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CONJUGACY CLASS SIZES IN FINITE GROUPS

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Abstract

We make several conjectures, and prove some results, pertaining to conjugacy classes of a given size in finite groups, especially in p-groups and 2-groups.

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Let G be a finite group, with conjugacy classes have sizes $n_1 = 1 < n_2 < \cdots < n_k$. Thus the classes of size n_1 consist of the central elements, and we refer to classes of size n_2 and to their elements as *minimal classes* and *minimal elements*. Ito initiated the study of groups with a small value of k. His main results are as follows.

(1) If $k = 2$, then $G = H \times K$, where H is abelian and K is a p-group [7].
(2) If $k = 3$, then G is soluble [8] (simplifications of the proof and further results are given in [17] and [2]).
(3) If $k = 4$, and G is simple, then $G \cong PSL(2, 2^n)$ [9].
(4) If $k = 5$, and G is simple, then $G \cong PSL(2, q)$, where $q > 5$ is odd [10].
(5) Some results on the case $k = 6$ are obtained in [11].

It should be noted that these results pre-date the classification of the finite simple groups, and possibly were motivated by the quest for that classification. In contrast, here we are interested in soluble groups, and in particular in p-groups. Result (1) shows that if $k = 2$, that is if all noncentral classes have the same size, we may assume that G is a p-group. For such groups it was proved in [7] that G contains a normal abelian subgroup N such that G/N has exponent p. This was generalized in [15], where it was shown that each p-group G contains a normal abelian subgroup $D(G)$ such that $G/D(G)$ has exponent dividing p^{k-1} (for $p = 2$, we can obtain an exponent that divides 2^{k-2}; the definition of $D(G)$ is given below). The case $k = 2$ was further improved by Isaacs [5], who showed that a p-group G with $k = 2$ satisfies $\exp(G/Z(G)) = p$ (here $Z(G)$ is the centre of G); this was reproved in [14] and [18], with each author being ignorant at the time of the previous proofs. Reference [18] also contains a result by Heineken that if G is metabelian, then it has nilpotence class 3 at most. A breakthrough was achieved by Ishikawa, who proved that a p-group in which $k = 2$ is of class 3 at most [4]. This bound is the best possible, except for $p = 2$, where Isaacs' result shows that the relevant groups are extensions of the centre by a group of exponent 2 and, since groups of exponent 2 are abelian, it follows that G is of class 2. Motivated by Ishikawa's result and by analogous results and conjectures regarding character degrees (for these, see for example the introduction to [13]), the present author made in [15] the following conjecture.

CONJECTURE A. *There exist functions $f(s)$ and $g(p)$ such that, if G is a finite p-group, and the numbers k, n_i are as above, then the subgroup H of G generated by the classes of sizes $n_1, \ldots, n_s$ has derived length* $\mathrm{dl}(H)$ *at most $f(s)$, and the subgroup $M(G)$ generated by the minimal elements has nilpotence class* $\mathrm{cl}(M(G))$ *at most $g(p)$.*

Note that we ask for a bound on the nilpotence class only for $s = 2$; this is because there are p-groups of arbitrary class containing an abelian maximal subgroup, and such groups satisfy $k = 3$. Thus we cannot bound the class of H if $s > 2$.

We note two more, successively weaker, conjectures.

CONJECTURE B. *If the p-group G is generated by the classes of sizes $n_1, \ldots, n_s$, then* $\mathrm{dl}(G) \leq f(s)$*, for some function of s.*

This is implied by Conjecture A, but is not equivalent to it, because if H is generated by the classes of the first s sizes, then the class sizes in H will usually be different from the class sizes in G of the elements of H.

CONJECTURE C. *The derived length of a p-group G with k class sizes is bounded by a function $f(k)$ of k.*

A variation on all three conjectures is obtained by allowing the functions $f(s)$ to depend also on p.

