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Approximation by Meromorphic Functions With Mittag-Leffler Type Constraints

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Abstract. Functions defined on closed sets are simultaneously approximated and interpolated by meromorphic functions with prescribed poles and zeros outside the set of approximation.

1 Introduction

Given a discrete set Z in the complex plane $\mathbb{C}$, the Mittag-Leffler theorem asserts the existence of a meromorphic function with prescribed singularities at the points of Z. For holomorphic functions, an analogous result allows one to prescribe the values of an entire function along with finitely many of its derivatives at the points of Z. By prescribing the "left tail" of the Laurent series at points of Z, it is possible [3] to combine these two classical results.

In the present paper, we further combine this process with simultaneous approximation on a closed set disjoint from the discrete set Z. As a consequence, we obtain a generalization of the main result in the recent paper of A. Sauer [9] on approximation by functions with prescribed zeros, poles and asymptotic behaviour.

2 Definitions and Basic Results

For $F \subset \mathbb{C}$, we denote by $\mathcal{H}(F)$ and $\mathcal{M}(F)$, the set of all holomorphic functions and meromorphic functions on F, respectively; we also denote the set of all functions continuous on F and holomorphic on F° by $\mathcal{A}(F)$, where F° is the interior of F. The Riemann sphere will be denoted by $\mathbb{C}_\infty$.

Definition 1 Let F be a closed subset of $\mathbb{C}$. A speed on F is a positive, continuous function on F. If ε is a speed on F, then F is called a *set of ε-approximation*, provided that for each $f \in \mathcal{A}(F)$ and each constant $\lambda > 0$, there is a function $g \in \mathcal{H}(\mathbb{C})$ such that for $z \in F$,

$$|f(z) - g(z)| < \lambda\varepsilon(z).$$

Definition 2 A closed subset F of $\mathbb{C}$ is called a *set of uniform approximation* if F is a set of ε-approximation for some (hence for each) positive constant ε.

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Note that, if F is a set of ε-approximation and the speed ε is bounded then F is, *a fortiori*, a set of uniform approximation. The following characterization of sets of uniform approximation is due to Arakelian [1].

Theorem 1 *A set F is a set of uniform approximation if and only if $\mathbb{C}_\infty \setminus F$ is connected and locally connected at ∞.*

Let us state another Theorem of Arakelian which will be useful in this paper (see [2, p. 39]).

Theorem 2 *Let $\varepsilon\colon [0,\infty) \to (0,\infty)$ be continuous and decreasing such that*

$$\int_1^\infty t^{-3/2} \log \varepsilon(t)\, dt > -\infty. \tag{1}$$

Then, for every set of uniform approximation F and for every function $f \in \mathcal{A}(F)$, there exists an entire function g such that

$$|f(z) - g(z)| < \varepsilon(|z|),$$

for all $z \in F$.

We may extend any continuous function $\varepsilon\colon [0,\infty) \to (0,\infty)$ to a function continuous on all of $\mathbb{C}$ by setting $\varepsilon(z) = \varepsilon(|z|)$. Let us call such a function ε satisfying the conditions in Theorem 2 a canonical speed. As an example, we can consider $\varepsilon\colon [0,\infty) \to (0,\infty)$ defined by

$$\varepsilon(t) := \exp(-t^{1/3}).$$

Then ε is a canonical speed satisfying $\varepsilon \leq 1$.

In the next example we will show that the decreasing condition in Theorem 2 can not be waived.

Example 1 Consider the set of uniform approximation

$$F = \{z \in \mathbb{C} : \Re z \geq 0\}.$$

We will construct $\varepsilon\colon [0,\infty) \to (0,1]$ continuous satisfying (1), with $\lim_{t\to\infty} \varepsilon(t) = 0$, in such a way that F is not a set of ε-approximation.

