Canad. Math. Bull. Vol. **50** (2), 2007 pp. 291–312

# Beurling's Theorem and Characterization of Heat Kernel for Riemannian Symmetric Spaces of Noncompact Type

Rudra P. Sarkar and Jyoti Sengupta

*Abstract.* We prove Beurling's theorem for rank 1 Riemannian symmetric spaces and relate its consequences with the characterization of the heat kernel of the symmetric space.

## 1 Introduction

The uncertainty principle in harmonic analysis reflects the inevitable tradeoff between the function and its Fourier transform as it says that both of them cannot decay very rapidly. This principle has several quantitative versions which were proved by Hardy, Morgan, Gelfand–Shilov, and Cowling–Price (see [7, 9, 25] and the references therein). In more recent times Hörmander [16] proved the following theorem, which is the strongest theorem in this genre in the sense that it implies the theorems of Hardy, Morgan, Gelfand–Shilov, and Cowling–Price.

***Theorem 1.1*** ( [16]) *Let $f \in L^1(\mathbb{R})$. Then $\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)||\widehat{f}(y)|e^{|x||y|}\,dxdy < \infty$ implies $f = 0$ almost everywhere.*

Hörmander attributes this theorem to A. Beurling.

As is well known in physics, the uncertainty in the momentum is smallest for a given uncertainty in the position if the wave function is the Gaussian $e^{-\frac{x^2}{4t}}$. In harmonic analysis this means that the tradeoff is optimal when the function is Gaussian. The quantitative versions of the uncertainty principle also accommodate this optimal situation. The above theorem of Hörmander was further generalized in [3], which takes care of this aspect of uncertainty.

***Theorem 1.2*** ( [3]) *Let $f \in L^2(\mathbb{R})$ and $N \geq 0$. Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\widehat{f}(y)|}{(1+|x|+|y|)^N} e^{|x||y|}\,dxdy < \infty$$

*implies $f(x) = P(x)e^{-tx^2}$, where $t > 0$ and $P$ is a polynomial with* $\deg P < \frac{N-1}{2}$.

Received by the editors February 5, 2005; revised March 31, 2006.
AMS subject classification: 22E30, 43A85.
Keywords: Beurling's Theorem, Riemannian symmetric spaces, uncertainty principle.

We will refer to Theorem 1.2 simply as *Beurling's theorem* for the sake of brevity.

The aim of this article is to prove the analogue of Theorem 1.2 for Riemannian symmetric spaces $X$ of the noncompact type of rank 1. We recall that such a space is of the form $G/K$ where $G$ is a noncompact connected semisimple Lie group of real rank 1 with finite centre and $K \subset G$ is a maximal compact subgroup.

The precise statement of the theorem and its proof appear in Section 3. In Section 4 we show that the estimate considered in the main theorem is the sharpest possible. In Section 5 we indicate how the theorems of Hardy, Morgan, Gelfand–Shilov, and Cowling–Price on symmetric spaces follow from our Beurling's theorem. The mutual dependencies of these uncertainty theorems can be schematically displayed as follows:

$$\begin{tikzcd}
\text{Beurling's} \arrow[r, Rightarrow] & \text{Gelfand–Shilov} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Cowling–Price} \arrow[d, Rightarrow] \\
 & \text{Morgan's} \arrow[r, Rightarrow] & \text{Hardy's.}
\end{tikzcd}$$

This shows that Beurling's theorem is the *master theorem*. Some of the latter theorems, which follow from Beurling's were proved independently on symmetric spaces in recent years by many authors (see [4, 6, 19–23, 25]).

After completing this work we had the opportunity to see Demange's thesis [5] in which he further generalized Theorem 1.2 (see Theorem 6.1). In Section 6 we have given the appropriate analogue of Demange's theorem on symmetric spaces.

It is unlikely that the method pursued here will generalize to the case when the rank of $X$ is greater than 1, since we utilize here the fact that for rank 1 spaces the Plancherel density $\mu$ is a proper map (see (3.9)).

## 2 Notation and Preliminaries

The pair $(G, K)$ is as described in the introduction. We let $G = KAN$ denote a fixed Iwasawa decomposition of $G$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of $G$, $K$, $A$ and $N$, respectively. We recall that dimension of $\mathfrak{a}$ is 1. We choose and keep fixed throughout a system of positive restricted roots, which we denote by $\Sigma^+$. Let $\gamma \in \Sigma^+$ denote the unique simple root, and let $H_\gamma \in \mathfrak{a}$ be the dual basis of $\mathfrak{a}$. Using $\gamma$ (respectively, $H_\gamma$) we can identify $\mathfrak{a}^*$ (respectively, $\mathfrak{a}$) with $\mathbb{R}$. That is, we identify $t$ with $tH_\gamma$ and $\lambda$ with $\lambda\gamma$. The complexification $\mathfrak{a}^*_{\mathbb{C}}$ of $\mathfrak{a}^*$ can then be identified with $\mathbb{C}$. Under this correspondence the half-sum of the elements of $\Sigma^+$, denoted by $\rho$ corresponds to the real number $\frac{1}{2}(m_\gamma + 2m_{2\gamma})$, where $m_\gamma$ (respectively $m_{2\gamma}$) is the multiplicity of the root $\gamma$ (respectively, $2\gamma$). We will frequently identify $\rho$ with this positive real number without further comment. Furthermore, the positive Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$ (respectively, $\mathfrak{a}^*_+ \subset \mathfrak{a}^*$) gets identified under this correspondence with the set of positive real numbers. We let $\exp tH_\gamma = a_t \in A$ for $t \in \mathbb{R}$. This identifies $A$ with $\mathbb{R}$. Let $\log a$ be the unique element in $\mathfrak{a}$ such that $\exp(\log a) = a$. Thus under the above identification $\log a_t = t$.

Let $H\colon G \to \mathfrak{a}$ be the Iwasawa projection associated to the Iwasawa decomposition, $G = KAN$. Then $H$ is left $K$-invariant and right $MN$-invariant where $M$ is the centralizer of $A$ in $K$. For $\lambda \in \mathfrak{a}^*$ (respectively, $H \in \mathfrak{a}$) we denote by $\lambda^+$ (respectively, $H^+$) the unique Weyl translate of $\lambda$ (respectively, $H$) that belongs to the closure of the positive Weyl chamber $\mathfrak{a}^*_+$ (respectively, $\mathfrak{a}_+$). We have $\lambda^+(H^+) = |\lambda(H)|$ where $|r|$ denotes the modulus of the real number $r$. Note that the Weyl group is isomorphic to $\mathbb{Z}_2$. The unique nontrivial element of the Weyl group takes an element $\lambda \in \mathfrak{a}^* \equiv \mathbb{R}$ (respectively, $H \in \mathfrak{a}$) to $-\lambda$ (respectively, $-H$). Therefore $\lambda^+$ (respectively, $H^+$) corresponds to $|\lambda|$ (respectively, $|H|$) under the above identification of $\mathfrak{a}^*$ (respectively, $\mathfrak{a}$) and $\mathbb{R}$.

We have the $\mathfrak{a}$-valued function $A(x, k)$ on $X \times K$ defined by $A(x, k) = -H(x^{-1}k)$, $x \in X, k \in K$. Note that $A$ descends to a function, also denoted by $A\colon X \times K/M \to \mathfrak{a}$, since $H$ is right $M$-invariant.

The Killing form $B$ of the Lie algebra $\mathfrak{g}$ restricted to $\mathfrak{a}$ is positive definite and gives a Weyl group equivariant isomorphism between $\mathfrak{a}$ and $\mathfrak{a}^*$. Using this isomorphism we get an inner product on $\mathfrak{a}^*$ which we will also denote by $B$. We will normalize the Killing form so that $B(\gamma, \gamma) = 1$. Then we have $B(H_\gamma, H_\gamma) = 1$. By abuse of notation we will denote this normalized Killing form by the same letter $B$. Henceforth we will always use this normalized $B$ and call it the Killing form. Note that with the above identification $B(\lambda, \lambda) = B(\lambda\gamma, \lambda\gamma) = \lambda^2$ and $B(\log a_t, \log a_t) = B(tH_\gamma, tH_\gamma) = t^2$. Thus $|t|$ and $|\lambda|$ respectively are the Killing norms of $\log a_t \in \mathfrak{a}$ and $\lambda \in \mathfrak{a}^*$.

For $x \in G$, we define $\sigma(x) = d(xK, K)$ where $d$ is the canonical distance function for $X = G/K$ coming from the Riemannian structure induced by the normalized Killing form restricted to $\mathfrak{p}$. Here $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (Cartan decomposition) and $\mathfrak{p}$ can be identified with the tangent space at $eK$ of $G/K$. The function $\sigma$ is $K$-biinvariant and continuous. Note that for $x = k_1 a_t k_2$, $k_1, k_2 \in K, a_t \in A$, $\sigma(x) = \sigma(a_t) = \sqrt{B(tH_\gamma, tH_\gamma)} = |t|$, *i.e.*, $\sigma(a) = |\log a|$, the Killing norm of $\log a$.

On $X$ we fix the measure $dx$ which is induced by the metric we obtain from $B$. As the metric is $G$-invariant, so is $dx$. On $G$ we fix the Haar measure $dg$ satisfying

$$\int_X f(x)\, dx = \int_G f(g)\, dg,$$

for every integrable function $f$ on $X$ which we also consider as a right $K$-invariant function on $G$. While dealing with functions on $X$, we may gloss over the difference between the two measures.

We normalize the Haar measure $da$ on $A$ so that $\int_A f(a)\, da = \int_{\mathbb{R}} f(a_t)\, dt$, where $dt$ is the Lebesgue measure on $\mathbb{R}$. As usual, on the compact group $K$ we fix the normalized Haar measure $dk$, *i.e.*, $\mathrm{vol}(K) = \int_K dk = 1$. Finally we fix the Haar measure $dn$ on $N$ by the condition that $\int_G f(g)\, dg = \int_A \int_N \int_K f(ank)\, dkdnda$ holds for every integrable function $f$ on $G$.

For an integrable function $f$ on $\mathbb{R}$, we define its Euclidean Fourier transform at $\lambda$ as $\int_{\mathbb{R}} f(x) e^{-i\lambda x}\, dx$, and we denote it by $\mathcal{F}f(\lambda)$. We follow the practice of using $C, C'$ etc. to denote constants whose values are not necessarily the same at each occurrence.

