1}^{m} \big\langle du\big(\mathrm{Ric}_M(e_i)\big), du(e_i)\big\rangle \\ &\qquad - \sum_{i=1,j=1}^{m} \big\langle {}^N R\big(du(e_j), du(e_i)\big) due_i, due_j\big\rangle \\ &\geq \langle \Delta du, du\rangle + |\nabla du|^2.\end{aligned} \tag{27}$$

Let η be a compactly supported nonnegative smooth function on M, and $\phi = \eta \cdot |du|^{\tilde q - 1}$, $\tilde q \geq p + 5$. As in Lemma 2.3, we have:

$$\begin{aligned}\int_M \phi^2 \cdot \langle \Delta du, du\rangle &= \frac{(p-2)(2\tilde q - p - 4)}{4} \cdot \int_M \eta^2 \cdot |du|^{2\tilde q - 6} \cdot \big|\langle d|du|^2, du\rangle\big|^2 \\ &\qquad + (p-2) \cdot \int_M \eta \cdot |du|^{2\tilde q - 4} \cdot \big\langle (\langle d|du|^2, du\rangle), (\langle d\eta, du\rangle)\big\rangle.\end{aligned} \tag{28}$$

By (27), (28), and as in Theorem 2.4, we can conclude the following theorem.

Theorem 3.1 *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and (N, h) be a complete Riemannian manifold with nonpositive sectional curvature, then each p-harmonic map u from M to N with $\int_M |du|^q \, dv_g < \infty$, $0 < q < \infty$ is a constant map $(p > 1)$.*

Remark Note that 2-harmonic maps are harmonic maps by the definition. Applying Theorem 3.1 with $p = 2$, one can obtain the result due to R. Schoen and S. T. Yau in [5].

Let M be a complete noncompact Riemannian manifold with nonegative Ricci curvature and N be a Riemannian manifold with nonpositive sectional curvature. Then each harmonic map u from M to N with finite energy has to be a constant map.

References

[1] R. Greene and H. Wu, *Harmonic forms on noncompact Riemannian and Kähler manifolds.* Michigan. Math. J. **28**(1981), 63–81.

[2] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds.* J. Differential Geom. **36**(1992), 417–450.

[3] J. Moser, *On Harnack's theorem for elliptic equations.* Comm. Pure Appl. Math. **14**(1961), 577–591.

[4] P. Li, *Lecture Notes on Geometric Analysis.* Lecture Notes Series **6**, Research Institute of Mathematics and Global Analysis Research Center, Seoul National University. Seoul, 1993.

[5] R. Schoen and S. T. Yau, *Harmonic Maps and the Topology of Stable Hypersurfaces and manifolds with nonnegative Ricci curvature.* Comment. Math. Helv. **39**(1976), 333–341.

[6] S. T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and the applications to geometry.* Indiana Math. J. **25**(1976), 659–679.

[7] E. Calabi, *On manifolds with nonnegative Ricci curvature.* Notices Amer. Math. Soc. **22**(1975), A205.

[8] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps.* CBMS Regional Conf. Series **50**, Amer. Math. Soc, 1983.

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