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# NORMALITY AND SHARED SETS

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**Abstract**

Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k+1$. Let $a$ and $b$ be distinct finite complex numbers, and let $k$ be a positive integer. If, for each pair of functions $f$ and $g$ in $\mathcal{F}$, $f^{(k)}$ and $g^{(k)}$ share the set $S=\{a, b\}$, then $\mathcal{F}$ is normal in $D$. The condition that the zeros of functions in $\mathcal{F}$ have multiplicity at least $k+1$ cannot be weakened.

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## 1. Introduction

A family $\mathcal{F}$ of functions meromorphic in the plane domain $D$ is said to be normal in $D$ if each sequence $\{f_n\}\subset\mathcal{F}$ has a subsequence $\{f_{n_j}\}$ that converges spherically locally uniformly in $D$ to a meromorphic function or $\infty$ (see [10, 15, 18]).

Let $f$ and $g$ be meromorphic functions on $D$, $a\in\mathbb{C}\cup\{\infty\}$, and let $S$ be a set of complex numbers. If $f(z)=a$ if and only if $g(z)=a$, we say that $f$ and $g$ share $a$ in $D$; if $f(z)\in S$ if and only if $g(z)\in S$, we say that $f$ and $g$ share $S$ in $D$.

In [13], Montel proved the following well-known normality criterion.

THEOREM A. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, and let $a$, $b$ and $c$ be three distinct values in the extended complex plane. If, for each function $f\in\mathcal{F}$, $f\neq a, b, c$, then $\mathcal{F}$ is normal in $D$.*

In [16], Sun extended Theorem A as follows.

THEOREM B. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$; and let $a$, $b$ and $c$ be three distinct values in the extended complex plane. If each pair of functions $f$ and $g$ in $\mathcal{F}$ share $a$, $b$ and $c$ in $D$, then $\mathcal{F}$ is normal in $D$.*

In [4], Fang and Hong extended Theorem B further.

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THEOREM C. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$; and let $a$, $b$ and $c$ be three distinct values in the extended complex plane. If each pair of functions $f$ and $g$ in $\mathcal{F}$ share the set $S = \{a, b, c\}$ in $D$, then $\mathcal{F}$ is normal in $D$.*

In this paper, we prove the following theorem.

THEOREM 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k + 1$, where $k$ is a positive integer. Let $a$ and $b$ be distinct (finite) complex numbers. If, for each pair of functions $f$ and $g$ in $\mathcal{F}$, $f^{(k)}$ and $g^{(k)}$ share the set $S = \{a, b\}$, then $\mathcal{F}$ is normal in $D$.*

EXAMPLE 1.2. Let $k$ be a positive integer. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where $f_n(z) = nz^k$, $n = 1, 2, 3, \ldots$. Then each function in $\mathcal{F}$ has a single zero of multiplicity $k$. Clearly, for each pair of functions $f_m$, $f_n$ in $\mathcal{F}$, $f_m^{(k)}$ and $f_n^{(k)}$ share the set $S = \{1/2, 1/3\}$ in $D$. But $\mathcal{F}$ clearly fails to be normal on any neighbourhood of 0. This shows that the condition in Theorem 1.1 that the zeros of functions in $\mathcal{F}$ have multiplicity at least $k + 1$ cannot be weakened.

The following result of Gu [9] is well known.

THEOREM D. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, let $k$ be a positive integer, and let $b$ be a nonzero complex number. If, for each function $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)} \neq b$ in $D$, then $\mathcal{F}$ is normal in $D$.*

Recently, we improved Theorem D as follows.

THEOREM E [7, Theorem 1]. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k + 2$, and let $b$ be a nonzero complex number. If each pair of functions $f$ and $g$ in $\mathcal{F}$ share 0, and $f^{(k)}$ and $g^{(k)}$ share $b$ in $D$, then $\mathcal{F}$ is normal in $D$.*

We also gave an example to show that the condition in Theorem E that the zeros of functions in $\mathcal{F}$ have multiplicity at least $k + 2$ cannot be weakened.

In this paper, we continue our investigations and prove the following results.

THEOREM 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k + 2$, where $k$ is a positive integer. Let $a$ and $b$ be two nonzero complex numbers. If each pair of functions $f$ and $g$ in $\mathcal{F}$ share $a$, and $f^{(k)}$ and $g^{(k)}$ share $b$ in $D$, then $\mathcal{F}$ is normal in $D$.*

EXAMPLE 1.4. Let

$$f_n(z) = \frac{(a_n z + 1)^{k+1}}{nz - 1},$$

where $a_n > 0$ satisfies $a_n^{k+1} k! = n$. Let $\mathcal{F} = \{f_n\}$ and $D = \{z : |z| < 1\}$. It can be shown, using Rouché's theorem, that, for sufficiently large $n$, $f_n(z) = -1$ has only the solution $z = 0$ in $D$. Thus, for each pair of functions $f_n$ and $f_m$ in $\mathcal{F}$:

(1) $f_n$ and $f_m$ share $-1$ for $n, m \geq N$, where $N$ is a positive integer depending only on $k$;
(2) all zeros of $f_n$ and $f_m$ have multiplicity $k+1$; and
(3) $f_n^{(k)}$ and $f_m^{(k)}$ share 1.

Clearly, $\mathcal{F}$ is not normal in $D$. This shows that the condition in Theorem 1.3 that 'all of whose zeros have multiplicity at least $k+2$' cannot be weakened.