For each $n \in \mathbb{N}_1 := \mathbb{N} \setminus \{1\}$, set $F_n = \{z \in \mathbb{C} : |z| \leq n, \Re z \geq 0\}$ and $\alpha_n = \{z \in \mathbb{C} : |z| = n, \Re z > 0\}$, by the Two Constants Theorem (see [5]) there exists a decreasing sequence $\{\varepsilon_n : 0 < \varepsilon_n < 1, n \in \mathbb{N}\}$, such that if $f \in \mathcal{A}(F_n)$, $|f| \leq 1$ and $|f| \leq \varepsilon_n$ on α_n. Then

$$\max_{z\in K} |f(z)| < \frac{1}{n},$$

where $K = \{z \in \mathbb{C} : |z-1| \leq \frac{1}{3}\}$. Therefore if $f \in \mathcal{A}(F)$, $|f| \leq \varepsilon_n$ on α_n and $|f| \leq 1$, then $f(z) = 0$ for every $z \in K$, so $f \equiv 0$. Hence if $\varepsilon\colon [0,\infty) \to (0,1]$ is any continuous function such that for $n \in \mathbb{N}_1$, $\varepsilon(n) \leq \varepsilon_n$, then $f \in \mathcal{A}(F)$ and $|f| \leq \varepsilon$ implies $f \equiv 0$. This shows that F is not a set of ε-approximation. Indeed, consider $f(z) := 1/(z+1)$, $f \in \mathcal{A}(F) \setminus \mathcal{H}(\mathbb{C})$ and suppose there exists $g \in \mathcal{H}(\mathbb{C})$ such that on F

$$|f-g| < \varepsilon.$$

Thus $f = g$ on F, so $f = g$ on $\mathbb{C} \setminus \{-1\}$ which is a contradiction.

Among all continuous functions $\varepsilon\colon [0,\infty) \to (0,1]$ with $\varepsilon(n) \leq \varepsilon_n$, $n \in \mathbb{N}_1$, we construct one which satisfies (1) and $\lim_{t\to\infty} \varepsilon(t) = 0$.

Let $\tilde{\varepsilon}\colon [0,\infty) \to (0,1]$ be a continuous decreasing function satisfying (1). For $n > 1$, choose ε_n as above, and decreasing so rapidly that $\varepsilon_n < \tilde{\varepsilon}(n+1)$, and choose $0 < \eta_n < 1/2$ such that

$$\int_{n-\eta_n}^{n+\eta_n} t^{-3/2} \log \varepsilon_n \, dt > -\frac{1}{2^n}.$$

Now, we define a continuous function ε as follows: on $[n, n+\eta_n]$, it is the segment from the point (n, ε_n) to the point $\big(n+\eta_n, \tilde{\varepsilon}(n+\eta_n)\big)$, on $[n+\eta_n, n+1-\eta_{n+1}]$, it is equal to $\tilde{\varepsilon}(t)$ and on $[n+1-\eta_{n+1}, n+1]$, it is the segment from the point $\big(n+1-\eta_{n+1}, \tilde{\varepsilon}(n+1-\eta_{n+1})\big)$ to the point $(n+1, \varepsilon_{n+1})$, for each $n \geq 1$. We may define ε on $[0,1]$ by $\varepsilon(t) = \varepsilon_1$. Thus, considering $I_n := [n-\eta_n, n+\eta_n]$, we deduce

$$\begin{aligned}
\int_1^\infty t^{-3/2} \log \varepsilon(t)\, dt &= \int_{[1,\infty)\setminus \bigcup_{n=1}^\infty I_n} t^{-3/2} \log \varepsilon(t)\, dt \\
&\quad + \int_{\bigcup_{n=1}^\infty I_n} t^{-3/2} \log \varepsilon(t)\, dt \\
&\geq \int_1^\infty t^{-3/2} \log \tilde{\varepsilon}(t)\, dt + \sum_{n=1}^\infty \int_{I_n} t^{-3/2} \log \varepsilon_n \, dt, \\
&> -\infty,
\end{aligned}$$

as required.

To prove the next theorem we need two lemmas.

Lemma 1 *Let F be a set of uniform approximation and U an open neighbourhood of F. Then, there exists a simply connected open neighbourhood U_s of F such that $F \subset U_s \subset U$.*

Proof Let $\mathcal{W} := \{W_j : j \in J\}$ be the class of all bounded components of $\mathbb{C} \setminus U$. Using triangulation we may assume that ∂U is a locally polygonal neighbourhood of F, so $\mathcal{W}$ is locally finite.

