ON THE ASYMPTOTIC BEHAVIOR OF FUNCTIONS HARMONIC IN A DISC

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Let D be the open unit disc, and let C be the unit circle in the complex plane. Let f be a (real-valued) function that is harmonic in D. A simple continuous curve $\beta : z(t)$ $(0 \leqq t < 1)$ contained in D such that $|z(t)| \to 1$ as $t \to 1$ is a *boundary path* with *end* $\bar{\beta} \cap C$ (the bar denotes closure). If also $f(z(t)) \to a(-\infty \leqq a \leqq +\infty)$ as $t \to 1$, then β is an *asymptotic path* of f for the value a, and f is said to have the *asymptotic value* a. If there is an asymptotic path of f, for a value $a(-\infty \leqq a \leqq +\infty)$, with end the point ζ of C, then f is said to have the asymptotic value a at ζ. Let A denote the set of points of C at which f has an asymptotic value. If $-\infty \leqq u < v \leqq +\infty$, set $A(u, v) = \{\zeta \in C$: there exists a real number a such that f has the asymptotic value a at ζ and $u < a < v\}$, and set $A' = A(-\infty, +\infty)$. We use repeatedly the fact that $A(u, v)$ *is a Borel set* (see [5, Theorem 7 (iii)]), and is therefore measurable. Let $d(z, S)$ denote the Euclidean distance from the point z to the set S in the plane. For a sequence $\{\Gamma_n\}$ of Jordan arcs in D and an arc γ in C, the symbol $\Gamma_n \to \gamma$ means that to each $\varepsilon > 0$ there corresponds a natural number n_ε such that if $n > n_\varepsilon$, then

$$\Gamma_n \subset \{z : d(z, \gamma) < \varepsilon\} \text{ and } \gamma \subset \{z : d(z, \Gamma_n) < \varepsilon\}.$$

The following theorem is closely related to the theorem [4, Theorem 1] for meromorphic functions.

THEOREM 1. *Let $\zeta \in C$ and suppose that there exists a sequence $\{z_n\} \subset D$ such that $z_n \to \zeta$ and $f(z_n) \to +\infty$. Then at least one of the following three statements holds.*

(i) *Each open arc containing ζ contains the end of an asymptotic path of f for the value $+\infty$.*

(ii) *ζ is one endpoint of an arc $\gamma \subset C$ such that there exist a sequence $\{a_n\}$ of real numbers and a sequence $\{\Gamma_n\}$ of Jordan arcs such that*

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$$a_n \uparrow +\infty, \; \Gamma_n \subset \{z \in D : f(z) = a_n\}, \; \Gamma_n \to \gamma.$$

(iii) *For each real number M and open arc γ containing ζ, the set $\gamma \cap A(M, +\infty)$ has positive Lebesgue measure.*

Remarks. At each $\zeta \in C$, the real part of the elliptic modular function satisfies neither (ii) nor (iii). The real part of a holomorphic function having the spiral asymptotic values 0 and 1 satisfies, at each $\zeta \in C$, neither (i) nor (iii). The real part f of a holomorphic function constructed by MacLane [2, p. 75] is such that A' is dense on C, f has neither of the asymptotic values $\pm\infty$, and there exists a set $E \subset C$ with positive measure such that f does not have an asymptotic value at any point of E. Let ζ be a point of C such that the intersection of E with each open arc containing ζ has positive measure. Then neither (i) nor (ii) holds, and for each open arc γ containing ζ, the measure of $\gamma \cap A'$ is less than the measure of γ.

We first prove

LEMMA. *Let λ be a real number, and suppose that Δ is a component of $\{z \in D : f(z) > \lambda\}$. Then either there exists an asymptotic path α of f for the value $+\infty$ such that $\alpha \subset \Delta$, or there exists a set $E \subset C$ with positive exterior Lebesgue measure such that each $e^{i\theta} \in E$ is the end of an asymptotic path α_θ, for a finite value, such that $\alpha_\theta \subset \Delta$.*

Remark 1. By making simple modifications in the following proof, an analogous lemma for holomorphic functions can be established. Thus, the proofs of the theorems [3, Theorem 2] and [4, Theorem 1], which are based on the lemma [3, Lemma 2], can be simplified.