EXAMPLE 1.5. Let $a \neq 0$ be a complex number, and let

$$f_n(z) = \frac{a}{naz+1}.$$

Let $\mathcal{F} = \{f_n\}$ and $D = \{z : |z| < 1\}$. Then, for each $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 0$ and $f(z) = a$ has only the solution $z = 0$ in $D$. Thus, for each pair of functions $f_n$ and $f_m$ in $\mathcal{F}$:

(1) $f_n$ and $f_m$ share $a$;
(2) all zeros of $f_n$ and $f_m$ have multiplicity at least $k+2$; and
(3) $f_n^{(k)}$ and $f_m^{(k)}$ share 0.

But $\mathcal{F}$ is not normal in $D$. This shows that $b \neq 0$ is necessary in Theorem 1.3.

THEOREM 1.6. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$; let $a$, $b$ and $c$ be complex numbers such that $a \neq b$, $c \neq 0$; and let $k$ be a positive integer. If, for each pair of functions $f, g \in \mathcal{F}$, $f$ and $g$ share the set $S = \{a, b\}$ and $f^{(k)}$ and $g^{(k)}$ share the value $c$, then $\mathcal{F}$ is normal in $D$.*

Example 1.5 also shows that $c \neq 0$ is necessary in Theorem 1.6.

THEOREM 1.7. *Let $k \geq 2$ be a positive integer; let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k$; and let $a$, $b$ and $c$ be complex numbers such that $a \neq b$, $c \neq 0$. If, for each pair of functions $f, g \in \mathcal{F}$, $f$ and $g$ share $c$ and $f^{(k)}$ and $g^{(k)}$ share the set $S = \{a, b\}$, then $\mathcal{F}$ is normal in $D$.*

Example 1.2 also shows that $c \neq 0$ is necessary in Theorem 1.7.

EXAMPLE 1.8. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where $f_n(z) = nz + c$. Then, for each pair of functions $f, g \in \mathcal{F}$, $f$ and $g$ share $c$, and $f'$ and $g'$ share the set $\{1/2, 1/3\}$ in $D$. Clearly, $\mathcal{F}$ is not normal in $D$. This shows that Theorem 1.7 is not valid for $k = 1$.

THEOREM 1.9. *Let $a$, $b$ and $c$ be complex numbers such that $bc \neq 0$; let $k$ and $m$ be positive integers with $k < m$; and let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k+1$. If, for each pair of functions $f, g \in \mathcal{F}$, $f$ and $g$ share $a$, $f^{(k)}$ and $g^{(k)}$ share $b$, and $f^{(m)}$ and $g^{(m)}$ share $c$, then $\mathcal{F}$ is normal in $D$.*

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Example 1.2 shows that the condition in Theorem 1.9 that all zeros have multiplicity at least $k+1$ cannot be weakened, and [7, Examples 1 and 2] show that $c \neq 0$ is necessary in Theorem 1.9.

## 2. Auxiliary results

For the proofs of Theorems 1.1, 1.3, 1.6, 1.7 and 1.9, we require the following auxiliary results.

LEMMA 2.1 [14, 19]. *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. Then, if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) *a number $0 < r < 1$,*
(b) *points $z_n$, $|z_n| < r$,*
(c) *functions $f_n \in \mathcal{F}$, and*
(d) *positive numbers $\rho_n \to 0$,*

*such that*

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} = g_n(\xi) \to g(\xi),$$

*locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$.*

LEMMA 2.2. *Let k be a positive integer and g a function meromorphic on $\mathbb{C}$ such that $g^{(k)}$ omits two values in $\mathbb{C}$. Then $g^{(k)}$ is constant.*

PROOF. Clearly, no nonconstant rational function omits two values on $\mathbb{C}$. On the other hand, if $g^{(k)}$ is transcendental, then it takes on every finite value with at most one exception infinitely often [10, Theorem 3.4]. □

LEMMA 2.3 [17, Theorem 7]. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+2$; and let b be a nonzero complex number. If $f^{(k)} \neq b$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

LEMMA 2.4 [6, Theorem 2]. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D; let a and b be nonzero complex numbers; and let k be a positive integer. If, for each $f \in \mathcal{F}$, all the zeros of f have multiplicity at least $k+1$ and $f(z) = a$ if and only if $f^{(k)}(z) = b$, then $\mathcal{F}$ is normal in D.*

LEMMA 2.5 [1, 8, 12, Corollary of Theorem 1]. *Let f be a nonconstant meromorphic function on the plane and $k \geq 2$ a positive integer. Suppose that $f(z) \neq 0$, and $f^{(k)}(z) \neq 0$ for all $z \in \mathbb{C}$. Then either $f(z) = e^{Az+B}$ or $f(z) = 1/(Az+B)^m$, where $A \neq 0$ and B are constants and m is a positive integer.*

LEMMA 2.6 [17, Lemma 8]. *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + q(z)/p(z)$, where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$, and q and p are two coprime*

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*polynomials, neither of which vanishes identically, with* $\deg q < \deg p$*; and let* $k$ *be a positive integer and* $b$ *a nonzero complex number. If* $f^{(k)} \neq b$*, and the zeros of* $f$ *all have multiplicity at least* $k+1$*, then*

$$f(z) = \frac{b(z-d)^{k+1}}{k!(z-c)},$$

*where* $c$ *and* $d$ *are distinct complex numbers.*

LEMMA 2.7 [5, Theorem 1]. *Let* $\mathcal{F}$ *be a family of meromorphic functions defined in* $D$*; let* $k$ *(where* $k \geq 2$*) be a positive integer; and let* $a$*,* $b$*,* $c$ *and* $d$ *be complex numbers such that* $b \neq a, 0$ *and* $c \neq 0$*. If, for each* $f \in \mathcal{F}$*, all zeros of* $f - d$ *have multiplicity at least* $k$*,* $f(z) = 0$ *if and only if* $f^{(k)}(z) = a$ *and* $f^{(k)}(z) = b$ *implies that* $f(z) = c$*, then* $\mathcal{F}$ *is normal in* $D$*.*