For each $j \in J$, let $\tilde{W}_j$ be the component of $\mathbb{C} \setminus F$ containing W_j. Each $\widetilde{W_j}$ is unbounded because $\mathbb{C}_\infty \setminus F$ is connected.

By Theorem 1, $\mathbb{C}_\infty \setminus F$ is locally connected at ∞, so by a characterization of the local connectedness of $\mathbb{C}_\infty \setminus F$ at ∞, for every neighbourhood G_1 of ∞ there exists a neighbourhood $G_2 \subset G_1$ of ∞ with the property that each point $z \in G_2 \setminus F$, $z \neq \infty$ can be connected to ∞ in $\mathbb{C}$ by a continuous curve $\gamma \subset G_1 \setminus F$. This means that the continuous function $\gamma\colon [0,1] \to G_1 \setminus F$ with $\gamma(0) = z$ has the property that for any given compact set $K \subset \mathbb{C}$ there is a t_K such that, for each $t > t_K$, $\gamma(t) \notin K$. Therefore there is a basis $\{V_j : j \in J\}$ of open neighbourhoods of ∞ such that, for each j, $V_{j+1} \subset V_j$ and each $w \in V_{j+1}$ can be connected to ∞ by a curve in $V_j \setminus F$.

Hence for each $j \in J$, there exists a curve σ_j in $\mathbb{C} \setminus F$ from a point $w_j \in W_j$ to ∞ and we may assume that the family $\{\sigma_j : j \in J\}$ is locally finite. Let B_j be a connected polygonal neighbourhood of σ_j which does not intersect F. We may also assume that $\{\bar{B}_j : j \in J\}$ is a locally finite family of closed sets. Hence $\bigcup_{J \in J} \bar{B}_j$ is closed. Set $U_s = U \setminus \bigcup_{j \in J} \bar{B}_j$, thus $F \subset U_s$. Then, $\mathbb{C}_\infty \setminus U = \bigcup_{j \in J} W_j \cup W_\infty$, where W_∞ is the component of $\mathbb{C}_\infty \setminus U$ which contains ∞, so

$$\mathbb{C}_\infty \setminus U_s = \Big(\bigcup_{j \in J} W_j \cup B_j \Big) \cup \{\infty\} \cup W_\infty,$$

which is connected, therefore U_s is simply connected. ■

Lemma 2 *Let F be a set of uniform approximation and $f \in \mathcal{A}(F)$ without zeros on F. Then there exists a branch of* $\ln f$ *in $\mathcal{A}(F)$.*

Proof Considering a continuous extension of f on $\mathbb{C}$ and applying the previous lemma, we can suppose the existence of a continuous nonvanishing extension of f on a simply connected neighbourhood U_s of F. Thus there exists $f_s\colon U_s \to \mathbb{C}^*$ continuous such that $f_s \circ i = f$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $i\colon F \to U_s$ is the identity map. Let $\tilde{\mathbb{C}}^* = \mathbb{C}$ be the universal covering of $\mathbb{C}^*$ and $\tilde{f}\colon U_s \to \tilde{\mathbb{C}}^*$ be the lift of f_s, so $\exp \circ \tilde{f} = f_s$ and $\exp \circ \tilde{f} \circ i = f$.

We shall prove that $\tilde{f}$ is holomorphic on F°. Let $z_0 \in F^\circ$, $p_0 = \tilde{f}(z_0)$ and $\tilde{U}_0$ a neighbourhood of p_0 such that $\exp(\tilde{U}_0)$ is biholomorphic to $U_0 \subset \mathbb{C}^*$. Suppose V_0 is a neighbourhood of z_0, $V_0 \subset F^\circ$ small enough such that $\tilde{f}(V_0) \subset \tilde{U}_0$ and $f(V_0) \subset U_0$. Hence $\exp \circ \tilde{f}|_{V_0} = f|_{V_0}$ and $\tilde{f}|_{V_0} = \exp|_{\tilde{U}_0}^{-1} \circ f|_{V_0}$. Therefore $\tilde{f}$ is holomorphic on F°. ■