***Definition 2.1*** For a function $f$ in $C_c^\infty(X)$, the Helgason Fourier transform $\widetilde{f}$ of $f$ is defined by

$$\widetilde{f}(\lambda,k)=\int_X e^{(-i\lambda+\rho)(A(x,k))}f(x)\,dx,\qquad \lambda\in\mathfrak{a}^*,k\in K.$$

Note that $\widetilde{f}$ descends to a function on $\mathfrak{a}^*\times K/M$. By abuse of notation we will continue to denote this function by $\widetilde{f}$. For $f\in L^1(X)$, there exists a subset $B$ of $K$ of full Haar measure, such that $\widetilde{f}(\lambda,k)$ exists for all $k\in B$ and $\lambda\in\mathbb{C}$ with $|\Im\lambda|\le\rho$, where $\Im\lambda$ is the imaginary part of $\lambda$. Indeed for each fixed $k\in B$, $\lambda\mapsto\widetilde{f}(\lambda,k)$ is holomorphic in the strip $\{\lambda\in\mathbb{C}\mid|\Im\lambda|<\rho\}$ and continuous on its boundary (see [14, 15] for proof, see also [18]).

***Definition 2.2*** For $f$ in $C_c^\infty(X)$, the Radon transform $\mathcal{R}f$ of $f$ is defined by

$$\mathcal{R}f(k,a)=e^{\rho(\log a)}\int_N f(kan)\,dn,\qquad k\in K,a\in A.$$

Then $\mathcal{R}f$ descends to a function on $K/M\times A$, and (as in the case of $\widetilde{f}$) we continue to denote this function by $\mathcal{R}f$. We will also use the notation $\mathcal{R}f(k,t)$ for $\mathcal{R}f(k,a_t)$, $t\in\mathbb{R}$.

For $f$ as above, the basic relation between $\mathcal{R}f$ and $\widetilde{f}$ is the following:

$$\widetilde{f}(\lambda,k)=\mathcal{F}\mathcal{R}f(k,\cdot)(\lambda),\tag{2.1}$$

where $\mathcal{F}$ denotes the Euclidean Fourier transform on $A\equiv\mathbb{R}$, *i.e.*, $\mathcal{F}\mathcal{R}f(k,\cdot)(\lambda)=\int_{\mathbb{R}}\mathcal{R}f(k,t)e^{-i\lambda t}\,dt$.

Let $\widehat{K}_0$ be the set of equivalence classes of irreducible unitary representations of $K$ which are class 1 with respect to $M$, *i.e.*, the irreducible unitary representations of $K$ which contain an $M$-fixed vector. Let $\delta\in\widehat{K}_0$ and let $f\in L^1(X)$ be $K$-finite of type $\delta$, *i.e.*, $d(\delta)\bar{\chi}_\delta*f=f$ where $d(\delta)$ (respectively, $\chi_\delta$) denotes the degree (respectively, character) of $\delta$ and $(d(\delta)\bar{\chi}_\delta*f)(x)=d(\delta)\int_K f(kx)\bar{\chi}_\delta(k)dk$ for $x\in X$. In particular, if $\delta$ is the trivial representation, then $f$ is a $K$-invariant function on $X$. For a function $f$ of type $\delta$ we have $|f(x)|\le C\int_K|f(kx)|\,dk$ where $C=d(\delta)\sup_{k\in K}|\chi_\delta(k)|=d(\delta)^2$. Let $g(x)=\int_K|f(kx)|\,dk$. Then $g\in L^1(X)$ and $g$ is $K$-invariant, that is $g\in L^1(G)$ and $g$ is $K$-biinvariant. We have $|f(x)|\le d(\delta)^2g(x)$.

***Definition 2.3*** For a $\delta$-type function $f$ in $L^1(X)$, the Abel transform $\mathcal{A}f$ of $f$ is defined by

$$\mathcal{A}f(a)=e^{\rho(\log a)}\int_N f(an)\,dn,\qquad a\in A.$$

It is well known that for a $K$-invariant function $g\in L^1(X)$, $\mathcal{A}g$ exists for almost every $a\in A$ and $\mathcal{A}g\in L^1(A)$ [10, p. 27]. Now since

$$|\mathcal{A}f(a)|\le e^{\rho(\log a)}\int_N|f(an)|\,dn\le d(\delta)^2e^{\rho(\log a)}\int_N g(an)\,dn=d(\delta)^2\mathcal{A}g(a)$$

for the $K$-invariant function $g$ constructed from $f$ as above, we conclude that $\mathcal{A}f \in L^1(A)$. We will write $\mathcal{A}f(t)$ for $\mathcal{A}f(a_t)$, $t \in \mathbb{R}$.

It is also well known that for $f \in L^1(X)$, $\mathcal{R}f \in L^1(K \times A, dkda)$. We include the proof here for the sake of completeness. For $f \in L^1(X)$, we construct the function $g(x) = \int_K |f(kx)|\, dk$. Then $g$ is a $K$-biinvariant function in $L^1(G)$ and hence, as mentioned above (recall the identification of $A$ and $\mathbb{R}$), $\mathcal{A}g(t) \in L^1(\mathbb{R})$. But

$$\begin{aligned}\mathcal{A}g(t) &= e^{\rho t}\int_N g(a_t n)\, dn = e^{\rho t}\int_N\int_K |f(ka_t n)|\, dkdn \\ &= \int_K e^{\rho t}\int_N |f(ka_t n)|\, dndk = \int_K \mathcal{R}|f|(k,t)\, dk.\end{aligned}$$

Since $\int_A \mathcal{A}g(t)\, dt < \infty$ we have $\int_A\int_K \mathcal{R}|f|(k,t)\, dkdt < \infty$. This proves our assertion. Thus for $f \in L^1(X)$, we can consider the Euclidean Fourier transform in the $A$-variable of $\mathcal{R}f(k,\cdot)$ for each fixed $k$ where it exists. Now, it can also be shown in a similar way that the relation (2.1) holds when $f \in L^1(X)$ for almost every $k \in K$.

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \equiv \mathbb{C}$, we denote by $\phi_\lambda$ the elementary spherical function with parameter $\lambda$. We have for all $x \in X$, $\phi_\lambda(x) = \phi_{-\lambda}(x) = \int_K e^{(\rho - i\lambda)(A(x,k))}\, dk$ [12, p. 418]. We will often regard $\phi_\lambda$ as a $K$-biinvariant function on $G$.

The following estimates on the growth of $\phi_\lambda$ are well known [8, Proposition 4.6.1; Theorems 4.6.4, 4.6.5], [13]. Let $\Xi(x) = \phi_0(x)$. Then

$$|\phi_\lambda(x)| \le 1 \text{ for } \lambda \in \mathbb{C}, |\Im\lambda| \le \rho, \tag{2.2a}$$

$$|\phi_\lambda(x)| \le e^{|\Im\lambda|\sigma(x)}\Xi(x) \text{ for all } \lambda \in \mathbb{C}, \tag{2.2b}$$

$$\Xi(a) \le C(1+\sigma(a))e^{-\rho(\log a)} \text{ and } \Xi(a) \ge e^{-\rho(\log a)} \text{ for } a \in \overline{A^+} = \exp\bar{\mathfrak{a}}_+ \text{ and a positive constant } C. \tag{2.2c}$$

We denote the spherical Plancherel measure on $\mathfrak{a}^*$ by $\mu(\lambda)d\lambda$, where $d\lambda$ is the Lebesgue measure and $\mu(\lambda) = |c(\lambda)|^{-2}$, $c(\cdot)$ is Harish-Chandra's $c$-function. It is well known that $\mu(\lambda)$ is real analytic on $\mathfrak{a}^* \equiv \mathbb{R}$ (see [2, p. 399] for an explicit expression of $\mu(\lambda)$).

Recall that the elements $\delta \in \widehat{K}_0$ can be labelled by a pair of integers $(r,s)$ with $s \ge r$ [13, pp. 344–347 ]. The trivial representation in $\widehat{K}_0$ corresponds to the pair $(0,0)$ in this setup. Note that if $m_{2\gamma} > 0$, then $r \ge 0, s \ge 0$ and if $m_{2\gamma} = 0$, then $r = 0$. Thus for both the cases $(r,s)$ is a pair of nonnegative integers.

It is known that for each $\delta \in \widehat{K}_0$, the $M$-fixed vector is unique upto a scalar multiple [17]. Let $(\delta, V_\delta) \in \widehat{K}_0$. Suppose $\{v_i \mid i = 1, \dots, d(\delta)\}$ is an orthonormal basis of $V_\delta$ of which $v_1$ is the $M$-fixed vector. Let $Y_{\delta,j}(k) = \langle v_j, \delta(k)v_1\rangle, 1 \le j \le d(\delta)$ and let $Y_0$ be the constant function $Y_0 \equiv 1$ on $K$. Note that $Y_{\delta,j}$ is right $M$-invariant, *i.e.*, it is a function on $K/M$. Recall that $L^2(K/M)$ is the carrier space of the spherical principal series representations $\pi_\lambda$, $\lambda \in \mathbb{C}$ in the compact picture and $\{\sqrt{d(\delta)}Y_{\delta,j} : 1 \le j \le d(\delta), \delta \in \widehat{K}_0\}$ is an orthonormal basis for $L^2(K/M)$ adapted to the decomposition $L^2(K/M) = \Sigma_{\delta\in\widehat{K}_0} V_\delta$. As the space $K/M$ can be identified with $S^{m_\gamma + m_{2\gamma}}$, this

https://doi.org/10.4153/CMB-2007-029-6 Published online by Cambridge University Press

decomposition can be viewed as the spherical harmonic decomposition and therefore $Y_{\delta,j}$'s can be considered as the *spherical harmonics*. The action of $\pi_\lambda$ is given by:

$$(\pi_\lambda(x)g)(k) = e^{(i\lambda+\rho)A(x,k)}g(\kappa(x^{-1}k)) \text{ for } x \in G, k \in K \text{ and } g \in L^2(K/M).$$

Here $\kappa(x)$ is the $K$-part of an element $x \in G$ in the Iwasawa decomposition $G = KAN$. The representation $\pi_\lambda$ is unitary for $\lambda \in \mathbb{R}$. For $f \in L^1(X)$, $\delta \in \widehat{K}_0$ and $1 \le j \le d(\delta)$ we define $f_{\delta,j}(x) = \int_K f(kx)\overline{Y_{\delta,j}(k)}\,dk$. It can be verified that $f_{\delta,j}$ is a function of type $\delta$.

We have $|f_{\delta,j}(x)| \le \|\overline{Y_{\delta,j}}\|_\infty \int_K |f(kx)|\,dk \le \int_K |f(kx)|\,dk$, since $\|\overline{Y_{\delta,j}}\|_\infty = \sup_{k\in K}|\overline{Y_{\delta,j}(k)}| \le 1$.