Thus Ishikawa's theorem establishes the case $k = 2$ of Conjecture C. However, it is easy to see that a minor variation on his argument actually shows that if G is generated by its minimal elements, then its class is at most 3, and thus the case $s = 2$ of Conjecture B is established. This is stated explicitly in [1], where the argument of [4] is streamlined. Finally, the case $s = 2$ of Conjecture A, that is the second part of the conjecture, was proved in [16], where we showed that $\mathrm{cl}(M(G)) \leq 3$. This is the best possible for odd primes, but for $p = 2$ it was already shown in [15] that $\mathrm{cl}(M(G)) \leq 2$. The only other known cases of the conjectures are that if $k = 3$ and $p = 2$ then G is

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metabelian [15, Corollary 2], and if $k = 3$ and $p = 3$ then $\mathrm{dl}(G) \le 4$ [16, Corollary 10]. These follow easily from the bounds for $\mathrm{cl}(M(G))$. The latter result was given a shorter proof by Isaacs [6], who showed that it holds also in some wider families of groups than p-groups, for example supersoluble groups. Here we first apply the method of [16] and [6] to show the following result.

THEOREM 1. *Let G be a finite group. Suppose that the subgroup $M(G)$ generated by the minimal elements is soluble, and contains a normal subgroup N with abelian Sylow subgroups such that $M(G)/N$ is supersoluble. Then $M(G)$ is nilpotent, of class 3 at most.*

Then we make another modest contribution towards a proof of the conjectures, establishing for $p = 2$ the case $s = 3$ of Conjecture A and the case $k = 4$ of Conjecture C. Let us call the classes of size n_3, and their elements, *almost minimal*. Recall that in a 2-group G we have $G^2 = \Phi(G)$, where G^2 is the subgroup generated by all squares in G and $\Phi(G)$ is the Frattini subgroup.

THEOREM 2. *Let G be a finite 2-group, and let H be the subgroup which is generated by the minimal and almost minimal classes. Then $\mathrm{cl}(H \cap G^2) \le 3$ and $\mathrm{dl}(H) \le 3$.*

COROLLARY 3. *Let G be a 2-group in which $k = 4$. Then $\mathrm{dl}(G) \le 3$.*

COROLLARY 4. *Let G be a 2-group with $k \ge 4$ class sizes. Then $\mathrm{cl}(G^{2^{k-3}}) \le 3$.*

The last corollary should be compared with the earlier quoted results from [15] and [16], according to which, with the same notation, $G^{2^{k-2}}$ is abelian, while if G is a p-group with k class sizes and p is odd, then $G^{p^{k-2}}$ has class 3 at most and $G^{p^{k-1}}$ is abelian.

We pass now to the proofs. $Z_i(G)$ denotes the ith term of the upper central series of G.

LEMMA A. *Let A be a normal abelian subgroup of G, let $a \in A$, and let $x \in G$ be a noncentral element. Then the number of conjugates of $[a, x]$ is less than the number of conjugates of x.*

For p-groups, this lemma is in [16, Theorem 1]. The general result is proved in [6].

PROOF OF THEOREM 1. Use the notation of the theorem, and let A be the Fitting subgroup of N, its maximal nilpotent normal subgroup. Since the Sylow subgroups of N are abelian, A is also abelian. Let $a \in A$, and let x be a minimal element of G. Then Lemma A shows that $[x, a] \in Z(G)$. Since elements like x generate $M(G)$, we have $[A, M(G)] \le Z(G)$, and thus $[A, M(G)] \le Z(M(G))$ and $A \le Z_2(M(G))$, and also $A \le Z_2(N)$. But if $A \ne N$, and K/A is a nilpotent normal subgroup of N/A, then from $A \le Z_2(N)$ it follows that K is also nilpotent, contradicting the maximality of A. Thus $A = N$, and $N \le Z_2(M(G))$. Therefore, $M(G)$ is also supersoluble, and Isaacs' result applies. □

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Next, recall that the *centralizer equality subgroup* $D(G)$ of a p-group G is generated by all elements x such that $C_G(x) = C_G(x^p)$.