A divisor on $\mathbb{C}$ is a function $D\colon \mathbb{C} \to \mathbb{Z}$, such that the set of points $\zeta \in \mathbb{C}$ where $D(\zeta) \neq 0$ is a discrete set. We denote a divisor D by a formal sum

$$D := \sum_{\zeta \in \mathbb{C}} D(\zeta)\zeta.$$

Suppose $\Omega \subseteq \mathbb{C}$, $f \in \mathcal{M}(\Omega)$ and $\zeta \in \Omega$, then the order of f at ζ, positive for a zero, negative for a pole, will be denoted by $\operatorname{ord}_\zeta(f)$. By the divisor of $f \in \mathcal{M}(\mathbb{C})$,

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$f \neq 0$, we mean the divisor

$$D = \sum_{\zeta \in \mathbb{C}} \operatorname{ord}_\zeta(f)\zeta.$$

We shall call a sequence $\{z_n\}$ (possibly finite or empty) of distinct points in $\mathbb{C}$ *admissible* (with respect to a set F) if $\{z_n\}$ has no finite accumulation point and all z_n are contained in $\mathbb{C} \backslash F$.

Theorem 3 *Let F be a set of ε-approximation, $\varepsilon \leq 1$, $\{z_n\}$ an admissible sequence and $\{o_n\}$ a sequence in $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. Further, let $\varphi \in \mathcal{A}(F)$ without zeros. Then, there exists $f \in \mathcal{M}(\mathbb{C})$ such that $D = \sum_n o_n z_n$ is the divisor of f and for each $z \in F$,*

$$|\varphi(z) - f(z)| < |\varphi(z)|\varepsilon(z).$$

Proof We remark that

$$|1 - e^w| \leq e|w|, \quad \text{if } |w| \leq 1.$$

Consider $h \in \mathcal{M}(\mathbb{C})$ such that the divisor of h is D (see [6]). Since F is a set of uniform approximation, by Lemma 2 there exists a simply connected neighbourhood of F containing no z_n and branches H and Φ of $\ln h$ and $\ln \varphi$ respectively, in $\mathcal{A}(F)$. By hypothesis there exists $G \in \mathcal{H}(\mathbb{C})$ such that on F,

$$|H - (G + \Phi)| < \frac{\varepsilon}{e} < 1.$$

Set $g = e^{-G}$ and $f = gh$, so $f \in \mathcal{M}(\mathbb{C})$, and the divisor of f is D. On F we have

$$\begin{aligned} |\varphi - f| &= |\varphi| \left| 1 - \frac{gh}{\varphi} \right| \\ &\leq |\varphi| \, |H - G - \Phi| e \\ &< |\varphi| \varepsilon. \end{aligned}$$

∎

Note that, if in addition to the hypotheses of the previous theorem, we assume the boundedness of φ on F, then for $z \in F$, the desired f satisfies,

$$|\varphi(z) - f(z)| < \varepsilon(z).$$

Corollary 1 *Let F, ε, $\{z_n\}$ be as in the previous theorem and $\{o_n\}$ a sequence in $\mathbb{N}$. Then there exists $f \in \mathcal{H}(\mathbb{C})$ with exactly the zeros z_n of order o_n and $|1 - f| < \varepsilon$.*

We wish to apply the results of approximation theory to a study of asymptotic expansions.

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Definition 3 Let F be an unbounded set in $\mathbb{C}$. A function $f\colon F \to \mathbb{C}$ has an asymptotic expansion in F if there exists a complex sequence $\{a_n\}$ such that for all $n \in \mathbb{N}$

$$z^n\Big(f(z) - \sum_{i=0}^{n-1} a_i z^{-i}\Big) \to a_n$$

as $z \to \infty$ in F. We denote $f(z) \sim \sum_{i=0}^{\infty} a_i z^{-i}$.

For $n = 1, 2, \dots,$ we set

$$R_n(f, z) := f(z) - \sum_{i=0}^{n-1} a_i z^{-i}.$$

Then $f(z) \sim \sum_{i=0}^{\infty} a_i z^{-i}$ is equivalent to $R_n(f, z) = O(|z|^{-n})$ for all $n \in \mathbb{N}$. Note that the asymptotic expansion of f need not converge and is therefore a formal power series in $1/z$. As a particular case if a is a constant, we have that $f \sim a$ if and only if $f(z) - a = O(|z|^{-n})$, for all $n \in \mathbb{N}$. Hence, for $a \neq 0$, the meaning we give to the expression $f \sim a$ is much stronger than the assertion "f is asymptotic to a", which merely means that $f(z)/a \to 1$.