LEMMA 2.8 [6, Theorem 1]. *Let* $\mathcal{F}$ *be a family of meromorphic functions defined in* $D$*; let* $k$ *(where* $k \geq 2$*) be a positive integer; and let* $a$*,* $b$*, and* $c$ *be complex numbers such that* $b \neq 0$ *and* $c \neq a$*. If, for each* $f \in \mathcal{F}$*,* $f$ *has only zeros of multiplicity at least* $k$*,* $f(z) = a$ *if and only if* $f^{(k)}(z) = b$ *and* $f^{(k)}(z) = 0$ *implies that* $f(z) = c$*, then* $\mathcal{F}$ *is normal in* $D$*.*

LEMMA 2.9 [2, 18, Lemma 2.4]. *Let* $T(r)$ *be a continuous, nondecreasing, nonnegative function and* $a(r)$ *a nonincreasing, nonnegative function on the interval* $(r_0, R)$*. If there exist constants* $b$ *and* $c$ *such that*

$$T(r) \leq a(r) + b \log^+ \frac{1}{\rho - r} + c \log^+ T(\rho)$$

*whenever* $r_0 < r < \rho < R$*, then*

$$T(r) \leq 2a(r) + B \log \frac{2}{R - r} + C,$$

*where* $B$ *and* $C$ *are constants depending only on* $b$ *and* $c$*.*

LEMMA 2.10 [11, 18, Lemma 4.3]. *Let* $f(z)$ *be meromorphic in* $|z| < R$ *(where* $R \leq \infty$*). If* $f(0) \neq 0, \infty$*, then, for every positive integer* $k$*,*

$$m\left(r, \frac{f^{(k)}}{f}\right) \leq C_k \left\{1 + \log^+ \log^+ \frac{1}{|f(0)|} + \log^+ \frac{1}{r} + \log^+ \frac{1}{\rho - r} + \log^+ \rho + \log^+ T(\rho, f)\right\},$$

*where* $0 < r < \rho < R$ *and* $C_k$ *is a constant depending only on* $k$*.*

LEMMA 2.11. *Let* $f$ *be meromorphic in* $D = \{|z| < R\}$*; let* $k$ *(where* $k \geq 2$*) be a positive integer; and let* $a$*,* $b$ *and* $c$ *be complex numbers such that* $b \neq a$ *and* $c \neq 0$*.*

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*Suppose that all zeros of $f$ have multiplicity at least $k$, $f(0) \neq 0, \infty$, $f^{(k+1)}(0) \neq 0, \infty$; that for some $z_0 \in D$, $z_0 \neq 0$, $f(z_0) = c$, but $f(z) \neq c$ for any $z \in D$, $z \neq z_0$; and that $f^{(k)}(z) \neq a, b$, for any $z \in D$. Then, for $0 < r < R$,*

$$\begin{aligned} T(r, f) \leq \frac{1}{k-1} \bigg\{ & k \log r + 2km\left(r, \frac{f'}{f}\right) + km\left(r, \frac{f'}{f-c}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) \\ & + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-b}\right) + \log \frac{|(f^{(k)}(0)-a)(f^{(k)}(0)-b)|}{|f^{(k+1)}(0)|} \\ & + \log \frac{1}{|f(0)|} + k \log \frac{|f(0)(f(0)-c)|}{|f'(0)|} + M \bigg\}, \end{aligned} \tag{2.1}$$

*where $M$ is a constant.*

PROOF. Starting from [10, (2.1)], we have by familiar properties of the functions of Nevanlinna theory (see [10, pp. 4–5, 56])

$$\begin{aligned} & m\left(r, \frac{1}{f^{(k)}-a}\right) + m\left(r, \frac{1}{f^{(k)}-b}\right) \\ & \leq m\left(r, \frac{1}{f^{(k)}-a} + \frac{1}{f^{(k)}-b}\right) + M_1 \\ & = m\left(r, \left(\frac{f^{(k+1)}}{f^{(k)}-a} + \frac{f^{(k+1)}}{f^{(k)}-b}\right) \frac{1}{f^{(k+1)}}\right) + M_1 \\ & \leq m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a} + \frac{f^{(k+1)}}{f^{(k)}-b}\right) + m\left(r, \frac{1}{f^{(k+1)}}\right) + M_1 \\ & \leq m\left(r, \frac{1}{f^{(k+1)}}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-b}\right) + M_1 + \log 2 \\ & \leq T(r, f^{(k+1)}) - N\left(r, \frac{1}{f^{(k+1)}}\right) + \log \frac{1}{|f^{(k+1)}(0)|} \\ & \quad + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-b}\right) + M_1 + \log 2 \\ & \leq T(r, f^{(k)}) + \overline{N}(r, f) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) \\ & \quad + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-b}\right) + \log \frac{1}{|f^{(k+1)}(0)|} + M_1 + \log 2. \end{aligned}$$

Since $f^{(k)} \neq a, b$, the extreme left-hand side of the inequality above is

$$T\left(r, \frac{1}{f^{(k)}-a}\right) + T\left(r, \frac{1}{f^{(k)}-b}\right).$$

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On the other hand (see [10, (1.10)–(1.11)]) we have

$$\begin{aligned}
&T\left(r, \frac{1}{f^{(k)}-a}\right) + T\left(r, \frac{1}{f^{(k)}-b}\right)\\
&\quad = T(r, f^{(k)}-a) + T(r, f^{(k)}-b) - \log|f^{(k)}(0)-a| - \log|f^{(k)}(0)-b|\\
&\quad \geq T(r, f^{(k)}) - \log^+|a| - \log 2 - \log|f^{(k)}(0)-a|\\
&\qquad + T(r, f^{(k)}) - \log^+|b| - \log 2 - \log|f^{(k)}(0)-b|\\
&\quad = 2T(r, f^{(k)}) - \log|(f^{(k)}(0)-a)(f^{(k)}(0)-b)|\\
&\qquad - \log^+|a| - \log^+|b| - 2\log 2.
\end{aligned}$$