LEMMA B. *Let G be a 2-group, and let x be a minimal element satisfying $x^2 \in Z(G)$. Then $x \in Z_2(G)$.*

See [15, Proposition 6].

LEMMA C. *In a p-group G, $D(G)$ is abelian.*

This is the first claim of [15, Theorem 7].

PROOF OF THEOREM 2. Let x be a minimal element of G, and let q be the largest power of 2 such that $C_G(x^q) = C_G(x)$. Then x^q is a minimal element whose square is central. By Lemma B, $x^q \in Z_2(G)$. It follows that, for each $u \in G$, the subgroup $\langle u, x^q\rangle$ has class at most 2, implying $[u^2, x^q] = [u, (x^q)^2] = 1$, and thus $C_G(x) = C_G(x^q) \geq G^2$, and $[M(G), G^2] = 1$ (this reproves that $\mathrm{cl}(M(G)) \leq 2$). Write $K = H \cap G^2$, and let A be maximal among the normal abelian subgroups of G that are contained in K. By Lemma A, $[A, H] \leq M(G)$, and thus $[A, K] \leq Z(K)$ and $A \leq Z_2(K)$. Therefore, $K' \leq C_K(Z_2(K)) \leq C_K(A) = A$, that is, $K' \leq Z_2(K)$ and $\mathrm{cl}(K) \leq 3$. Our claims follow, since $H^2 \leq K$, and H/H^2, of exponent 2, is abelian.

PROOF OF COROLLARY 3. Write again $K = G^2 \cap H$, and let $D = D(G)$. Since D is abelian, we have, by Lemma A, $[D, H] \leq M(G)$. We saw in the previous proof that $M(G)$ centralizes G^2, and thus $[D, K, K] \leq [D, H, G^2] = 1$. Since D itself is abelian, it follows that $D \leq Z_2(DK)$, and since $\mathrm{cl}(K) \leq 3$, any commutator of weight 4 in the elements of D and K is trivial, that is, $\mathrm{cl}(DK) \leq 3$. Let $x \notin D$, then x^2 has fewer conjugates than x, and since $k = 4$, it follows that x^2 is central, minimal, or almost minimal, anyway $x^2 \in H$, and therefore $x^2 \in K$. Thus G/DK has exponent 2, and therefore it is abelian, and $\mathrm{dl}(G) \leq 3$.

PROOF OF COROLLARY 4. Let $x \in G$. Then either one of the elements $x, x^2, \ldots, x^{2^{k-3}}$ belongs to D, or $x^{2^{k-3}}$ has at most n_3 conjugates. In either case $x^{2^{k-3}} \in DK$, and thus $\mathrm{cl}(G^{2^{k-3}}) \leq \mathrm{cl}(DK) \leq 3$.

In [12, Theorem 13], it is shown that for $p \geq 5$ the free groups of rank at least 3 in the variety of groups of exponent p and class 4 have four class sizes, and the proof shows that these groups have derived length 3. However, we do not have examples of 2-groups, or 3-groups, with these values for k and $\mathrm{dl}(G)$. Other examples for large primes can be obtained from Blackburn's construction of exceptional p-groups of maximal class [3, III.14.24]. These groups are constructed for a prime $p \geq 5$ and an integer r such that $6 \leq 2r \leq p + 1$. They have order p^{2r}, class $2r - 1$, contain an extra special maximal subgroup E, and satisfy $C_G(Z_2(G)) = H$, where H is a maximal subgroup different from E. If $x \in E - Z_2(G)$, then $C_G(x) \leq E$. It follows that the class sizes of G are $1, p, p^2, p^{2r-3}, p^{2r-2}$. Let K be a maximal subgroup different from both E and H. Then the class sizes in K are $1, p, p^2, p^{2r-3}$, and $K' = \gamma_3(G)$

has order p^{2r-3} and is not abelian, provided that $2r \geq 8$, and then $\mathrm{dl}(K) = 3$. Note that the proviso $2r \geq 8$ forces $p \geq 7$.

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