For $\delta \in \widehat{K}_0$, $1 \le j \le d(\delta)$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $x \in X$, we define

$$\Phi^j_{\lambda,\delta}(x) = \int_{K/M} e^{(i\lambda+\rho)(A(x,kM))}\overline{Y_{\delta,j}(kM)}\,dk. \tag{2.3}$$

We have $\Phi^j_{\lambda,\delta}(x) = \langle \pi_\lambda(x)Y_0, Y_{\delta,j}\rangle$, that is, $\Phi^j_{\lambda,\delta}$ is a matrix coefficient of the spherical principal series representation. It follows that for each fixed $x \in X$, $\lambda \to \Phi^j_{\lambda,\delta}(x)$ is holomorphic in $\lambda$. Let $\Delta$ be the Laplace–Beltrami operator of $X$. It is a negative self adjoint operator. It is well known [11, p. 333] that $\Phi^j_{\lambda,\delta}$'s are eigenfunctions of $\Delta$ with eigenvalues $-(\lambda^2+\rho^2)$. When $\delta = \delta_0$ is trivial, then $Y_{\delta_0,j} = Y_{\delta_0,1} = Y_0$ and $\Phi^1_{\lambda,\delta_0}$ is obviously the elementary spherical function $\phi_\lambda(x)$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $x = ka_tK \in X$ and $1 \le j \le d(\delta)$ [13, p. 344]

$$\Phi^j_{\lambda,\delta}(x) = \overline{Y_{\delta,j}(kM)}\Phi^1_{\lambda,\delta}(a_t). \tag{2.4}$$

Then $\Phi^1_{\lambda,\delta}$ is related to $\Phi^1_{-\lambda,\delta}$ by

$$\Phi^1_{\lambda,\delta} = \frac{Q_\delta(\lambda)}{Q_\delta(-\lambda)}\Phi^1_{-\lambda,\delta}, \tag{2.5}$$

where $Q_\delta$ are Kostant's polynomials [13, p. 348, (13)]. Kostant's polynomials $Q_\delta$ are given by $Q_\delta(\lambda) = p_{r,s}(\lambda)q_{r,s}(\lambda)$. For explicit expression of $p_{r,s}$ and $q_{r,s}$ see [13, p. 345, (7), (8); p. 348, (15), (17)]. Note that our $Q_\delta(\lambda)$ is $Q^\delta(-\lambda)$ in [13]. Notice also that for both the cases $m_{2\gamma} = 0$ and $m_{2\gamma} > 0$, $Q_\delta$ is a polynomial and that $Q_\delta(\lambda)$ and $Q_\delta(-\lambda)$ are relatively prime [13, p. 348]. It is clear [13, pp. 344–345] that $\deg Q_\delta = s$. Indeed $Q_\delta(\lambda)$ is the polynomial factor of $\Phi^1_{\lambda,\delta}$ and hence of $\Phi^j_{\lambda,\delta}$ for $1 \le j \le d(\delta)$ [13, p. 344, (5), (6)].

Because of the relation (2.4) above, we have $\Phi^j_{\lambda,\delta} = \frac{Q_\delta(\lambda)}{Q_\delta(-\lambda)}\Phi^j_{-\lambda,\delta}$.

Let

$$\widehat{f}(\lambda)_{\delta,j} = \int_X f(x)\Phi^j_{-\lambda,\delta}(x)\,dx \quad \text{for } \lambda \in \mathfrak{a}^*.$$

It is clear that $\widehat{f}(\lambda)_{\delta,j}$ is the $(\delta, j)$-th matrix coefficient of the operator valued Fourier transform $\widehat{f}(\lambda) = \int_G f(x)\pi_{-\lambda}(x)\,dx$. From above we see that $\widehat{f}_{\delta,j}(\lambda)$ has the polynomial factor $Q_\delta(-\lambda)$.

Note that

$$
\begin{aligned}
|\Phi^j_{-\lambda,\delta}(x)| &\leq \int_{K/M} |e^{(-i\lambda+\rho)A(x,kM)}| \overline{Y_{\delta,j}(k)} dk \\
&\leq \int_{K/M} e^{(\Im\lambda+\rho)A(x,kM)}\, dk \qquad (\text{as } \|\overline{Y_{\delta,j}}\|_\infty \leq 1) \\
&= \phi_{\Im\lambda}(x).
\end{aligned}
$$

Therefore from the estimate (2.2a) we have $|\Phi^j_{\lambda,\delta}(x)| \leq 1$ for all $\lambda \in \mathbb{C}$ with $|\Im\lambda| \leq \rho$. From the above uniform estimate of $\Phi^j_{\lambda,\delta}$ and the fact that $\lambda \mapsto \Phi^j_{\lambda,\delta}$ is a holomorphic function in $\lambda$ (see the comment following equation (2.3)), it follows by a standard use of Morera's theorem in conjunction with Fubini's theorem that for $f \in L^1(X)$, $\widehat{f}(\lambda)_{\delta,j}$ is holomorphic on the open strip $\{\lambda \in \mathbb{C} \bigm| |\Im\lambda| < \rho\}$ and in particular it is real analytic on $\mathfrak{a}^* \equiv \mathbb{R}$.

Let $f \in L^1(X) \cap L^2(X)$. Then for every $\lambda \in \mathfrak{a}^*$,

$$
\|\widehat{f}(\lambda)\|_2^2 = \sum_{\delta\in\widehat{K}_0} \sum_{1\leq j\leq d(\delta)} |\widehat{f}(\lambda)_{\delta,j}|^2, \tag{2.6}
$$

where $\|\cdot\|_2$ is the Hilbert–Schmidt norm.

Also by (2.5)

$$
\widehat{f}(-\lambda)_{\delta,j} = \frac{Q_\delta(\lambda)}{Q_\delta(-\lambda)} \widehat{f}(\lambda)_{\delta,j}. \tag{2.5$'$}
$$

Note that for $\lambda \in \mathfrak{a}^*$, $\overline{Q_\delta(\lambda)} = Q_\delta(-\bar{\lambda}) = Q_\delta(-\lambda)$ [13, p. 348]. Consequently, $|\widehat{f}(\lambda)_{\delta,j}| = |\widehat{f}(-\lambda)_{\delta,j}|$ for $\lambda \in \mathfrak{a}^*$.

The following is also easy to see:

$$
\begin{aligned}
(2.7) \qquad & \int_K \widetilde{f}(\lambda,k)\overline{Y_{\delta,j}(k)}\, dk \\
&= \int_X \int_K f(x) e^{(-i\lambda+\rho)A(x,k)} \overline{Y_{\delta,j}(k)}\, dk dx \qquad (\text{by Fubini's theorem}) \\
&= \int_X f(x)\Phi^j_{-\lambda,\delta}(x)\, dx \\
&= \widehat{f}(\lambda)_{\delta,j}.
\end{aligned}
$$

Starting from the relation (2.1) and using (2.7) we have

$$
\int_K \mathcal{F}(\mathcal{R}(f)(k,\,\cdot\,))(\lambda)\overline{Y_{\delta,j}(k)}\, dk = \widehat{f}(\lambda)_{\delta,j}.
$$

Now the left-hand side is (recall that $\mathfrak{a}^* \equiv \mathbb{R}$):

$$\begin{aligned}
\int_K \int_{\mathbb{R}} \mathcal{R}(f)(k,t) e^{-i\lambda t}\, dt \overline{Y_{\delta,j}(k)}\, dk
&= \int_K \int_{\mathbb{R}} e^{\rho t} \int_N f(ka_t n)\, dn e^{-i\lambda t}\, dt \overline{Y_{\delta,j}(k)}\, dk \\
&= \int_{\mathbb{R}} e^{\rho t} \int_N f_{\delta,j}(a_t n) dn\, e^{-i\lambda t}\, dt \qquad \text{(by Fubini's theorem)} \\
&= \int_{\mathbb{R}} \mathcal{A}(f_{\delta,j})(t) e^{-i\lambda t}\, dt \\
&= \mathcal{F}(\mathcal{A}(f_{\delta,j}))(\lambda).
\end{aligned}$$

Therefore,

$$\mathcal{F}(\mathcal{A}(f_{\delta,j}))(\lambda) = \widehat{f}(\lambda)_{\delta,j}. \tag{2.8}$$

Note that from above it is also clear that

$$\int_K \mathcal{R}(f)(k,t) \overline{Y_{\delta,j}(k)}\, dk = \mathcal{A}(f_{\delta,j})(t), \tag{2.9}$$

and hence

$$\begin{aligned}
|\mathcal{A}(f_{\delta,j})(t)| &= \Big| \int_K \mathcal{R}(f)(k,t) \overline{Y_{\delta,j}(k)}\, dk \Big| \\
&\le \int_K |\mathcal{R}(f)(k,t)|\, dk \le \int_K \mathcal{R}|f|(k,t)\, dk
\end{aligned} \tag{2.10}$$

since $\|\overline{Y_{\delta,j}}\|_\infty \le 1$.

We will conclude this section with a description of the *heat-kernel* of the symmetric space $X$. The heat kernel on $X$ is an appropriate analogue of the Gauss kernel $p_t$ on $\mathbb{R}^n$, where $p_t(x) = (4\pi t)^{-n/2} e^{-\|x\|^2/4t}$, $t > 0$.

Recall that $\Delta$ is the Laplace–Beltrami operator of $X$. Then (see [24, Ch. V]) $T_t = e^{t\Delta}$, $t > 0$, defines a semigroup (heat-diffusion semigroup) of operators such that for any $\phi \in C_c^\infty(X)$, $T_t\phi$ is a solution of $\Delta u = \frac{\partial u}{\partial t}$ and $T_t\phi \to \phi$ a.e. as $t \to 0$. For every $t > 0$, $T_t$ is an integral operator with kernel $h_t$, that is, for any $\phi \in C_c^\infty(X)$, $T_t\phi = \phi * h_t$. The $h_t, t > 0$ are $K$-biinvariant functions on $G$ having the following properties:

(i) $h(x,t) = h_t(x)$ is in $C^\infty(G \times \mathbb{R}^+)$.
(ii) $\{h_t : t > 0\}$ form a semigroup under convolution $*$. That is $h_t * h_s = h_{t+s}$ for $t, s > 0$.
(iii) $h_t$ is a fundamental solution of $\Delta u = \frac{\partial u}{\partial t}$.
(iv) $h_t \in L^1(G) \cap L^\infty(G)$ for every $t > 0$.
(v) $\int_X h_t(x)\, dx = 1$ for every $t > 0$.

https://doi.org/10.4153/CMB-2007-029-6 Published online by Cambridge University Press

Thus we see that the heat kernel $h_t$ on $X$ retains all the nice properties of the Gauss kernel. It is well known that $h_t$ is given by [1]:

$$h_t(x) = \frac{1}{2} \int_{\mathfrak{a}^*} e^{-t(\lambda^2+\rho^2)} \phi_\lambda(x) \mu(\lambda)\, d\lambda. \tag{2.11}$$

That is, the spherical Fourier transform of $h_t$ is $\widehat{h_t}(\lambda) = e^{-t(\lambda^2+\rho^2)}$. It has been proved [1, Theorem 3.1 (i)] that for any $t > 0$, there exists $C > 0$ depending only on $X$ such that

$$h_t(\exp H) \le C t^{-\frac{1}{2}} e^{-\rho^2 t - \langle \rho, H\rangle - \frac{|H|^2}{4t}} (1 + |H|^2)^{\frac{d_X - 1}{2}} \tag{2.12}$$

for $H \in \overline{\mathfrak{a}^+}$, where $d_X = m_\gamma + m_{2\gamma} + 1 = \dim X$.