It now follows that

$$\begin{aligned}
T(r, f^{(k)}) \leq \overline{N}(r, f) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-b}\right)\\
+ \log\frac{|(f^{(k)}(0)-a)(f^{(k)}(0)-b)|}{|f^{(k+1)}(0)|} + M_2, \qquad (2.2)
\end{aligned}$$

where $M_2$ is a constant depending only on $a$, $b$ and $k$.

Since

$$T(r, f^{(k)}) = m(r, f^{(k)}) + N(r, f^{(k)}) \geq N(r, f) + k\overline{N}(r, f) \geq (k+1)\overline{N}(r, f), \qquad (2.3)$$

it follows from (2.2) and (2.3) that

$$\begin{aligned}
k\overline{N}(r, f) \leq m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-b}\right)\\
+ \log\frac{|(f^{(k)}(0)-a)(f^{(k)}(0)-b)|}{|f^{(k+1)}(0)|} + M_2. \qquad (2.4)
\end{aligned}$$

Using reasoning similar to that used to obtain (2.2),

$$\begin{aligned}
&m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f-c}\right) \leq m\left(r, \frac{1}{f} + \frac{1}{f-c}\right) + M_1\\
&\quad \leq m\left(r, \frac{1}{f'}\right) + m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f-c}\right) + M_1 + \log 2\\
&\quad \leq T(r, f') - N\left(r, \frac{1}{f'}\right) + m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f-c}\right)\\
&\qquad + \log\frac{1}{|f'(0)|} + M_1 + \log 2
\end{aligned}$$

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$$\leq T(r, f) + \overline{N}(r, f) - N\left(r, \frac{1}{f'}\right) + 2m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f-c}\right) + \log\frac{1}{|f'(0)|} + M_1 + \log 2.$$

Thus

$$\begin{aligned} T\left(r, \frac{1}{f}\right) &+ T\left(r, \frac{1}{f-c}\right) \\ &\leq T(r, f) + \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-c}\right) - N\left(r, \frac{1}{f'}\right) \\ &\quad + 2m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f-c}\right) + \log\frac{1}{|f'(0)|} + M_1 + \log 2 \\ &\leq T(r, f) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-c}\right) \\ &\quad + 2m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f-c}\right) + \log\frac{1}{|f'(0)|} + M_1 + \log 2. \end{aligned}$$

But

$$\begin{aligned} T\left(r, \frac{1}{f}\right) &+ T\left(r, \frac{1}{f-c}\right) \\ &= T(r, f) - \log|f(0)| + T(r, f-c) - \log|f(0) - c| \\ &\geq 2T(r, f) - \log|f(0)| - \log|f(0) - c| - \log^+|c| - \log 2. \end{aligned}$$

Hence

$$\begin{aligned} T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-c}\right) + 2m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f-c}\right) \\ + \log\left|\frac{f(0)(f(0)-c)}{f'(0)}\right| + M_3. \end{aligned}$$

Since all the zeros of $f$ have multiplicity at least $k$ and $f - c$ has only a single zero in $D$, we obtain

$$\begin{aligned} T(r, f) \leq \overline{N}(r, f) &+ \frac{1}{k}N\left(r, \frac{1}{f}\right) + \log r + 2m\left(r, \frac{f'}{f}\right) \\ &+ m\left(r, \frac{f'}{f-c}\right) + \log\frac{|f(0)(f(0)-c)|}{|f'(0)|} + M_3, \end{aligned} \tag{2.5}$$

where $M_3$ is a constant depending only on $c$.

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Together with (2.4), this yields

$$\begin{aligned}
kT(r, f) \le\ & k\overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + k\log r + 2km\left(r, \frac{f'}{f}\right)\\
&+ km\left(r, \frac{f'}{f-c}\right) + k\log\frac{|f(0)(f(0)-c)|}{|f'(0)|} + kM_3\\
\le\ & m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right)\\
&+ m\left(r, \frac{f^{(k+1)}}{f^{(k)}-b}\right) + 2km\left(r, \frac{f'}{f}\right)\\
&+ km\left(r, \frac{f'}{f-c}\right) + T(r, f) + \log\frac{1}{|f(0)|} + k\log\frac{|f(0)(f(0)-c)|}{|f'(0)|}\\
&+ \log\frac{|(f^{(k)}(0)-a)(f^{(k)}(0)-b)|}{|f^{(k+1)}(0)|} + k\log r + (kM_3 + M_2).
\end{aligned}$$

Thus (2.1) follows, so Lemma 2.11 is proved. □

LEMMA 2.12 [3]. *For $k$ a positive integer, let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, all of whose zeros have multiplicity at least $k+1$; and let $\mathcal{G} = \{f^{(k)} : f \in \mathcal{F}\}$. If $\mathcal{G}$ is normal in $D$, then $\mathcal{F}$ is also normal in $D$.*

## 3. Proofs of Theorems 1.1–1.9

PROOF OF THEOREM 1.1. Fix $z_0 \in D$. We show that $\mathcal{F}$ is normal at $z_0$. Let $f \in \mathcal{F}$. We consider two cases.

