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AN ANALYTICAL MODEL OF BOND RISK DIFFERENTIALS: A COMMENT

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In a paper published in the December 1975 issue of this Journal,[1] Bierman and Hass (BH) construct a steam roller for the purpose of cracking a nut.[2] BH's paper is essentially an attempt to use subjective probabilities to set yields on new bond issues. I am concerned primarily with the first two-thirds of their paper (pp. 757-67), which, in my view, contains a number of statements that are seriously misleading. The first section of this comment will briefly summarize those portions of pp. 757-67 of their paper. The second section contains the comment itself, plus a few observations on the final portion of their paper.

I.

(1) Assuming a risk-neutral buyer of debt issues and *given* what they call the "probability of survival" (P),[3] BH show how to obtain the "required" yield on a new risky perpetuity (their equation 4, p. 759). They use a perpetuity in order to avoid, at the outset, the complications created by the fact that (risky) borrowers must also, in every case, make not only interest payments but also payments on principal. They then state as their conclusions to this initial section of their paper that:

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[1]Harold Bierman, Jr. and Jerome E. Hass, "An Analytical Model of Bond Risk Differentials," *Journal of Financial and Quantitative Analysis* (December 1975), pp. 757-773.

[2]Which in fact had already been cracked. See [2, pp. 281-336] and [3, pp. 14-18]. BH state that they are building on Fisher's 1957 article. Fisher does talk briefly about the probability of default but his article runs essentially in terms of risk premiums, not probabilities. BH are actually "building," although they do not realize it, on Macaulay [8] and [2] and [3].

[3]Macaulay called this the probability of payment of interest and principal in full and on time. See [8, p. 61].

(a) If P is 1 ("the interest payments are certain")[4], the "required" rate on the new issue would be the riskless rate (!).

(b) As the "probability of survival" increases, the required rate declines (curve M-M, Figure 1), and vice versa.

(2) They then discuss "the relationship between the (dollar) amount of interest that has to be paid and the probability of survival" (p. 759). Given some probability distribution over EBIT, the probability of survival can be related to the dollar volume of interest payments which, of course, is the product of r (the required rate) and B_o. But BH do not make clear whether B_o is the amount of a new issue or the total amount of debt the issuer has outstanding.

(3) BH conclude this section of the paper by drawing curve Q-Q (Figure III) which they call "the interest rate-survival probability feasible set." The difference between M-M and Q-Q is unclear, except that M-M is the investor's curve and Q-Q, the firm's. Both curves would have to have been derived in much the same way.

(4) BH then suggest that the curve M-M (Figure 1)--which relates P and the lender's required rate of return and seems entirely subjective (a kind of indifference curve)--can be juxtaposed on curve Q-Q to obtain r^* which is defined as "the minimum interest rate required by the market for a fixed amount of debt," B_o.

II.

My comments follow:

(1) What are BH saying, esssntially? They are saying essentially that lenders who are risk-neutral can fix the "appropriate" yield on a new issue by dividing the return on a riskless security by the firm's "probability of survival" (their equation 4). At bottom, this is what the paraphernalia summarized in their Figure 2 means.

But given their definition of P (probability of survival), it (P) cannot be determined by reference to a probability distribution over EBIT, unless B_o is the total amount of debt the firm will have outstanding after the new issue has been sold. In any event, their statement that "from the probability distribution of earnings . . . for all possible levels of interest payments"[5] must assume one state of nature over the life of the bond(s?), for otherwise, we would need families of distributions for each coupon period. The statement

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[4] P. 759.

[5] BH, pp. 759-61.

also assumes that the same distribution would apply to each coupon period. Both assumptions are dubious.

One referee points out that "we normally will not take issue with a theoretical treatment (model) if . . . it predicts or explains," however unrealistic the assumptions on which it rests may be. Quite so. But, he adds, BH offer no empirical evidence whatever in support of their model.

What might they have done? They might, for example, have used their equation 4 (p. 759) to "predict" the yields on a sample of new issues. But this would have required a P for each issue, a P they did not have. To obtain a P they might have used a variation on their equation 4, thus: $P = \frac{1+i}{1+r}$, (see [2]) but then they could not have used P, which would have been derived in part from r, to predict r.