## 3 Statement and Proof of the Theorem

***Theorem 3.1*** *Let $f \in L^2(X)$ satisfy*

$$\int_X \int_{\mathfrak{a}^*} \frac{|f(x)| \|\widehat{f}(\lambda)\|_2\, e^{\sigma(x)|\lambda|}\, \Xi(x)}{(1 + \sigma(x) + |\lambda|)^d}\, dx \mu(\lambda) d\lambda < \infty \tag{3.1}$$

*for some nonnegative integer $d$. Then $f$ is a $K$-finite function of the form $f = \sum_{\delta \in F} h_\delta$ where $F = \{\delta \in \widehat{K}_0 \mid s < \frac{d - d_X}{2}\}$ is a finite set of $K$-types and $h_\delta$ is a function of type $\delta$ having Fourier coefficients $\widehat{h}_{\delta,j}(\lambda) = P'_{\delta,j}(\lambda^2) Q_\delta(-\lambda) e^{-\alpha\lambda^2}$ for $1 \le j \le d(\delta)$. Here $\alpha$ is a positive constant and $P'_{\delta,j}$ a polynomial which depends on $\delta$ and $j$.*

*In particular if $d \le d_X$, then $f = 0$ almost everywhere.*

***Remark 3.2*** In Section 2 we discussed the correspondence of elements of $\widehat{K}_0$ with a pair of integers, which we have used in the above statement. As $s \ge 0$, only finitely many $s$ can satisfy $s < \frac{d - d_X}{2}$. Again $r \ge 0$ and $r \le s$. Therefore for a given $s$, there can only be finitely many $r$ such that $(r, s)$ corresponds to an element of $\widehat{K}_0$. Hence there are only finitely many elements in $F$.

**Proof** We have divided the proof into several steps for the convenience of the reader. We will use Fubini's theorem freely throughout the proof without explicitly mentioning it.

**Step 1:** In this step we will show that $f \in L^1(X)$. It is given that $f \in L^2(X)$. Hence $f$ is a locally integrable function on $X$. We will first show that $\widehat{f}$ cannot be supported on a set of finite measure.

From (3.1) we have

$$\int_X \frac{|f(x)| e^{\sigma(x)|\lambda|} \Xi(x)}{(1 + \sigma(x) + |\lambda|)^d}\, dx < \infty$$

for almost every $\lambda \in \mathfrak{a}^*$. Since $\widehat{f} \not\equiv 0$, there exists $\lambda_0 \neq 0$ such that $\widehat{f}(\lambda_0) \neq 0$ and the inequality above holds for $\lambda = \lambda_0$. Suppose $|\lambda_0| = r > 0$. Thus we have

$$\int_X \frac{|f(x)| e^{r\sigma(x)} \Xi(x)}{(1 + \sigma(x) + r)^d} \, dx < \infty.$$

As $f \in L^1_{\mathrm{loc}}(X)$, for $0 < r' < r$, $\int_X |f(x)| e^{r'\sigma(x)} \Xi(x) \, dx < \infty$. Recall that $\widehat{f}(\lambda)_{\delta,j} = \int_X f(x) \Phi^j_{-\lambda,\delta}(x) \, dx$, for $\delta \in \widehat{K}_0$ and $1 \leq j \leq d(\delta)$. Now,

$$\begin{aligned} \Big| \int_X f(x) \Phi^j_{-\lambda,\delta}(x) \, dx \Big| &\leq \int_X |f(x)| |\Phi^j_{-\lambda,\delta}(x)| \, dx \\ &\leq \int_X |f(x)| e^{|\Im\lambda| \sigma(x)} \Xi(x) \\ &\leq \int_X |f(x)| e^{r'\sigma(x)} \Xi(x) e^{(|\Im\lambda| - r')\sigma(x)} \, dx. \end{aligned}$$

This shows that $\widehat{f}(\,\cdot\,)_{\delta,j}$ is holomorphic in the open strip $|\Im\lambda| < r'$ in $\mathfrak{a}^*_{\mathbb{C}}$. Therefore $\widehat{f}(\,\cdot\,)_{\delta,j}$ and hence $\widehat{f}$ cannot be supported on a set of finite measure.

Now since $\widehat{f}$ is supported on a set of infinite measure and as $\mu(\lambda)$ is real analytic, from (3.1) we see that for some $\lambda_1 \in \mathfrak{a}^*$ with $|\lambda_1| > 2|\rho|$,

$$\int_X \frac{|f(x)| \Xi(x) e^{|\lambda_1| \sigma(x)}}{(1 + \sigma(x) + |\lambda_1|)^d} \, dx < \infty.$$

Now from (2.2c) we have for $x = k_1 a k_2$, $\Xi(x)^{-1} \leq e^{\rho(\log a)} \leq e^{|\rho| |\log a|} = e^{|\rho| \sigma(x)}$. Therefore

$$\Xi(x)^{-1} e^{-|\lambda_1| \sigma(x)} (1 + \sigma(x) + |\lambda_1|)^d \leq e^{-|\rho| \sigma(x)} (1 + \sigma(x) + |\lambda_1|)^d.$$

The function $e^{-|\rho|\sigma(x)}(1 + \sigma(x) + |\lambda_1|)^d$ is continuous and bounded. Hence

$$\int_X |f(x)| \, dx = \int_X \frac{|f(x)| \Xi(x) e^{|\lambda_1| \sigma(x)}}{(1 + \sigma(x) + |\lambda_1|)^d} \big( \Xi(x)^{-1} e^{-|\lambda_1| \sigma(x)} (1 + \sigma(x) + |\lambda_1|)^d \big) \, dx < \infty.$$

That is, $f \in L^1(X)$.

**Step 2:** In this step we will show that (3.1) implies the condition:

$$\int_K \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mathcal{R}(|f|)(k,t) |\widehat{f}(\lambda)_{\delta,j}| e^{|\lambda||t|}}{(1 + |t| + |\lambda|)^d} \, dk dt \mu(\lambda) d\lambda < \infty, \tag{3.2}$$

for every fixed $\delta \in \widehat{K}_0$ and $1 \leq j \leq d(\delta)$.

Since all the terms of the integrand in (3.1) are right $K$-invariant, it is equivalent to

$$\int_G \int_{\mathfrak{a}^*} \frac{|f(x)| \|\widehat{f}(\lambda)_{\delta,j}\|_2 e^{\sigma(x)|\lambda|} \Xi(x)}{(1 + \sigma(x) + |\lambda|)^d} \, dx \mu(\lambda) d\lambda < \infty.$$

https://doi.org/10.4153/CMB-2007-029-6 Published online by Cambridge University Press

Also as

$$|\widehat{f}(\lambda)_{\delta,j}| \leq \sqrt{\sum_{\delta'\in\widehat{K}_0}\sum_{j'=1}^{d(\delta')}|\widehat{f}(\lambda)_{\delta',j'}|^2} = \|\widehat{f}(\lambda)\|_2$$

we have

$$\int_G\int_{\mathfrak{a}^*}\frac{|f(x)||\widehat{f}(\lambda)_{\delta,j}|e^{\sigma(x)|\lambda|}\Xi(x)}{(1+\sigma(x)+|\lambda|)^d}\,dx\mu(\lambda)d\lambda<\infty. \tag{3.3}$$

By definition, $\Xi(x) = \int_K e^{-\rho(H((k^{-1}x)^{-1})}\,dk$. Plugging this into (3.3) and using $K$-invariance of the measure $dx$ and of $\sigma(x)$, we obtain

$$\int_K\int_X\int_{\mathfrak{a}^*}\frac{|f(kx)||\widehat{f}(\lambda)_{\delta,j}|e^{\sigma(x)|\lambda|-\rho(H(x^{-1}))}}{(1+\sigma(x)+|\lambda|)^d}\,\mu(\lambda)\,d\lambda\,dxdk<\infty.$$

We write the integral $dx$ over $X$ as $da\,dn$ over $AN$ and use the fact that $H((an)^{-1}) = -\log a$ to get,

$$\int_K\int_A\int_N\int_{\mathfrak{a}^*}\frac{|f(kan)|\,|\widehat{f}(\lambda)_{\delta,j}|\,e^{\sigma(an)|\lambda|+\rho(\log a))}}{(1+\sigma(an)+|\lambda|)^d}\,\mu(\lambda)\,d\lambda\,dadndk<\infty. \tag{3.4}$$

Assuming $f\neq 0$, this implies that there exists $\alpha>0$ such that

$$\int_{\mathfrak{a}^*}|\widehat{f}(\lambda)_{\delta,j}|e^{\alpha|\lambda|}\mu(\lambda)\,d\lambda<\infty.$$

In particular it follows that $\widehat{f}(\lambda)_{\delta,j}\in L^1(\mathfrak{a}^*,\mu(\lambda)d\lambda)$.

For $l>0$ the function

$$g_l(t)=\frac{e^{lt}}{(1+t+l)^d}$$

is monotonically decreasing for $0<t<\frac{d}{l}-1-l$ and increasing for $t>\frac{d}{l}-1-l$. Therefore for $a\in A$ with $\sigma(a)>\frac{d}{|\lambda|}-1-|\lambda|$, one has

$$e^{\sigma(an)|\lambda|}(1+\sigma(an)+|\lambda|)^d\geq e^{\sigma(a)|\lambda|}(1+\sigma(a)+|\lambda|)^d,$$

as $\sigma(an\geq\sigma(a)$. Hence by (3.4) one concludes

$$\int_K\int_{\mathfrak{a}^*}\int_{\{a\in A|\sigma(a)>\frac{d}{|\lambda|}-1-|\lambda|\}}\int_N\frac{|f(kan)||\widehat{f}(\lambda)_{\delta,j}|e^{\sigma(a)|\lambda|+\rho(\log a))}}{(1+\sigma(a)+|\lambda|)^d}\,dnda\,\mu(\lambda)d\lambda\,dk<\infty.$$

That is,

$$\int_K\int_{\mathfrak{a}^*}\int_{\{a\in A|\sigma(a)>\frac{d}{|\lambda|}-1-|\lambda|\}}\frac{\mathcal{R}(|f|)(k,a)|\,|\widehat{f}(\lambda)_{\delta,j}|\,e^{\sigma(a)|\lambda|)}}{(1+\sigma(a)+|\lambda|)^d}\,da\mu(\lambda)d\lambda dk<\infty. \tag{3.5}$$

https://doi.org/10.4153/CMB-2007-029-6 Published online by Cambridge University Press

Note that when $|\lambda|$ is large so that $\frac{d}{|\lambda|} - 1 - |\lambda| < 0$ then $\{a \in A \mid \sigma(a) > \frac{d}{|\lambda|} - 1 - |\lambda|\} = A$ as $\sigma(a)$ is nonnegative. This shows that $\mathcal{R}(|f|)(k,a)$ is finite almost everywhere. Further assuming that $\widehat{f}(\lambda)_{\delta,j}$ is nonzero, it is nonzero for $\lambda$ with arbitrary large $|\lambda|$ as it is a holomorphic function in $\lambda$. This implies that $\mathcal{R}(|f|)(k,a)$ is in $L^1(A \times K)$.