Case 1. $f^{(k)}(z_0) \ne a, b$. Then there exists a disc $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f^{(k)} \ne a, b$ in $D_\delta$. Thus, for each $g \in \mathcal{F}$, the zeros of $g$ have multiplicity at least $k+1$ and $g^{(k)} \ne a, b$ in $D_\delta$.

We claim that $\mathcal{F}$ is normal in $D_\delta$. For notational simplicity, we may assume that $D_\delta$ is the unit disc $\Delta$.

Suppose, on the contrary, that $\mathcal{F}$ is not normal in $\Delta$. Then by Lemma 2.1, we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$ and $\rho_n \to 0^+$ such that $g_n(\xi) = \rho_n^{-k} f_n(z_n + \rho_n\xi)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $g$ on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$. By Hurwitz's theorem, either $g^{(k)} \ne a, b$ or $g^{(k)} \equiv a$ or $g^{(k)} \equiv b$. In either of the latter cases, $g$ would be a polynomial of degree at most $k$, contradicting the fact that $g$ is nonconstant and all zeros of $g$ have multiplicity at least $k+1$. Hence $g^{(k)} \ne a, b$. But then from Lemma 2.2, it follows that $g^{(k)}$ is a constant. As before, since the zeros of $g$ have multiplicity at least $k+1$, it follows that $g$ is a constant, a contradiction. Hence $\mathcal{F}$ is normal in $\Delta$, and so $\mathcal{F}$ is normal at $z_0$.

Case 2. $f^{(k)}(z_0) = a$ or $b$. Then there exists $\delta > 0$ such that $f^{(k)} \ne a, b$ in $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$.

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Let $\mathcal{G} = \{f_n\}$ be a sequence in $\mathcal{F}$. Then, without loss of generality, we may assume that there exists a subsequence of $\{f_n\}$ (which we again denote by $\{f_n\}$) such that $f_n^{(k)}(z_0) = a$. Thus, by the condition of the theorem, $f_n^{(k)} \neq a, b$ in $D_\delta^o$ and $f_n^{(k)}(z_0) = a$. As before, we may assume that $z_0 = 0$ and $\delta = 1$.

We claim that $\mathcal{G}$ is normal in $\Delta$.

Suppose, on the contrary, that $\mathcal{G}$ is not normal in $\Delta$. Then by Lemma 2.1, we can find a subsequence of $\mathcal{G}$, which we again denote by $\{f_n\}$, $z_n \in \Delta$, $z_n \to 0$ and $\rho_n \to 0^+$ such that $g_n(\xi) = \rho_n^{-k} f_n(z_n + \rho_n \xi)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $g$ on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$.

Clearly, $g^{(k)} \neq b$. Now we consider two subcases.

Case 2.1. $z_n/\rho_n \to \infty$. Then $g^{(k)} \neq a$, so, again by Lemma 2.2, $g$ is constant, a contradiction.

Case 2.2. $z_n/\rho_n \to -\alpha$. Then, it is easy to see that $g^{(k)}(\xi) \neq a$ for $\xi \neq \alpha$ and $g^{(k)}(\alpha) = a$. We consider a further two subcases.

Case 2.2.1. $b \neq 0$. Then by Lemma 2.6, it follows that

$$g(\xi) = \frac{b(\xi - d)^{k+1}}{k!(\xi - c)},$$

where $c$ and $d$ are distinct complex numbers. Thus

$$g^{(k)}(\xi) = b + \frac{A}{(\xi - c)^{k+1}},$$

where $A$ is a nonzero complex number.

Obviously, $g^{(k)}(\xi) = a$ has $k+1$ distinct solutions, which contradicts the fact that $g^{(k)}(\xi) = a$ has only the solution $\xi = \alpha$.

Case 2.2.2. $b = 0$. Then $g^{(k)} \neq 0$. Since all the zeros of $g$ have multiplicity at least $k+1$, it follows that $g \neq 0$.

If $k \geq 2$, then by Lemma 2.5, either $g(\xi) = e^{A\xi + B}$, or $g(\xi) = 1/(A\xi + B)^n$, where $A$ and $B$ are complex numbers, $A \neq 0$, and $n$ is a positive integer. Clearly, for functions of this form, $g^{(k)}(\xi) = a$ has more than a single solution, contradicting what has been shown above.

If $k = 1$, then $g \neq 0$ and $g'(\xi) = a$ has only one solution. It follows from Hayman's alternative [10, Corollary to Theorem 3.5] that $g$ is a rational function. This together with $g \neq 0$ and $g' \neq 0$ yields $g(\xi) = 1/(A\xi + B)^n$, where $A$ and $B$ are complex numbers, $A \neq 0$, and $n$ is a positive integer. Again $g'(\xi) = a$ has more than a single solution, a contradiction.

Hence $\mathcal{G}$ is normal in $\Delta$. Thus a subsequence of $\{f_n\}$ converges locally uniformly with respect to the spherical metric to a meromorphic function or $\infty$. Hence $\mathcal{F}$ is normal at $z_0$, and so $\mathcal{F}$ is normal in $D$. The proof of Theorem 1.1 is complete. □

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PROOF OF THEOREM 1.3. Let $z_0 \in D$. We show that $\mathcal{F}$ is normal at $z_0$. Let $f \in \mathcal{F}$. We consider two cases.

Case 1. $f^{(k)}(z_0) \neq b$. Then there exists a disc $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f^{(k)} \neq b$ in $D_\delta$. Thus, for each $g \in \mathcal{F}$, the zeros of $g$ have multiplicity at least $k + 2$ and $g^{(k)} \neq b$ in $D_\delta$. By Lemma 2.3, $\mathcal{F}$ is normal in $D_\delta$. Hence, $\mathcal{F}$ is normal at $z_0$.