(2) As indicated above, their P is not conceptually clear unless their B is the total amount of bonds the firm will have outstanding after the new bond is issued. But their final paragraph, p. 758, suggests virtually unmistakably, that both *B* and *P* refer to the new bond, not to the total amount of bonds the firm has outstanding.

(3) The derivation of Q-Q (Figures 2 and 3) is far from clear. All one can say is that, given the rate of interest at which the firm wishes to borrow and the amount it wishes to borrow, one can find the total amount of interest (I) $(-r_oB_o)$ the firm would be required to pay (Figure 2, IV). Then, *given* a curve relating I and P (Figures 2 and 3), one can find P $(=P_o)$. One can then relate P_o to r via Figure 2, quadrants II and I $(=r_o)$. r_o then becomes the first point on the "feasible set" Q-Q. Now change r, hold B_o constant, and another "feasible" point can be obtained, and so forth--until the entire curve Q-Q (Figure 3) is developed. *Note that the relationship between P and I cannot be derived without an implicit (or explicit) probability distribution over EBIT.* Note also that the shapes of the curves in quadrants I and III and in Figure 3 are unexplained and are not obvious.

(4) The statement in paragraph (4) of Section I above assumes that every investor would apply the same P to every B_or combination, or in other words that the P's are "objective." Moreover, to the naked eye, the real difference between M-M (Figure 1) and Q-Q (Figure 3) is far from clear: both relate P to r. And their juxtaposition, to obtain an equilibrium point, is possible only because of the way the curves are drawn: M-M is convex throughout while Q-Q fluctuates between being concave and convex. In fact, both should have the same general shape and the same elasticity, because both relate *P* and r and therefore reflect

the same probability distributions over EBIT. There is no technical reason why M-M and Q-Q should intersect or be tangent to one another.

(5) BH then give us a numerical illustration the outcome of which is simply implicit in the assumption that the probability distribution over EBIT is rectangular. Further, BH make the illustration heavier going than need be. If the probability distribution is given, P can be obtained directly from it. If it is .983, r would simply be 1.04/.983 = 110580, or precisely their result. And 1.058 or 5.80 percent would be points on both M-M and Q-Q. But we should note again that the critical input is the distribution over EBIT, *without which no result would be forthcoming*. At worst, no such distribution can be obtained--or at best it would be so subjective, over any reasonable period (15-20 years) as to be virtually worthless.[6]

(6) In the remainder of the paper, BH allow for principal repayment and for revision of the probabilities, conditional upon past survival. By and large, all the comments made above apply but several additional points are worth making.

BH say (p. 768) that "equation 4, in fact, holds for a bond of any . . . maturity, *if the probability of survival is constant over time*."[7] This statement is incorrect. Equation 4 holds whether P is a constant or not--provided only that it is the n^{th} root of whatever the product of the n P's may be--in other words, that it is a geometric average of the n P's. This is very different from saying that P must be a constant. The individual P's can take on any time shape whatsoever.

The proof that P need not be a constant is as follows:

Start with the most general form of the relevant present value equation:

$$(1) \qquad 1 = \frac{r_1 P_1}{(1+g_1)} + \frac{r_2 P_1 P_2}{(1+g_1)\ (1+g_2)} + \frac{(1+r_n) P_1 P_2 \ . \ . \ . \ P_n}{(1+g_1)\ (1+g_2)\ . \ . \ . \ (1+g_n)}$$

where r_1 is the one-period yield on a risky bond sold at par; r_2 is the two-period yield on a risky bond sold at par, etc. $r_1 \neq r_2$, etc.

P_1 is the probability that r_1 will be paid *in full and on time*. P_2 is the probability that r_2 will be paid in full and on time, and so forth $P_1 \neq P_2$, etc.

$(1+g_1)$ is the one-period rate on riskless bonds; $(1+g_2)$ is the two-period rate on riskless bonds, and so forth $1+g_1 \neq 1+g_2$, etc.