Finally when $\sigma(a) \leq \frac{d}{|\lambda|} - 1 - |\lambda|$ then $e^{\sigma(a)|\lambda|} \leq e^d$. Therefore,

$$\frac{\mathcal{R}(|f|)(k,a)|\,|\widehat{f}(\lambda)_{\delta,j}|e^{\sigma(a)|\lambda|)}}{(1+\sigma(a)+|\lambda|)^d} \leq e^d \mathcal{R}(|f|)(k,a)|\widehat{f}(\lambda)_{\delta,j}|.$$

As the right-hand side is integrable with respect to $\mu(\lambda)d\lambda da dk$, we have

$$(3.6)\quad \int_K \int_{\mathfrak{a}^*} \int_{\{a\in A|\sigma(a)\leq \frac{d}{|\lambda|}-1-|\lambda|\}} \frac{\mathcal{R}(|f|)(k,a)|\,|\widehat{f}(\lambda)_{\delta,j}|\,e^{\sigma(a)|\lambda|)}}{(1+\sigma(a)+|\lambda|)^d}\, da\,\mu(\lambda)d\lambda\, dk < \infty.$$

The inequalities (3.5) and (3.6) together establish (3.2).

**Step 3:** From (3.2) and (2.10) we have

$$(3.7)\qquad \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\mathcal{A}(f_{\delta',j'})(t)||\widehat{f}(\lambda)_{\delta,j}|e^{|\lambda||t|}}{(1+|t|+|\lambda|)^d}\, dt\mu(\lambda)d\lambda < \infty$$

for $\delta, \delta' \in \widehat{K}_0$, $1 \leq j \leq d(\delta)$, $1 \leq j' \leq d(\delta')$.

In particular we can take $\delta = \delta'$ and $j = j'$ to obtain

$$(3.8)\qquad \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\mathcal{A}(f_{\delta,j})(t)||\widehat{f}(\lambda)_{\delta,j}|e^{|\lambda||t|}}{(1+|t|+|\lambda|)^d}\, dt\mu(\lambda)d\lambda < \infty.$$

**Step 4:** Now we will show that in (3.8) $\mu(\lambda)d\lambda$ can be replaced by $d\lambda$. We have the following asymptotic estimate of the spherical Plancherel density [2]

$$(3.9)\qquad \mu(\lambda) = |c(\lambda)|^{-2} \asymp \langle\lambda,\gamma\rangle^2(1+|\langle\lambda,\gamma\rangle|)^{m_\gamma+m_{2\gamma}-2},$$

where $m_\gamma, m_{2\gamma}$ are as defined in Section 2. Here $f \asymp g$ means $c_1 g(\lambda) \leq f(\lambda) \leq c_2 g(\lambda)$ for two positive constants $c_1, c_2$ and $\lambda \in \mathfrak{a}^*$, $|\lambda|$ large.

As $\widehat{f}(\lambda)_{\delta,j}$ is holomorphic, hence continuous, this immediately implies our assertion, that is, we get

$$(3.10)\qquad \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\mathcal{A}(f_{\delta,j})(t)||\widehat{f}(\lambda)_{\delta,j}|e^{|\lambda||t|}}{(1+|t|+|\lambda|)^d}\, dtd\lambda < \infty.$$

**Step 5:** In this step we will deduce that $\widehat{f}(\lambda)_{\delta,j} = P(\lambda)e^{-\alpha\lambda^2}$, where $P$ is a polynomial which depends on $\delta$, $j$ and $\alpha$ is a positive constant, which is independent of $\delta$, $j$.

We claim that $\mathcal{A}(f_{\delta,j}) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Recall that $f \in L^1(X) \cap L^2(X)$. Therefore by Plancherel theorem $\int_{\mathbb{R}} \|\widehat{f}(\lambda)\|_2^2 \mu(\lambda) d\lambda < \infty$ and in particular $\widehat{f}(\lambda)_{\delta,j}$ is in $L^2(\mathbb{R}, \mu(\lambda)d\lambda)$. On the other hand, $f$ being an $L^1$ function, $\widehat{f}(\lambda)_{\delta,j}$ is a continuous function and hence locally integrable. Now since the Plancherel density $\mu(\lambda)$ tends to $\infty$ as $|\lambda|$ tends to $\infty$, $\widehat{f}(\lambda)_{\delta,j} \in L^2(\mathbb{R})$. But $\widehat{f}(\lambda)_{\delta,j}$ is also the Euclidean Fourier transform of $\mathcal{A}(f_{\delta,j})$ (see (2.8)). Therefore, we can use the Euclidean Plancherel theorem to conclude that $\mathcal{A}(f_{\delta,j})$ is in $L^2(\mathbb{R})$. Again since $f$ is in $L^1(X)$, so is $f_{\delta,j}$, which is a function of type $\delta$ on $X$. We have shown (following Definition 2.3) that $\mathcal{A}(f_{\delta,j})$ is in $L^1(A) = L^1(\mathbb{R})$. Thus the claim is established.

In view of (2.8) we can apply Theorem 1.2 to obtain $\widehat{f}(\lambda)_{\delta,j} = P(\lambda)e^{-\alpha\lambda^2}$. *A priori* the polynomial $P$ as well as the constant $\alpha$ depend on $\delta, j$. We will see that the constant $\alpha$ is actually independent of $\delta, j$.

Suppose for $\delta_1, \delta_2 \in \widehat{K}_0$ and $1 \le j_1 \le d(\delta_1)$, $1 \le j_2 \le d(\delta_2)$,

$$\widehat{f}(\lambda)_{\delta_1,j_1} = P_1(\lambda)e^{-\alpha_1\lambda^2}, \tag{3.11}$$

$$\widehat{f}(\lambda)_{\delta_2,j_2} = P_2(\lambda)e^{-\alpha_2\lambda^2}, \tag{3.12}$$

where $P_1, P_2$ are two polynomials, $\alpha_1, \alpha_2$ are positive constants. Suppose $\alpha_1 \neq \alpha_2$. Without loss of generality we can assume that $\alpha_1 < \alpha_2$. From (3.12) above we have

$$\mathcal{A}(f_{\delta_2,j_2})(t) = P_2(t)e^{-\frac{1}{4\alpha_2}t^2}. \tag{3.13}$$

Substituting (3.11) and (3.13) in (3.7) we see that the integrand in (3.7) is

$$\frac{|P_1(\lambda)||P_2(t)|e^{-(\sqrt{\alpha_1}|\lambda| - \frac{1}{2\sqrt{\alpha_2}}|t|)^2} e^{A|\lambda||t|}}{(1+|t|+|\lambda|)^d},$$

where $A = 1 - \sqrt{\alpha_1/\alpha_2} > 0$ as $\alpha_1/\alpha_2 < 1$. Therefore the integrand in (3.7) grows very rapidly in the neighbourhood of the hyperplane (pair of straight lines) $\sqrt{\alpha_1}|\lambda| = \frac{1}{2\sqrt{\alpha_2}}|t|$ and the integral diverges. This establishes that the positive constant $\alpha$ is independent of $\delta$ and $j$.

**Step 6:** This is our final step wherein we conclude the proof of the theorem. From the previous step we know that $\widehat{f}(\lambda)_{\delta,j} = P(\lambda)e^{-\alpha\lambda^2}$. From (2.5′) we get

$$\frac{Q_\delta(\lambda)}{Q_\delta(-\lambda)} = \frac{P(-\lambda)}{P(\lambda)}.$$

Since $Q_\delta(\lambda)$ and $Q_\delta(-\lambda)$ are relatively prime (see §2), $Q_\delta(-\lambda)$ divides $P(\lambda)$ and hence $P(\lambda) = P'(\lambda^2)Q_\delta(-\lambda)$ where $P'(\lambda^2)$ is a polynomial in $\lambda^2$. As $\deg Q_\delta = s$ we see that $\deg P(\lambda) \ge s$.

On the other hand, noting that $\mathcal{A}(f_{\delta,j})(t) = P(t)e^{-\frac{1}{4\alpha}t^2}$, substituting $\widehat{f}(\lambda)_{\delta,j}$ and $\mathcal{A}(f_{\delta,j})(t)$ back in (3.8) and using (3.9), it is easy to verify that $\deg P < \frac{d'-1}{2}$ where $d' = d - (m_\gamma + m_{2\gamma})$ as otherwise the integral in (3.8) diverges.

Therefore if $s \geq \frac{d'-1}{2}$, then $f_{\delta,j} = 0$ almost everywhere. As $s \geq r$ and $s, r \geq 0$, we conclude that only for finitely many $\delta \in \widehat{K}_0$ which corresponds to the pair of integers $(r, s)$ with $s < \frac{d'-1}{2}$, $f_{\delta,j}$ will satisfy (3.8). We thus conclude that $f$ is a $K$-finite function of the form described in the statement of the theorem.

In particular if $d' \leq 1$, that is, if $d \leq 1 + m_\gamma + m_{2\gamma} = d_X$, then there is no $s$ satisfying $s < \frac{d'-1}{2}$ and hence in that case $f = 0$ almost everywhere. ■

***Remark 3.3*** If in the above theorem we add the condition that $f$ is a $K$-biinvariant function, then the theorem concludes that the spherical Fourier transform of $f$, *i.e.,* $\int_X f(x)\phi_{-\lambda}(x)\,dx = P'(\lambda^2)e^{-\alpha\lambda^2}$ for some polynomial $P$ and a positive constant $\alpha$. From this it follows that $f$ is a derivative of the heat-kernel $h_\alpha$.