Case 2. $f^{(k)}(z_0) = b$. Then there exists $\delta > 0$ such that $f^{(k)} \neq b$ in the punctured disc $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$. Hence, for each $g \in \mathcal{F}$, $g^{(k)}(z) \neq b$, $z \in D_\delta^o$. We consider two subcases.

Case 2.1. $f(z_0) \neq a$. Then there exists a disc $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f \neq a$ in $D_\delta$. Hence, for each pair of functions $f, g \in \mathcal{F}$, $f \neq a$, $g \neq a$, and $f^{(k)}$ and $g^{(k)}$ share $b$ in $D_\delta$. It follows from Theorem E that $\mathcal{F}$ is normal in $D_\delta$ and hence normal at $z_0$.

Case 2.2. $f(z_0) = a$. Then there exists $\delta > 0$ such that $f \neq a$ in the punctured disc $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$. Hence, for each $g \in \mathcal{F}$, all zeros of $g$ have multiplicity at least $k + 2$, and $g(z) = a$ if and only if $g^{(k)}(z) = b$, $z \in D_\delta$. Thus, by Lemma 2.4, $\mathcal{F}$ is normal in $D_\delta$, and so $\mathcal{F}$ is normal at $z_0$.

Therefore, $\mathcal{F}$ is normal in $D$. The proof of Theorem 1.3 is complete. □

PROOF OF THEOREM 1.6. Let $z_0 \in D$. We show that $\mathcal{F}$ is normal at $z_0$. Let $f \in \mathcal{F}$. We consider two cases.

Case 1. $f(z_0) \neq a, b$. Then there exists a disc $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f \neq a, b$ in $D_\delta$. Thus, for each $g \in \mathcal{F}$, $g \neq a, b$ in $D_\delta$. Consider the family of functions $\mathcal{F}_b = \{g - b : g \in \mathcal{F}\}$ on $D_\delta$. Clearly, if $h \in \mathcal{F}_b$, then $h \neq 0$ on $D_\delta$. Furthermore, if $h, \tilde{h} \in \mathcal{F}_b$, then $h$ and $\tilde{h}$ share $a - b$ while $h^{(k)}$ and $\tilde{h}^{(k)}$ share $c$. Thus by Theorem 1.3, $\mathcal{F}_b$ and hence $\mathcal{F}$ is normal in $D_\delta$, so $\mathcal{F}$ is normal at $z_0$.

Case 2. $f(z_0) = a$ or $b$. Then there exists $\delta > 0$ such that $f \neq a, b$ in $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$. Let $\{f_n\}$ be a sequence of $\mathcal{F}$. Then, for each $f_n$, $f_n \neq a, b$ in $D_\delta^o$ and $f_n(z_0) = a$ or $b$. Thus there exists a subsequence, which we continue to denote by $\{f_n\}$, such that (say) $f_n(z_0) = a$. Hence, for each $f_n$, $f_n \neq b$ in $D_\delta$; and for each pair of functions $f_n$ and $f_m$, $f_n^{(k)}$ and $f_m^{(k)}$ share $c$. As in Case 1, it follows from Theorem 1.3 that $\{f_n\}$ is normal in $D_\delta$, so $\mathcal{F}$ is normal at $z_0$.

Therefore, $\mathcal{F}$ is normal in $D$. The proof of Theorem 1.6 is complete. □

PROOF OF THEOREM 1.7. Let $z_0 \in D$. We show that $\mathcal{F}$ is normal at $z_0$. Let $f \in \mathcal{F}$. We consider two cases.

Case 1. $f(z_0) \neq c$. Then there exists a disc $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f \neq c$ in $D_\delta$. Thus, for each $g \in \mathcal{F}$, $g \neq c$ in $D_\delta$. Applying Theorem 1.1 to the family $\mathcal{F}_c = \{g - c : g \in \mathcal{F}\}$, we see that $\mathcal{F}_c$, and hence $\mathcal{F}$, is normal in $D_\delta$. Thus, $\mathcal{F}$ is normal at $z_0$.

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Case 2. $f(z_0) = c$. Then there exists $\delta > 0$ such that $f \neq c$ in the punctured disc $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$. Hence, $\mathcal{F}$ is normal in $D_\delta^o$. Next we consider two subcases.

Case 2.1. $f^{(k)}(z_0) = a$ or $b$. Then there exists $\delta > 0$ such that $f^{(k)} \neq a, b$ in $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$.

Let $\{f_n\}$ be a sequence of $\mathcal{F}$. Then for each $f_n$, $f_n^{(k)} \neq a, b$ in $D_\delta^o$ and $f_n^{(k)}(z_0) = a$ or $b$. Then there exists a subsequence, which we again call $\{f_n\}$, such that (say) $f_n^{(k)}(z_0) = a$ for all $n$. Hence, for each $f_n$, all of whose zeros have multiplicity at least $k$, $f_n^{(k)} \neq b$ and $f_n(z) = c$ if and only if $f_n^{(k)}(z) = a$ in $D_\delta$. Thus, by Lemmas 2.7 and 2.8, $\{f_n\}$ is normal in $D_\delta$, so there exists a subsequence of $\{f_n\}$ that converges to a meromorphic function or $\infty$ in $D_\delta$. Hence $\mathcal{F}$ is normal in $D_\delta$, so $\mathcal{F}$ is normal at $z_0$.

Case 2.2. $f^{(k)}(z_0) \neq a, b$. Then there exists a disc $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f^{(k)} \neq a, b$ in $D_\delta$. Without loss of generality, we assume that $z_0 = 0$ and $D_\delta$ is the unit disc $\Delta$.