The next corollary is the main result of [9].

Corollary 2 *Let F be a set of uniform approximation, $\{z_n\}$ an admissible sequence and $\{o_n\}$ a sequence in $\mathbb{N}$. Then there exists $f \in \mathcal{H}(\mathbb{C})$ with exactly the zeros z_n of order o_n and $f \sim 1$ on F.*

Proof Taking the canonical speed $\varepsilon(z) := e^{-|z|^{1/3}}$ on F and applying Corollary 1 implies $f \sim 1$. ■

Theorem 4 *Let F be a set of uniform approximation, $\{z_n\}$ an admissible sequence, $\{o_n\}$ a sequence in $\mathbb{Z}^*$ and $\varepsilon \leq 1$ a canonical speed. Then, there exists $f \in \mathcal{M}(\mathbb{C})$ such that $D := \sum_n o_n z_n$ is the divisor of f and $|f| < \varepsilon$ on F.*

Proof Since F is a set of uniform approximation and ε is a canonical speed (see [2], p. 40), there exists a nonvanishing function $\varphi \in \mathcal{H}(\mathbb{C})$ such that

$$|\varphi(z)| < \frac{1}{2}\varepsilon(z),$$

for all $z \in F$.

By Theorem 3, there exists $f \in \mathcal{M}(\mathbb{C})$ with divisor D such that for $z \in F$,

$$|\varphi(z) - f(z)| < \frac{1}{2}\varepsilon(z),$$

so

$$|f(z)| < \varepsilon(z),$$

for all $z \in F$. ■

Corollary 3 *Let F be an unbounded set of uniform approximation,* $\{z_n\}$ *an admissible sequence and* $\{o_n\}$ *a sequence in* $\mathbb{Z}^*$. *Then there exists* $f \in \mathcal{M}(\mathbb{C})$ *such that the divisor of* f *is* $\sum_n o_n z_n$ *and* $f \sim 0$ *on F.*

By a left tail at $\zeta \in \mathbb{C}$ (see [3]), we mean a series of the form

$$\sum_{j=-\infty}^{J} a_j (z-\zeta)^j,$$

for some integer J which is convergent in some deleted neighbourhood of ζ. If the coefficients of a left tail at ζ coincide with the corresponding Laurent coefficients of a function f holomorphic in a deleted neighbourhood of ζ, then we say that the left tail is a left tail of f at the point ζ. For the special case where $J = -1$, we call it a p-tail (p for principal part).

Lemma 3 *Let F be a set of* ε*-approximation and* $Z := \{z_n\}$ *an admissible sequence. Moreover, for each n let* t_n *be a left* p*-tail at* z_n. *Then for* $f \in \mathcal{A}(F)$, *there exists a function g holomorphic in* $\mathbb{C} \setminus Z$ *such that* t_n *is a left tail of g at* z_n *and* $|f-g| < \varepsilon$ *on F.*

Proof By Theorem 4 in [3], there exists a function f_∞ holomorphic on $\mathbb{C}$ except for isolated (possibly artificial) singularities at the points of Z such that for each n, t_n is a left tail of f_∞ at z_n. Since Z is an admissible sequence, $f - f_\infty \in \mathcal{A}(F)$. On the other hand F is a set of ε-approximation so there exists $g_0 \in \mathcal{H}(\mathbb{C})$ such that on F,

$$|f - f_\infty - g_0| < \varepsilon.$$

Set $g := f_\infty + g_0$. Then g is holomorphic on $\mathbb{C}$ except for isolated singularities at the points of Z, such that for each n, t_n is a left tail of g at z_n. ■