## 4 Sharpness of the Estimate

In order to complete the picture we investigate the optimality of the condition used in Theorem 3.1. More precisely, suppose a function $f \in L^1(X) \cap L^2(X)$ satisfies

$$\int_X \int_{\mathfrak{a}^*} \frac{|f(x)| \|\widehat{f}(\lambda)\|_2 e^{c\sigma(x)|\lambda|}\Xi(x)^{1-\varepsilon}}{(1+\sigma(x)+|\lambda|)^d}\, dx\mu(\lambda)d\lambda < \infty \tag{4.1}$$

for some nonnegative integer $d$ and $c, \varepsilon \in \mathbb{R}$.

(i) We will see that if $\{c > 1 \text{ and } \varepsilon \geq 0\}$ or if $\{c \geq 1 \text{ and } \varepsilon > 0\}$ in (4.1), then $f = 0$ almost everywhere.

(ii) We will find a symmetric space $X$ on which there can be infinitely many linearly independent functions in $L^1(X) \cap L^2(X)$ satisfying the estimate (4.1) with $\{c < 1 \text{ and } \varepsilon \leq 0\}$ and with $\{c \leq 1 \text{ and } \varepsilon < 0\}$. These functions are not of the form characterized in Theorem 3.1.

In case (i) as $c > 1$ and $\Xi^{-\varepsilon} \geq 1$, $f$ satisfies the condition (3.1) in Theorem 3.1 and hence $\widehat{f}(\lambda)_{\delta,j} = P_{\delta,j}(\lambda)e^{-\alpha\lambda^2}$. Therefore $\mathcal{A}(f_{\delta,j})(t) = P_{\delta,j}(t)e^{-\beta t^2}$ where $\alpha\beta = \frac{1}{4}$, since $\mathcal{A}(f_{\delta,j})$ is the Euclidean Fourier inverse of $\widehat{f}(\cdot)_{\delta,j}$.

On the other hand starting from the condition (4.1) and following the steps of the proof of Theorem 3.1 we obtain finally,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\mathcal{A}(f_{\delta,j})(t)||\widehat{f}(\lambda)_{\delta,j}|e^{c|\lambda||t|}e^{\varepsilon\rho t}}{(1+|t|+|\lambda|)^d}\, dtd\lambda < \infty. \tag{4.2}$$

Substituting $\mathcal{A}(f_{\delta,j})(t)$ and $\widehat{f}_{\delta,j}(\lambda)$ as obtained above in this inequality we see that it demands

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-(\sqrt{\alpha}|\lambda|-\sqrt{\beta}|t|)^2}e^{(c-1)|\lambda||t|}e^{\varepsilon\rho t}}{(1+|t|+|\lambda|)^d}\, dtd\lambda < \infty.$$

But around the hyperplane $\sqrt{\alpha}|\lambda| = \sqrt{\beta}|t|$, the integrand grows rapidly as $|t| \to \infty$, since $c-1 > 0$ or $\varepsilon > 0$. Hence the integral becomes infinite, which contradicts (4.2).

Next we consider the case (ii), that is, we will find a symmetric space $X$ and functions $f$ on $X$ which satisfy

$$\int_X \int_{\mathfrak{a}^*} \frac{|f(x)| \|\widehat{f}(\lambda)\|_2 e^{c\sigma(x)|\lambda|} \Xi(x)^{1+\varepsilon'}}{(1+\sigma(x)+|\lambda|)^d} \, dx \mu(\lambda) d\lambda < \infty \tag{4.3}$$

for some nonnegative integer $d$, and either $c < 1, \varepsilon' \geq 0$ or $c \leq 1, \varepsilon' > 0$.

Let $G = SL(2, \mathbb{C})$ considered as a real Lie group and $K = SU(2)$. Consider the symmetric space $X = SL(2, \mathbb{C}) / SU(2)$. Let

$$A = \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Then $\phi_\lambda(a_t) = \frac{\sin(\lambda t)}{\lambda \sinh t}$, and the Plancherel measure is $\mu(\lambda) = \lambda^2$ [12, p. 432]. We define a $K$-biinvariant function $g$ on $X$ by prescribing its spherical Fourier transform $\widehat{g}(\lambda) = \int_G g(x) \phi_{-\lambda}(x) \, dx = \mathcal{F}(\psi)(\lambda) e^{-\lambda^2/4} P(\lambda)$ for $\lambda \in \mathbb{R}$, where $\psi$ is an even function in $C_c^\infty(\mathbb{R})$ with support $[-\zeta, \zeta]$ for some $\zeta > 0$, $\mathcal{F}(\psi)$ is its Euclidean Fourier transform, and $P$ is an even polynomial in $\mathbb{R}$. This means that $g$ is the convolution (in $G$) of a smooth compactly supported $K$-biinvariant function on $G$ with a (invariant) derivative of the heat kernel of $X$. Indeed it is clear from the Paley–Wiener theorem that $\mathcal{F}(\psi)$ is also the spherical Fourier transform of a $K$-biinvariant smooth function on $G$ supported in a ball of radius $\zeta$.

It follows that $g$ is a $K$-biinvariant function of the $L^2$-Schwartz space of $G$. By the inversion formula for the spherical Fourier transform, we have

$$\begin{aligned} g(a_t) &= C \int_{\mathbb{R}} \widehat{g}(\lambda) \phi_\lambda(a_t) \mu(\lambda) \, d\lambda \\ &= \frac{C}{\sinh t} \int_{\mathbb{R}} \mathcal{F}(\psi)(\lambda) e^{-\frac{\lambda^2}{4}} \lambda P(\lambda) \sin \lambda t \, d\lambda. \end{aligned}$$

Using Fourier inversion on $\mathbb{R}$, we see that $g(a_t) = \frac{C}{\sinh t}(\psi_1 *_{\mathbb{R}} h)(t)$, where $*_{\mathbb{R}}$ is the convolution in $\mathbb{R}$, $\psi_1$ is a derivative of $\psi$ and hence a function in $C_c^\infty(\mathbb{R})$ with support contained in $[-\zeta, \zeta]$, $h(t) = e^{-t^2}$. An easy computation shows that

$$|g(a_t)| \leq C e^{-t^2} e^{4\zeta t - t} \leq C e^{-\sigma(a_t)^2} \Xi(a_t)^{1-4\zeta}.$$

If we choose $\zeta > 0$ such that $l = 1 - 4\zeta > 0$, then we see that the function $g$ on $X$ satisfies

$$|g(x)| \leq C e^{-\sigma(x)^2} \Xi^l(x) (1+\sigma(x))^M \text{ for all } x \in X$$

for $M > 0$ and $l \in (0, 1)$, and its spherical Fourier transform $\widehat{g}$ satisfies

$$|\widehat{g}(\lambda)| \leq C' e^{-\frac{\lambda^2}{4}} (1+|\lambda|)^N \text{ for all } \lambda \in \mathbb{R}$$

for some $N > 0$.

Thus we can find a function $g$ on $X$ which satisfies the above estimate for any given $l \in (0, 1)$. Suppose $\varepsilon' > 0$. We choose $l$ (that is, choose $\zeta$) so that $l + \varepsilon' \geq 1$. Then it is easy to verify that $g$ satisfies the estimate (4.3) with $c \leq 1$ for any suitable large $d$.

Now suppose $c < 1$ and $\varepsilon' \geq 0$. If $c \leq 0$, the above function $g$ clearly satisfies (4.3). We need only therefore consider the case when $0 < c < 1$. Notice that we can choose $\alpha, \beta \in \mathbb{R}^+$, $\alpha < 1$ and $\beta < \frac{1}{4}$ satisfying the constraint $4\alpha\beta = c^2$ such that the above function $g$ and its spherical Fourier transform $\widehat{g}$ satisfy

$$|g(x)| \leq Ce^{-\alpha\sigma(x)^2}\Xi(x) \qquad \text{for all } x \in X,$$

$$|\widehat{g}(\lambda)| \leq C'e^{-\beta\lambda^2} \qquad \text{for all } \lambda \in \mathbb{R}.$$

Clearly the pair $(g, \widehat{g})$ satisfy (4.3).

From the construction of $g$ it is clear that there are infinitely many linearly independent functions satisfying the estimate in case (ii). This example is a modification of the example given in [23].

## 5 Consequences of Beurling's Theorem

In this section we will justify our claim made in the introduction that this extension of the Beurling–Hörmander theorem is the "master theorem", that is, all other theorems of this genre follow from Theorem 3.1. All the theorems in this section also characterize the heat-kernel described in Section 2. First we consider the Gelfand–Shilov theorem.

***Theorem 5.1*** (Gelfand–Shilov) *Let $f \in L^2(X)$. Suppose $f$ satisfies*

$$\int_X \frac{|f(x)|e^{\frac{(\alpha\sigma(x))^p}{p}}\Xi(x)}{(1+\sigma(x))^N}\,dx < \infty, \tag{5.1}$$

$$\int_{\mathfrak{a}^*} \frac{\|\widehat{f}(\lambda)\|_2, e^{\frac{(\beta|\lambda|)^q}{q}}}{(1+|\lambda|)^N}\mu(\lambda)\,d\lambda < \infty, \tag{5.2}$$

*where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $N$ is a nonnegative integer.*

(i) *If $\alpha\beta > 1$, then $f = 0$ almost everywhere.*
(ii) *If $\alpha\beta = 1$ and $p \neq 2$ (and hence $q \neq 2$), then $f = 0$ almost everywhere.*
(iii) *If $\alpha\beta = 1$, $p = q = 2$ and $N < d_X + 1$, then $f = 0$ almost everywhere*
(iv) *If $\alpha\beta = 1$, $p = q = 2$ and $N \geq d_X + 1$, then $f$ is a $K$-finite function of the form described in Theorem 3.1. In particular if $N = d_X + 1$, then $f$ is a constant multiple of the heat kernel $h_t$ for some $t > 0$.*

**Proof** (i) Since $\frac{\alpha^p}{p}\sigma(x)^p + \frac{\beta^q}{q}|\lambda|^q \geq \alpha\beta\sigma(x)|\lambda|$ and $(1+\sigma(x)+|\lambda|)^{2N} \geq (1+\sigma(x))^N(1+|\lambda|)^N$, from the assumptions (5.1) and (5.2), we obtain:

$$\int_X\int_{\mathfrak{a}^*} \frac{|f(x)|\|\widehat{f}(\lambda)\|_2 e^{\alpha\beta\sigma(x)|\lambda|}\Xi(x)}{(1+\sigma(x)+|\lambda|)^{2N}}\,dx\mu(\lambda)d\lambda < \infty. \tag{5.3}$$

https://doi.org/10.4153/CMB-2007-029-6 Published online by Cambridge University Press

But as $\alpha\beta > 1$, we conclude that $f = 0$ almost everywhere (see §4).