Let $\{f_n\}$ be a sequence of $\mathcal{F}$. Then, for each $f_n$, $f_n(0) = c$ and $f_n(z) \neq c$ for $z \neq 0$, $z \in \Delta$. Hence, either there exists a subsequence (which we continue to denote by $\{f_n\}$) such that all zeros of $f_n(z) - c$ have multiplicity at least $k + 1$, or there exists a subsequence such that all zeros of $f_n(z) - c$ have the same multiplicity $l$, $1 \leq l \leq k$. If all zeros of $f_n(z) - c$ have multiplicity at least $k + 1$, then, by Theorem 1.1, $\{f_n\}$ is normal in $\Delta$. Suppose, then, that all zeros of $f_n(z) - c$ have the same multiplicity $l$, $1 \leq l \leq k$; we prove that $\mathcal{G} = \{f_n\}$ is normal at $z = 0$.

Suppose, on the contrary, that $\mathcal{G}$ is not normal at $z = 0$. By Lemma 2.1, there exist sequences $f_n \in \mathcal{G}$, $z_n \to 0$ and $\rho_n \to 0^+$ such that

$$g_n(\xi) = f_n(z_n + \rho_n \xi)$$

converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $g$ in $\mathbb{C}$, whose zeros all have multiplicity at least $k$.

Since $g_n(\xi) = c$ has the unique solution $\xi = -z_n/\rho_n$ with multiplicity $l$ for all $n$, it follows by the argument principle that $g(\xi) = c$ has at most one solution in $\mathbb{C}$.

We claim that $g^{(k+1)} \not\equiv 0$. In fact, if $g^{(k+1)} \equiv 0$, then $g(\xi) = A(\xi - \xi_0)^k$, where $A$ is a nonzero constant. Thus $g(\xi) = c$ has $k$ (where $k \geq 2$) distinct solutions, which contradicts the fact that $g(\xi) = c$ has at most one solution.

Now, we consider two subcases.

Case 2.2.1. There is a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that $f_{n_j}^{(k+1)}(z_{n_j} + \rho_{n_j}\xi) \equiv 0$. Then

$$g_{n_j}^{(k+1)}(\xi) = \rho_{n_j}^{k+1} f_{n_j}^{(k+1)}(z_{n_j} + \rho_{n_j}\xi) \equiv 0.$$

Letting $j \to \infty$, we obtain $g^{(k+1)}(\xi) \equiv 0$, a contradiction.

Case 2.2.2. There are only finitely many $f_n$ such that $f_n^{(k+1)}(\xi) \equiv 0$. We may assume that $f_n^{(k+1)}(\xi) \not\equiv 0$ for all $n$. Take $\xi_0 \in \mathbb{C}$ such that

$$g(\xi_0) \neq 0,\, c,\, \infty, \quad g'(\xi_0) \neq 0, \quad g^{(k)}(\xi_0) \neq 0, \quad g^{(k+1)}(\xi_0) \neq 0.$$

Then

$$\begin{aligned}
&\left|\frac{1}{\rho_n}\frac{(f_n(z_n+\rho_n\xi_0))^{k-1}(f_n(z_n+\rho_n\xi_0)-c)^k}{(f_n'(z_n+\rho_n\xi_0))^k}\right| \\
&\qquad \times \left|\frac{(f_n^{(k)}(z_n+\rho_n\xi_0)-a)(f_n^{(k)}(z_n+\rho_n\xi_0)-b)}{f_n^{(k+1)}(z_n+\rho_n\xi_0)}\right| \\
&\quad = \left|\frac{g_n^{k-1}(\xi_0)(g_n(\xi_0)-c)^k}{(g_n'(\xi_0))^k}\frac{(g_n^{(k)}(\xi_0)-a\rho_n^k)(g_n^{(k)}(\xi_0)-b\rho_n^k)}{g_n^{(k+1)}(\xi_0)}\right| \\
&\quad \to \left|\frac{g^{k-1}(\xi_0)(g(\xi_0)-c)^k}{(g'(\xi_0))^k}\frac{(g^{(k)}(\xi_0))^2}{g^{(k+1)}(\xi_0)}\right| \quad \text{as } n \to \infty.
\end{aligned}$$

It follows that

$$\begin{aligned}
&k\log\frac{|f_n(z_n+\rho_n\xi_0)(f_n(z_n+\rho_n\xi_0)-c)|}{|f_n'(z_n+\rho_n\xi_0)|} + \log\frac{1}{|f_n(z_n+\rho_n\xi_0)|} \\
&\quad + \log\frac{|(f_n^{(k)}(z_n+\rho_n\xi_0)-a)(f_n^{(k)}(z_n+\rho_n\xi_0)-b)|}{|f_n^{(k+1)}(z_n+\rho_n\xi_0)|} \\
&\quad = \log\left|\rho_n\frac{g_n^{k-1}(\xi_0)(g_n(\xi_0)-c)^k}{(g_n'(\xi_0))^k}\frac{(g_n^{(k)}(\xi_0)-a\rho_n^k)(g_n^{(k)}(\xi_0)-b\rho_n^k)}{g_n^{(k+1)}(\xi_0)}\right| \\
&\quad \to -\infty, \quad \text{as } n \to \infty. \qquad (3.1)
\end{aligned}$$