Theorem 5 *Let F be a set of* ε*-approximation,* $\varepsilon \leq 1$, $Z = \{z_n\}$ *an admissible sequence, and*

$$t_n(z) := \sum_{j=-\infty}^{j_n} a_{nj}(z - z_n)^j,$$

a left tail at z_n. *Then, for* $f \in \mathcal{A}(F)$ *there exists a function g holomorphic in* $\mathbb{C} \setminus Z$ *such that,* t_n *is a left tail of g at* z_n *and for* $z \in F$,

$$|f(z) - g(z)| < \varepsilon(z). \tag{2}$$

Proof Corollary 1 of Theorem 3 implies that there exists an entire function $\tilde{f}$ with zeros exactly at z_n of order $o_n > j_n$ and on F,

$$|1 - \tilde{f}(z)| < \frac{\varepsilon(z)}{4}.$$

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For each n let g_n be the p-tail of the function $t_n/\tilde{f}$ at z_n so $t_n/\tilde{f} = g_n + \varphi_n$ locally at z_n with φ_n holomorphic at z_n.

By Lemma 3, there exists a function γ holomorphic on $\mathbb{C} \setminus Z$ such that g_n is a left tail of γ at z_n and for $z \in F$,

$$|\gamma(z)| < \frac{\varepsilon(z)}{4},$$

for all z in F.

Define $h := \gamma\tilde{f}$. Since $\tilde{f}$ is an entire function, locally $h = (g_n + q_n)\tilde{f}$ with q_n holomorphic at z_n. In a neighbourhood of z_n,

$$\begin{aligned} h = (g_n + q_n)\tilde{f} &= \left(\frac{t_n}{\tilde{f}} + q_n - \varphi_n\right)\tilde{f} \\ &= t_n + (q_n - \varphi_n)\tilde{f}. \end{aligned}$$

Since $(q_n - \varphi_n)\tilde{f}$ is holomorphic at z_n with zero of order at least o_n, it follows that t_n is a left tail of h at z_n.

On F we have

$$\begin{aligned} |h(z)| &= |\tilde{f}(z)\gamma(z)| \\ &\le \big(|\tilde{f}(z) - 1| + 1\big)\,|\gamma(z)| \\ &< \frac{\varepsilon(z)}{4} + \frac{\varepsilon(z)}{4} \\ &< \frac{\varepsilon(z)}{2}. \end{aligned}$$

By Corollary 1, there exists an entire function ω having zeros of order o_n at z_n and near 1 on F. Multiplying by a constant, we may assume that ω is bounded on F and $|\omega| > 1$. Since F is a set of ε-approximation, there is an entire function ψ such that for $z \in F$,

$$\left|\psi(z) - \frac{f}{\omega}(z)\right| < \frac{\varepsilon}{2|\omega(z)|}.$$

Set $\tilde{g} := \omega\psi$. Then $|\tilde{g} - f| < \varepsilon/2$ on F and $\tilde{g}$ has zeros of order at least o_n at z_n. Set $g := h + \tilde{g}$. Then g is a holomorphic function on $\mathbb{C} \setminus Z$ such that for each n, t_n is a left tail of g at z_n and for $z \in F$ it satisfies (2). ■

Corollary 4 *Let F be a set of ε-approximation, $\varepsilon \le 1$ and $\{z_n\}$ an admissible sequence. Further, let a_{nj}, $n \in \mathbb{N}$, $j = 0, 1, 2, \dots, j_n$, be complex numbers. Then for $f \in \mathcal{A}(F)$ there exists $g \in \mathcal{H}(\mathbb{C})$ such that, for each $z \in F$,*

$$|f(z) - g(z)| < \varepsilon(z),$$

and for $j = 0, 1, 2, \dots, j_n$, $n \in \mathbb{N}$,

$$g^{(j)}(z_n) = a_{nj}.$$

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Proof By the previous theorem, for

$$t_n = a_{n0} + a_{n1}(z - z_n) + \frac{1}{2!}a_{n2}(z - z_n)^2 + \cdots + \frac{1}{j_n!}a_{nj_n}(z - z_n)^{j_n},$$

we can find $g \in \mathcal{H}(\mathbb{C})$ which satisfies the desired properties. ■

Applying Theorem 2 in Lemma 3 and Theorem 5, analogous results can be deduced for sets of uniform approximation F and canonical speeds ε.

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