Remark 2. The proof of this lemma involves a combination of ideas from the papers [1], [2] and [3].

Proof of the lemma. Suppose that there does not exist an asymptotic path α of f for the value $+\infty$ such that $\alpha \subset \Delta$. We prove that there exist a (finite) number $\lambda' \geqq \lambda$ and a component Δ' of $\{z \in \Delta : f(z) > \lambda'\}$ such that f is bounded in Δ'. If this were not the case, we could choose $\lambda_n \uparrow +\infty$ ($\lambda_n > \lambda$), let Δ_1 be a component of $\{z \in \Delta : f(z) > \lambda_1\}$, let Δ_2 be a component of $\{z \in \Delta_1 : f(z) > \lambda_2\}$, and in this way define a sequence $\{\Delta_n\}$ such that Δ_{n+1} is a component of $\{z \in \Delta_n : f(z) > \lambda_{n+1}\}$. Let $\alpha : z(t)$ $(0 \leqq t < 1)$ be a boundary path that is eventually in each Δ_n; that is, to each n there corresponds t_n $(0\leqq t_n<1)$ such that $z(t)\in\Delta_n$ if $t_n<t<1$. Since α is an asymptotic path of f for the value $+\infty$ and $\alpha\subset\Delta$, we have a contradiction; and we have established the existence of λ' and Δ' with the stated properties.

By the minimum principle, Δ' is simply connected. Let $D_w=\{|w|<1\}$, and let φ be a conformal mapping of D_w onto Δ'. Set $F(w)=f(\varphi(w))(w\in D_w)$. The radial limit $F(e^{i\theta})$ of the bounded harmonic function F at $e^{i\theta}$ exists for almost all $e^{i\theta}$, and F has a Poisson integral representation in terms of the radial limits $F(e^{i\theta})$. Since $F(w)>\lambda'(w\in D_w)$, there exists a set $E'_w\subset\{|w|=1\}$, with positive measure, such that $F(e^{i\theta})>\lambda'$ if $e^{i\theta}\in E'_w$. Let E_w be a subset, with positive measure, of E'_w such that the radial limit $\varphi(e^{i\theta})$ of φ at $e^{i\theta}$ exists for each $e^{i\theta}\in E_w$. If $\varphi(e^{i\theta})\in D$, then $F(e^{i\theta})=f(\varphi(e^{i\theta}))=\lambda'$, so $\varphi(e^{i\theta})\in C$ if $e^{i\theta}\in E_w$. Set $E_z=\{\varphi(e^{i\theta}) : e^{i\theta}\in E_w\}$. By an extension of Löwner's lemma [6, p. 34], E_z has positive exterior measure. But if $\zeta\in E_z$, $\zeta=\varphi(e^{i\theta})$, the set $\{\varphi(re^{i\theta}) : 0\leqq r<1\}$ is a boundary path, with end ζ, that is contained in Δ' and on which f has the limit $F(e^{i\theta})$ at ζ. This completes the proof of the lemma.

Proof of Theorem 1. We clearly may suppose that $f(0)<f(z_n)-1$ $(n\geqq1)$. Let Δ_n be the component of $\{z\in D : f(z)>f(z_n)-1\}$ that contains z_n. Since $f(z_n)\to+\infty$, each disc $\{|z|\leqq r\}$ $(0<r<1)$ intersects only finitely many Δ_n. Since $0\notin\Delta_n$, there exists a level curve L_n on the boundary of Δ_n such that 0 and Δ_n are contained in different components of $D-L_n$. Thus, it is easy to see that if (ii) does not hold, then the diameter of Δ_n tends to zero as $n\to\infty$. We suppose now that neither (i) nor (ii) holds. We wish to prove that (iii) holds, and we let M be a real number and γ an open arc containing ζ. Let γ' be an open subarc of γ that contains ζ and is such that no asymptotic path of f for the value $+\infty$ has end contained in γ'. Let n_0 be such that $f(z_{n_0})-1>M$ and $\bar{\Delta}_{n_0}\cap C\subset\gamma'$. It follows from the lemma that $\gamma\cap A(M,+\infty)$ has positive exterior measure. Thus, since $A(M,+\infty)$ is measurable, (iii) holds; and the proof of Theorem 1 is complete.