Set $h_n(z) = f_n(z_n+\rho_n\xi_0+z)$, $n = 1, 2, 3, \ldots$. Then

$$\begin{aligned}
h_n(0) &= f_n(z_n+\rho_n\xi_0) = g_n(\xi_0) \to g(\xi_0) \neq 0,\, c,\, \infty, \\
h_n'(0) &= f_n'(z_n+\rho_n\xi_0) = \frac{g_n'(\xi_0)}{\rho_n} \to \infty, \\
h_n^{(k)}(0) &= f_n^{(k)}(z_n+\rho_n\xi_0) = \frac{g_n^{(k)}(\xi_0)}{\rho_n^k} \to \infty, \\
h_n^{(k+1)}(0) &= f_n^{(k+1)}(z_n+\rho_n\xi_0) = \frac{g_n^{(k+1)}(\xi_0)}{\rho_n^{k+1}} \to \infty. \qquad (3.2)
\end{aligned}$$

Now for $R = 1$, $|z| < \frac{1}{2}$, $z_n + \rho_n\xi_0 + z \in \tilde{D} = \{w : |w| < R\}$ for sufficiently large $n$. Hence, by (3.1), (3.2) and Lemma 2.11, for $r < R$ and sufficiently large $n$,

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$$
\begin{aligned}
T(r, h_n) \le \frac{1}{k-1}\bigg[ & k \log r + 2km\left(r, \frac{h_n'}{h_n}\right) + km\left(r, \frac{h_n'}{h_n - c}\right) \\
& + m\left(r, \frac{h_n^{(k+1)}}{h_n^{(k)}}\right) + m\left(r, \frac{h_n^{(k+1)}}{h_n^{(k)} - a}\right) + m\left(r, \frac{h_n^{(k+1)}}{h_n^{(k)} - b}\right) \\
& + \log \frac{|(h_n^{(k)}(0) - a)(h_n^{(k)}(0) - b)|}{|h_n^{(k+1)}(0)|} + \log \frac{1}{|h_n(0)|} \\
& + k \log \frac{|h_n(0)(h_n(0) - c)|}{|h_n'(0)|} + M \bigg] \\
\le \ & C\bigg[ 2km\left(r, \frac{h_n'}{h_n}\right) + km\left(r, \frac{h_n'}{h_n - c}\right) + m\left(r, \frac{h_n^{(k+1)}}{h_n^{(k)}}\right) \\
& + m\left(r, \frac{h_n^{(k+1)}}{h_n^{(k)} - a}\right) + m\left(r, \frac{h_n^{(k+1)}}{h_n^{(k)} - b}\right) + 1 \bigg]. \qquad (3.3)
\end{aligned}
$$

By Lemma 2.10, we have

$$T(r, h_n) \le C_1\left[ \log^+ T(\rho, h_n) + \log \frac{1}{\rho - r} + 1 \right],$$

where $0 < r < \rho < R$.

Thus, by Lemma 2.9, we obtain

$$T(r, h_n) \le C_2\left[ \log \frac{2}{1 - r} + 1 \right],$$

where $C_2$ does not depend on $h_n$.

Let $b_n$ be a pole of $h_n$ with $|b_n| < \frac{1}{2}$. Then

$$\log \frac{1/2}{|b_n|} \le N\left(\frac{1}{2}, h_n\right) \le T\left(\frac{1}{2}, h_n\right) \le C_3,$$

so that $|b_n| > 1/(2e^{C_3})$.

Thus $f_n$ is holomorphic in $|z| < 1/(2e^{C_3})$, and hence (see [10, Theorem 1.6])

$$\log M\left(\frac{1}{4e^{C_3}}, f_n\right) \le 3T\left(\frac{1}{2e^{C_3}}, f_n\right) \le 3C_2\left[ \log \frac{2}{1 - 1/(2e^{C_3})} + 1 \right].$$

Therefore, $\mathcal{G}$ is normal at the origin, so $\mathcal{F}$ is normal in $D$. The proof of Theorem 1.7 is complete. □

PROOF OF THEOREM 1.9. Let $z_0 \in D$. We show that $\mathcal{F}$ is normal at $z_0$. Let $f \in \mathcal{F}$. We consider several cases.

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Case 1. $f(z_0) \neq a$. Then there exists a disc $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f \neq a$ in $D_\delta$. Thus, for each $g \in \mathcal{F}$, $g \neq a$ in $D_\delta$. Since $\mathcal{F}_a = \{g - a : g \in \mathcal{F}\}$ satisfies the hypotheses of Theorem E, $\mathcal{F}_a$ and hence $\mathcal{F}$ is normal in $D_\delta$; and so $\mathcal{F}$ is normal at $z_0$.

Case 2. $f(z_0) = a$. Then there exists a disc $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$ such that $f \neq a$ in $D_\delta^o$. Hence, $\mathcal{F}$ is normal in $D_\delta^o$. We consider two subcases.

Case 2.1. $f^{(k)}(z_0) \neq b$. Then there exists a disc $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f^{(k)} \neq b$ in $D_\delta$. Set $\mathcal{G} = \{f^{(k)} : f \in \mathcal{F}\}$. Then for each pair of functions $f, g \in \mathcal{G}$, $f \neq b$ and $g \neq b$, and $f^{(m-k)}$ and $g^{(m-k)}$ share $c$ in $D_\delta$. As before, Theorem E shows that $\mathcal{G}$ is normal in $D_\delta$. Thus by Lemma 2.12, $\mathcal{F}$ is normal in $D_\delta$ and so at $z_0$.

Case 2.2. $f^{(k)}(z_0) = b$. Then clearly there exists $\delta > 0$ such that $f^{(k)} \neq b$ in $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$. Thus, for each function $f \in \mathcal{F}$, $f(z) = a$ if and only if $f^{(k)}(z) = b$ in $D_\delta$, so by Lemma 2.4, $\mathcal{F}$ is normal in $D_\delta$. Thus $\mathcal{F}$ is normal at $z_0$.

This completes the proof of Theorem 1.9. □

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