(ii) Fix a $\delta \in \widehat{K}_0$ and an integer $j$ such that $1 \leq j \leq d(\delta)$. We will show that $f_{\delta,j} = 0$ almost everywhere. Note that conditions (5.1) and (5.2) of the theorem can be reduced, respectively, to

$$\int_X \frac{|f_{\delta,j}(x)|e^{\frac{(\alpha\sigma(x))^p}{p}}\Xi(x)}{(1+\sigma(x))^N}\,dx < \infty, \tag{5.4}$$

$$\int_{\mathfrak{a}^*} \frac{|\widehat{f}(\lambda)_{\delta,j}|e^{\frac{(\beta|\lambda|)^q}{q}}}{(1+|\lambda|)^N}\mu(\lambda)\,d\lambda < \infty. \tag{5.5}$$

Therefore we can confine ourselves to the $(\delta, j)$-th component of the function. Using $\alpha\beta = 1$, we can argue as in (i) and show that

$$\int_X\int_{\mathfrak{a}^*} \frac{|f_{\delta,j}(x)|\widehat{f}(\lambda)_{\delta,j}|e^{\sigma(x)|\lambda|}\Xi(x)}{(1+\sigma(x)+|\lambda|)^{2N}}\,dx\mu(\lambda)d\lambda < \infty, \tag{5.6}$$

and thereby conclude from Theorem 3.1 that $\widehat{f}(\,\cdot\,)_{\delta,j}$ is either identically zero or of the form $P_{\delta,j}(\lambda)e^{-\beta_0\lambda^2}$ for some $\beta_0 > 0$.

Now, if we consider the case when $1 < p < 2$, then we see that unless $\widehat{f}(\lambda)_{\delta,j} = 0$ for almost every $\lambda$, it cannot satisfy (5.5) because $q > 2$.

Next we take up the case when $p > 2$ and hence $1 < q < 2$. Since $\mu(\lambda)$ has polynomial growth (3.9), $\widehat{f}(\lambda)_{\delta,j} = P_{\delta,j}e^{-\beta_0\lambda^2}$ satisfies

$$\int_{\mathfrak{a}^*} \frac{|\widehat{f}(\lambda)_{\delta,j}|e^{\frac{(\gamma_0|\lambda|)^2}{2}}}{(1+|\lambda|)^M}\mu(\lambda)\,d\lambda < \infty,$$

where $\gamma_0 = \sqrt{2\beta_0}$, for some suitable $M > 0$. We choose $\alpha_0$ such that $\alpha_0\gamma_0 > 1$. Since $p > 2$ and $f_{\delta,j} \in L^1(X)$, we see from (5.4) that

$$\int_X \frac{|f_{\delta,j}(x)|e^{\frac{(\alpha_{\delta,j}\sigma(x))^2}{2}}\Xi(x)}{(1+\sigma(x))^N}\,dx < \infty.$$

But then from (i) it follows that $f_{\delta,j} = 0$ almost everywhere.

(iii)–(iv) By the above argument, $\widehat{f}(\lambda)_{\delta,j} = P_{\delta,j}(\lambda)e^{-\beta_0\lambda^2}$. It follows from (5.5) with $q = 2$ that $\sqrt{2\beta_0} \geq \beta$. But if $2\beta_0 > \beta^2$, then $\alpha\sqrt{2\beta_0} > 1$. On the other hand, $\widehat{f}_{\delta,j}$ satisfies (5.5) with $q = 2$ and with $\beta$ replaced by $\sqrt{2\beta_0}$ for a suitably large $N$. Therefore by (i), $f_{\delta,j} = 0$ almost everywhere. Hence $\widehat{f}_{\delta,j}(\lambda) = P_{\delta,j}(\lambda)e^{-\beta^2\lambda^2/2}$. Now, as noted in the proof of Theorem 3.1, $Q_\delta(-\lambda)$ is a factor of $P_{\delta,j}(\lambda)$, and hence $\deg P_{\delta,j} \geq \deg Q_\delta = s$. Therefore, only for finitely many $\delta \in \widehat{K}_0$ can $\widehat{f}_{\delta,j}$ satisfy (5.5). This proves the first statement in (iv). Substituting $\widehat{f}_{\delta,j}$ back in (5.5) and using (3.9) it is now easy to verify that if $N < 2 + m_\gamma + m_{2\gamma} = d_X + 1$, then $\widehat{f}_{\delta,j} \equiv 0$, and if $N = 2 + m_\gamma + m_{2\gamma} = d_X + 1$, then $\deg P_{\delta,j} = 0$ and hence $\widehat{f}_{\delta,j}(\lambda) = Ce^{-\beta^2\lambda^2/2}$.

But that is possible only when $\delta$ is trivial. Indeed from (2.3) it follows that $\Phi^j_{i\rho,\delta} \equiv 0$ when $\delta \in \widehat{K}_0$ is nontrivial and $1 \leq j \leq d(\delta)$. Hence for such a $\delta$, $\widehat{f}_{\delta,j}(i\rho) = 0$, which is not possible if $\widehat{f}_{\delta,j}(\lambda) = Ce^{-\beta^2\lambda^2/2}$. Thus $f$ is a constant multiple of the heat kernel $h_t$ where $t = \frac{\beta^2}{2}$. ∎

We will see below that the theorems of Morgan, Hardy and Cowling–Price follow from the Gelfand–Shilov theorem proved above.

***Theorem 5.2*** (Morgan's theorem) *Let $f \colon X \to \mathbb{C}$ be measurable and assume that*

$$|f(x)| \leq C_1 e^{-a\sigma(x)^p} \Xi(x)(1+\sigma(x))^n \quad \text{for all } x \in X, \tag{5.7}$$

$$\|\widehat{f}(\lambda)\|_2 \leq C_2 e^{-b|\lambda|^q} \quad \text{for all } \lambda \in \mathfrak{a}^* \equiv \mathbb{R}, \tag{5.8}$$

*where $C_1, C_2$ and $a, b$ are positive constants, $n$ is a nonnegative integer, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $(ap)^{\frac{1}{p}}(bq)^{\frac{1}{q}} > 1$, then $f = 0$ almost everywhere.*
(ii) *If $(ap)^{\frac{1}{p}}(bq)^{\frac{1}{q}} = 1$ and $p \neq 2$, then $f = 0$ almost everywhere.*
(iii) *If $p = q = 2$ and $(ap)^{\frac{1}{p}}(bq)^{\frac{1}{q}} = 1$, that is $ab = \frac{1}{4}$, then $f$ is a constant multiple of the heat kernel.*

**Proof** Let $a = \frac{\alpha^p}{p}$ and $b = \frac{\beta^q}{q}$. Then $f$ and $\widehat{f}$ satisfies Theorem 5.1 for some suitable $N$. The condition $(ap)^{1/p}(bq)^{1/q} \geq 1$ translates as $\alpha\beta \geq 1$. Thus (i) and (ii) follow from Theorem 5.1(i) and (ii). For (iii) again we use the proof of Theorem 5.1(iii)–(iv) to conclude that $\widehat{f}_{\delta,j}(\lambda) = P_{\delta,j}(\lambda)e^{-b\lambda^2}$. But because of the condition (5.8) of this theorem, $P_{\delta,j}$ is a constant. But this implies that only for trivial $\delta = \delta_0$, can $\widehat{f}_{\delta,j}$ be nonzero, and the spherical Fourier transform of $f$ is $Ce^{-b\lambda^2}$ (see the argument at the end of the proof of Theorem 5.1, (iii)–(iv)). That is, $f$ is a constant multiple of the heat kernel at $t = b$. ∎

Morgan's theorem implies the well-known Hardy's theorem as a particular case $(p = q = 2)$. To stress this point we will write it as a separate theorem.

***Theorem 5.3*** (Hardy's theorem) *Let $f \colon X \to \mathbb{C}$ be measurable and assume that,*

$$|f(x)| \leq C_1 e^{-a\sigma(x)^2} \Xi(x)(1+\sigma(x))^n \quad \text{for all } x \in X, \tag{5.9}$$

$$\|\widehat{f}(\lambda)\|_2 \leq C_2 e^{-b|\lambda|^2} \quad \text{for all } \lambda \in \mathfrak{a}^* \equiv \mathbb{R}, \tag{5.10}$$

*where $C_1, C_2$ and $a, b$ are positive constants, $n$ is a nonnegative integer.*

(i) *If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.*
(ii) *If $ab = \frac{1}{4}$, then $f$ is a constant multiple of the heat kernel.*

***Theorem 5.4*** (Cowling–Price) *Let $f\colon X \to \mathbb{C}$ be measurable and assume that for positive constants $a, b$ and nonnegative integers $m, n$,*

$$\int_X \frac{(|f(x)|e^{a\sigma(x)^2}\Xi(x)^{\frac{2}{p_1}-1})^{p_1}}{(1+\sigma(x))^m}\,dx < \infty \tag{5.11}$$

$$\int_{\mathfrak{a}^*} \frac{(\|\widehat{f}(\lambda)\|_2 e^{b|\lambda|^2})^{p_2}}{(1+|\lambda|)^n}\mu(\lambda)\,d\lambda < \infty, \tag{5.12}$$

*where $1 \leq p_1, p_2 < \infty$,*

(i) *If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.*
(ii) *If $ab = \frac{1}{4}$, then $f$ is a $K$-finite function of the form described in Theorem* 3.1. *In particular, if $d_X < n \leq d_X + p_2$, then $f$ is a constant multiple of the heat kernel.*

**Proof** Let us first assume $p_1$ and $p_2$ are greater than 1. Let $q_1$ and $q_2$ be respectively the conjugates of $p_1$ and $p_2$, that is $\frac{1}{p_i} + \frac{1}{q_i} = 1, i = 1, 2$. Using the estimate of $\Xi(x)$ given in Section 2, we note that

$$\frac{\Xi(x)^{\frac{2}{q_1}}}{(1+\sigma(x))^{\frac{m'}{q_1}}}$$

is in $L^{q_1}(X)$ if $m' > 3$. Therefore it follows from condition (5.11) in the hypothesis that

$$\int_X \frac{|f(x)|e^{a\sigma(x)^2}\Xi(x)^{\frac{2}{p_1}-1}}{(1+\sigma(x))^{\frac{m}{p_1}}} \frac{\Xi(x)^{\frac{2}{q_1}}}{(1+\sigma(x))^{\frac{m'}{q_1}}}\,dx = \int_X \frac{|f(x)|e^{a\sigma(x)^2}\Xi(x)}{(1+\sigma(x))^{N_1}}\,dx < \infty,$$

where $N_1 = \frac{m}{p_1} + \frac{m'}{q_1}$. Similarly using (3.9), we see that if $n' > 1 + m_\gamma + m_{2\gamma}$, then

$$\int_{\mathfrak{a}^*} \frac{|\widehat{f}(\lambda)e^{b|\lambda|^2}}{(1+|\lambda|)^{N_2}}\mu(\lambda)\,d\lambda < \infty$$

where $N_2 = \frac{n}{p_2} + \frac{n'}{q_2}$.