It is well known that if f is bounded above in a neighborhood of the point ζ of C, then f has finite radial limits at almost all points of some open arc containing ζ. Thus, as a consequence of Theorem 1 we have

COROLLARY. *Let γ be an arc in C. Suppose that there exists a set $S\subset\gamma$, that is dense on γ* (i.e. $\gamma\subset\bar{S}$), *such that to each $\zeta\in S$ there corresponds a boundary* *path with end ζ on which f is bounded above. Then for each subarc γ' of γ, either f has the asymptotic value $+\infty$ at a point of γ', or $\gamma' \cap A'$ has positive measure.*

As a simple application of Theorem 1, we prove

THEOREM 2. *Let γ be an arc in C. Suppose that there exists a set $S \subset \gamma$, that is dense on γ, such that to each $\zeta \in S$ there corresponds a boundary path with end ζ on which f is either bounded above or bounded below. Then for each subarc γ' of γ, either f has an infinite asymptotic value at a point of γ', or $\gamma' \cap A'$ has positive measure. In particular, A is dense on γ.*

Remark. This result is closely related to theorems of MacLane [2, pp. 10, 25] for holomorphic functions.

Proof of Theorem 2. Suppose that there exists a subarc γ' of γ such that f does not have an infinite asymptotic value at a point of γ', and $\gamma' \cap A'$ does not have positive measure. Then, since A' is measurable, $\gamma' \cap A'$ has measure zero. Let ζ be an interior point of γ' and apply Theorem 1. Either there exists an asymptotic path of f for the value $+\infty$ whose end is a subarc of γ', or (ii) holds. In either case there exist a sequence $\{\Gamma_n\}$ of Jordan arcs in D and a subarc γ_0 of γ' such that $\Gamma_n \to \gamma_0$ and the minimum value of f on Γ_n tends to $+\infty$ as $n \to \infty$. Now let ζ be an interior point of γ_0 and apply Theorem 1 to the function $-f$. It follows as before that there exist a sequence $\{\Gamma'_n\}$ of Jordan arcs in D and a subarc γ_1 of γ_0 such that $\Gamma'_n \to \gamma_1$ and the maximum value of f on Γ'_n tends to $-\infty$ as $n \to \infty$. With this contradiction the proof of Theorem 2 is complete.

Similarly, we obtain

THEOREM 3. *Let γ be an arc in C. Suppose that there exists a set $S \subset \gamma$, that is dense on γ, such that to each $\zeta \in S$ there corresponds a boundary path with end ζ on which f is bounded. Then for each subarc γ' of γ, either f has both of the asymptotic values $+\infty$ and $-\infty$ at points of γ', or $\gamma' \cap A'$ has positive measure.*

The following global result is an immediate consequence of the lemma.

THEOREM 4. *Suppose that f is not bounded above. Then either f has the asymptotic value $+\infty$, or for each real number M, the set $A(M, +\infty)$ has positive measure.*

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Remark. The real part f of a holomorphic function constructed by MacLane [2, p. 71] is such that f has neither of the asymptotic values $\pm\infty$, and at each $\zeta \in C$,

$$\limsup_{z\to\zeta} f(z) = +\infty \text{ and } \liminf_{z\to\zeta} f(z) = -\infty.$$

It is now easy to see that we also have

THEOREM 5. *Either f has both $+\infty$ and $-\infty$ as asymptotic values, or A' has positive measure.*

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