[6]As noted above, we would need a separate joint distribution over every possible state of nature *for each coupon period*.

[7]P. 768, underlining supplied.

Now extract geometric averages of the P's, the r's, and the (1+g)'s. Call these averages $\bar{P}$, $\bar{r}$, and $(1+\bar{g})$.

Then rewrite equation (1) as follows:

$$(2) \qquad 1 = \frac{\bar{r}\bar{P}}{(1+\bar{g})} + \frac{\bar{r}\bar{P}^2}{(1+\bar{g})^2} + \ . \ . \ . \ + \frac{(1+\bar{r})\bar{P}^n}{(1+\bar{g})^n} .$$

Now solve (2) for $\bar{P}$:

$$(3) \qquad \bar{P} = \frac{1+\bar{g}}{1+\bar{r}}\text{[8]}$$

$\bar{P}$ is identical to BG's P.

We now know that if we divide the return (1 + yield) on a risky bond into the return on a riskless security *of the same maturity*, we will obtain the *average* probability that each payment will be made in full and on time, *provided* the lender is risk neutral and provided also that the riskless bond and the risky bond are the *same in every other way*--marketability, callability, and so forth.[9] But we know also that if we can, in some way, obtain the probability that payment of interest and principal will be made in full and on time of a risky bond, we can estimate an appropriate return and yield on that risky bond, thus:

$$(4) \qquad \frac{1+g}{P} = 1+r.$$

This equation is identical to BH's equation 4.

(7) BH say also that "the risk differential is *invariant with the life of the bond*" when investors are risk neutral.[10] This statement is also incorrect.

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[8]As proof:

$$1 = \frac{\bar{r}}{(1+\bar{g})} \cdot \frac{(1+\bar{g})}{1+\bar{r}} + \frac{\bar{r}}{(1+\bar{g})^2} \cdot \frac{(1+\bar{g})^2}{(1+\bar{r})^2} + \ . \ . \ . \ + \frac{(1+\bar{r})}{(1+\bar{g})^n} \cdot \frac{(1+\bar{g})^n}{1+\bar{r}^n}$$

$$= \frac{\bar{r}}{(1+\bar{r})} + \frac{\bar{r}}{(1+\bar{r})^2} + \ . \ . \ . \ + \frac{}{(1+\bar{r})^n}$$

$$= 1.$$

[9]BH do not touch at all on these "minor" difficulties.

[10]P. 769, underlining supplied.

As indicated above, r_o is a geometric *average* of the various n period rates from 1 to n, given by the term structure(s) of risky rates. And what falls out of their equation 4 is in fact this average. If the r's are different, then the "risk differentials" would be "constant" only by accident.

(8) BH obtain a probability by assumption, a method which is somewhat less than fully satisfactory. More satisfactory methods are available. One such method is illustrated in [2] and [4]. In [2], an exploratory paper published in 1971, P's for some 3000 direct placements were derived by using equation (3) above. Each of the 3000 P's was adjusted for callability and for demand and supply conditions in the market for lower grade risky debt issues.[11] In (4), these P's so adjusted were regressed, in various ways, on the variables life insurance companies take into account in fixing the yield on a risky issue--two of which, by the way, were (a) a five-year average, adjusted for trend, of EBIT and (b) the coefficient of variation of that average. In any event, the result was a regression equation which could be used to predict P. And as we have seen, given P and the rate on a (comparable) riskless security of the same maturity, an estimate of the yield on some given risky security can be obtained.[12]

(9) The BH model, as with so many other models being constructed these days in finance, requires inputs which are simply not available, and except in the most tenuous way, cannot be constructed. The BH model requires a probability distribution over EBIT. BH imply that they mean one distribution. But what they really need would be a joint distribution over all possible states of nature *separately for each coupon period*.[13] To put the requirement in these terms is, obviously, to suggest that it cannot be met.

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[11]See [2], pp. 309-322.

[12]The conceptual basis of equation (4) above, was discussed in [3, pp. 14-17].

[13]See [3, p. 14].

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