When either $p = 1$ or $p_2 = 1$, then the above two inequalities are evident. Thus this becomes a particular case of Theorem 5.1 when $p = q = 2$, $N = \max\{N_1, N_2\}$ and $a = \frac{\alpha^2}{2}, b = \frac{\beta^2}{2}$. Note that the conditions $ab > \frac{1}{4}$ and $ab = \frac{1}{4}$ in the hypothesis translate as $\alpha\beta > 1$ and $\alpha\beta = 1$ respectively, when we fit them in Theorem 5.1. The result now follows from Theorem 5.1(i), (iii) and (iv) in a fashion similar to what was used in the previous theorems in this section. We omit the details to avoid repetition. ■

In the above theorem, we may take either $p_1$ or $p_2$ or both to be infinity. The condition (5.11) with $p_1 = \infty$ means that $g(x) = |f(x)|e^{a\sigma(x)^2}\Xi(x)^{-1}(1+\sigma(x)^{-m}$ is

https://doi.org/10.4153/CMB-2007-029-6 Published online by Cambridge University Press

a bounded function on $X$ for $m$ as above. Hence $g(x)\Xi(x)^2/(1+\sigma(x))^{N_1}$ is integrable where $N_1 = m' > 3$ as described above. That is, as above,

$$\int_X \frac{|f(x)|e^{a\sigma(x)^2}\Xi(x)}{(1+\sigma(x))^{N_1}}\,dx < \infty.$$

Similarly for $p_2 = \infty$ we arrive at

$$\int_{\mathfrak{a}^*} \frac{|\widehat{f}(\lambda)|e^{b|\lambda|^2}}{(1+|\lambda|)^{N_2}}\mu(\lambda)\,d\lambda < \infty$$

for $N_2 = n' > 1 + m_\gamma + m_{2\gamma}$, since $|\widehat{f}(\lambda)|e^{b|\lambda|^2}$ is bounded on $\mathfrak{a}^*$. Note that the case $p_1 = p_2 = \infty$ of the Cowling–Price theorem implies Hardy's theorem.

Some parts of these theorems were proved independently on symmetric spaces. Part (i) of Hardy's theorem was proved in [4, 6, 23], while part (ii) was proved in [19, 25]. Part (i) of Cowling–Price theorem was proved in [20, 22] and part (ii) was proved in [21]. Part (i) of Morgan's theorem was proved in [22].

## 6 Concluding Remarks

In his thesis, Demange [5] further generalized Theorem 1.2:

***Theorem 6.1*** (Demange) *For two nonzero functions $f_1, f_2 \in L^2(\mathbb{R})$, if*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_1(x)||\widehat{f_2}(\lambda)|e^{|x||\lambda|}}{(1+|x|+|\lambda|)^d}\,dxd\lambda < \infty,$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_2(x)||\widehat{f_1}(\lambda)|e^{|x||\lambda|}}{(1+|x|+|\lambda|)^d}\,dxd\lambda < \infty,$$

*then $f_1(x) = P_1(x)e^{-\alpha x^2}$ and $f_2(x) = P_2(x)e^{-\alpha x^2}$ for some positive constant $\alpha$ and polynomials $P_1, P_2$.*

A careful reader will observe that using our technique, this theorem can be extended to symmetric spaces, with the following interesting consequence:

Consider two rank 1 symmetric spaces $X_1 = G_1/K_1$ and $X_2 = G_2/K_2$. Let $dx$ and $dy$ be the $G_1$ and $G_2$ invariant measures on $X_1$ and $X_2$ respectively. Let $\mu_i(\lambda)d\lambda$ be the corresponding Plancherel measures for $X_i$ and let $\sigma_i, \Xi_i$ be the $\sigma$ and $\Xi$ functions on $X_i$, for $i = 1, 2$. Let $f_1 \in L^2(X_1)$ and $f_2 \in L^2(X_2)$ be two nonzero functions.

***Theorem 6.2*** *Let $f_1$ and $f_2$ as above satisfy*

$$\int_{X_1}\int_{\mathfrak{a}_2^*} \frac{|f_1(x)|\|\widehat{f_2}(\nu)\|_2 e^{\sigma_1(x)|\nu|}\Xi_1(x)}{(1+\sigma_1(x)+|\nu|)^d}\,dx\mu_2(\nu)d\nu < \infty,$$
$$\int_{X_2}\int_{\mathfrak{a}_1^*} \frac{|f_2(y)|\|\widehat{f_1}(\lambda)\|_2 e^{\sigma_2(y)|\lambda|}\Xi_2(y)}{(1+\sigma_2(y)+|\lambda|)^d}\,dy\mu_1(\lambda)d\lambda < \infty.$$

https://doi.org/10.4153/CMB-2007-029-6 Published online by Cambridge University Press

*Then $f_1$ is a derivative of the heat kernel $h^1_\alpha$ of $X_1$ and $f_2$ is a derivative of the heat kernel $h^2_\alpha$ of $X_2$ for some instant $\alpha > 0$.*

We take $X_1 = X_2 = X$ and obtain the following corollary.

***Corollary 6.3*** *Let two nonzero functions $f_1, f_2 \in L^2(X)$ satisfy*

$$\int_X \int_{\mathfrak{a}^*} \frac{|f_1(x)| \|\widehat{f_2}(\lambda)\|_2 e^{\sigma(x)|\lambda|} \Xi(x)}{(1+\sigma(x)+|\lambda|)^d} \, dx\mu(\lambda) d\lambda < \infty,$$
$$\int_X \int_{\mathfrak{a}^*} \frac{|f_2(x)| \|\widehat{f_1}(\lambda)\|_2 e^{\sigma(x)|\lambda|} \Xi(x)}{(1+\sigma(x)+|\lambda|)^d} \, dx\mu(\lambda) d\lambda < \infty.$$

*Then $f_1$ (respectively, $f_2$) is a derivative of the heat kernel $h_\alpha$ for some instant $\alpha > 0$.*

The proof of this theorem proceeds along entirely similar lines to that of the proof of the main theorem of this article; we therefore omit it.

**Acknowledgement** We thank the referee for making several suggestions which improved the exposition.

## References


[1] J.-P.Anker, *Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces.* Duke Math. J. **65**(1992), no. 2, 257–297.

[2] ———, *A basic inequality for scattering theory on Riemannian symmetric spaces of the noncompact type*, Amer. J. Math. **113**(1991), no. 3, 391–398.

[3] A. Bonami, B. Demange, and P. Jaming, *Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms.* Rev. Mat. Iberoamericana **19**(2003), no. 1, 23–55.

[4] M. Cowling, A. Sitaram, and M. Sundari, *Hardy's uncertainty principle on semisimple groups*. Pacific J. Math. **192**(2000), no. 2, 293–296.

[5] B. Demange, *Principes d'incertitude associés à des formes quadratiques non dgénérés.* Thèse, Université d'Orléans, 2004.

[6] M. Ebata, M. Eguchi, S. Koizumi, and K. Kumahara, *A generalization of the Hardy theorem to semisimple Lie groups*. Proc. Japan Acad. Ser. A Math. Sci. **75**(1999), no. 7, 113–114.

[7] G. B. Folland and A. Sitaram, *The uncertainty principle: a mathematical survey*. J. Fourier Anal. Appl. **3**(1997), no. 3, 207–238.

[8] R. Gangolli and V. S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete 101, Springer-Verlag, Berlin, 1988.

[9] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis.* Ergebnisse der Mathematik und ihrer Grenzgebiete 28, Springer-Verlag, Berlin, 1994.

[10] S. Helgason, *A duality for symmetric spaces with applications to group representations.* Advances in Math. **5**(1970), 1–154.

[11] ———, *Eigenspaces of the Laplacian; integral representations and irreducibility*. J. Functional Analysis **17**(1974), 328–353.

[12] ———, *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*. Pure and Applied Mathematics 113, Academic Press, Orlando, FL, 1984.

[13] ———, *Geometric Analysis on Symmetric Spaces.* Mathematical Surveys and Monographs 39, American Mathematical Society, Providence, RI, 1994.

[14] ———, *The Abel, Fourier and Radon transforms on symmetric spaces.* Indag. Math. **16**(2005), 531–551.

[15] S. Helgason, R. Rawat, J. Sengupta, and A. Sitaram, *Some remarks on the Fourier transform on a symmetric space.* Technical Report January 1998, Indian Statistical Institute, Bangalore.

[16] L. Hörmander, *A uniqueness theorem of Beurling for Fourier transform pairs*. Ark. Mat. **29**(1991), no. 2, 237–240.

[17] B. Kostant, *On the existence and irreducibility of certain series of representations.* Bull. Amer. Math. Soc. **75**(1969), 627–642.
[18] P. Mohanty, S. K. Ray, R. P. Sarkar, and A. Sitaram, *The Helgason-Fourier transform for symmetric spaces. II.* J. Lie Theory **14**(2004), no. 1, 227–242.
[19] E. K. Narayanan and S. K. Ray, *The heat kernel and Hardy's theorem on symmetric spaces of noncompact type*. Proc. Indian Acad. Sci. Math. Sci. **112**(2002), no. 2, 321–330.
[20] ———, *$L^p$ version of Hardy's theorem on semisimple Lie groups*. Proc. Amer. Math. Soc. **130**(2002), no. 6, 1859–1866
[21] S. K. Ray and R. P. Sarkar, *Cowling-Price theorem and characterization of heat kernel on symmetric spaces*. Proc. Indian Acad. Sci. Math. Sci. **114**(2004), no. 2, 159–180.
[22] J. Sengupta, *The uncertainty principle on Riemannian symmetric spaces of the noncompact type*. Proc. Amer. Math. Soc. 130 (2002), no. 4, 1009–1017 (electronic).
[23] A. Sitaram and M. Sundari, *An analogue of Hardy's theorem for very rapidly decreasing functions on semi-simple Lie groups*. Pacific J. Math. **177**(1997), no. 1, 187–200.
[24] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*. Annals of Mathematics Studies 63, Princeton University Press, Princeton, 1970.
[25] S. Thangavelu, *Hardy's theorem for the Helgason Fourier transform on noncompact rank one symmetric spaces*. Colloq. Math. **94**(2002), no. 2, 263–280.

*Stat-Math Unit*
*Indian Statistical Institute*
*203 B. T. Road*
*Calcutta 700108*
*India*
*e-mail: rudra@isical.ac.in*

*School of Mathematics*
*Tata Institute of Fundamental Research*
*Homi Bhabha Rd.*
*Mumbai 400005*
*India*
*e-mail: sengupta@math.tifr.res.in*