THE EXPONENT OF A PRIMITIVE MATRIX*

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1. Introduction. Let A be an n by n matrix with non-negative entries. If a permutation matrix P exists such that $P^{-1} A P$ is of the form

$$P^{-1} A P = \begin{pmatrix} A & O \\ B & C \end{pmatrix}$$

where A and C are square matrices and O is a zero-matrix then A is said to be reducible. Otherwise, A is irreducible. If A is irreducible, then A is said to be primitive if there is an integer k such that $A^k > 0$, i.e., A^k has no zero entries. If A is primitive, the least integer m for which $A^m > 0$ is called the exponent of A. In (4), Wielandt has shown that for n by n primitive matrices the exponent is at most $(n-1)^2 + 1$. In this paper, the theory of directed graphs is used to determine conditions under which the exponent is less than $(n-1)^2 + 1$. The following results are obtained. If A contains r non-zero entries along its main diagonal then its exponent is at most $2n - r - 1$, and this result is the best possible. If all the diagonal entries of A are zero but its graph K_A (see next section for definition) contains a cycle of length d, then the exponent of A is at most $d(2n-d-1)$. For the case where $d < \frac{n}{2}$ this improves Wielandt's result.

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2. The graph of a matrix. A finite directed graph K consists of a set of n vertices denoted by the integers 1, 2, ..., n together with a set of edges. Each edge is an ordered pair (i, j) of vertices. If (i, i) is an edge it is called a loop, and i is called a loop vertex. A path of length k from vertex i to vertex j is a set of k edges of the graph (i, p_1), (p_1, p_2), (p_2, p_3), ... (p_{k-1}, j). This is sometimes denoted by $i \rightarrow p_1 \rightarrow p_2 \ldots \rightarrow j$. If i = j the path is called a cycle.

A path from i to j is called non-repetitive if each vertex which appears in an edge of the path, is the first member of at most one edge. If there is a path from i to j then there is a non-repetitive path. For, if in a path from i to j, a vertex appears twice as the first member of an edge, then by deleting all edges between the first and second appearance of this vertex the remaining set of edges still form a path. A non-repetitive cycle is called a circuit. The maximum length of a circuit is n while the maximum length of a non-repetitive path from i to j, where $i \neq j$, is n - 1. A graph K is said to be strongly connected if for every pair of vertices i and j, $i \neq j$, there is a path from i to j. In particular, if K is strongly connected and contains at least two vertices, then every vertex appears in at least one circuit. Let A be an n by n matrix whose entries a_{ij} are non-negative. With A we associate a graph K_A whose vertices are the integers 1, 2, ..., n and whose edges are the pairs (i, j) for which $a_{ij} > 0$. The following are known results:

(1) A is irreducible if and only if K_A is strongly connected (see (1)).

(2) A is primitive, if and only if A is irreducible and the greatest common divisor of the lengths of cycles of the graph K_A is 1 (see (2)).

(3) The entry in the ith row and jth column of A^t is non-zero if and only if K_A contains a path of length t from i to j.

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3. Bound for the exponent.

THEOREM 1. Let A be a non-negative irreducible n by n matrix with $r > 0$ non-zero entries along the main diagonal. Then A is primitive of exponent at most $2n - r - 1$, and this result is the best possible.

Proof. First note that if there is a path of length k from i to j which contains a loop vertex p, there is a path of length t from i to j for all $t \geq k$. This path is obtained by inserting the edge (p, p) into the path $k - t$ times at the appropriate place. Let i and j be any two vertices, not necessarily distinct. Let d be the length of the shortest path joining i to some loop vertex. Then $d \leq n - r$. Indeed, any non-repetitive path of length d contains $d + 1$ vertices and the totality of non loop vertices is $n - r$. If this minimal path is from i to p, choose a non-repetitive path from p to j. This path is of length at most $n - 1$. Combining the two paths, there is a path of length at most $n - r + n - 1$ from i to j, containing a loop vertex. Hence, $A^{2n-r-1} > 0$. That this result is the best possible is seen from the fact that the matrix A with entries $a_{ij} = 1$ if $j \equiv i + 1 \bmod n$, $a_{ii} = 1$ for $i = 1, 2, \ldots, r$ and all other $a_{ij} = 0$, has exponent equal to $2n - r - 1$.

If we put $r = n$ in theorem 1, we obtain a result of Herstein (3), that if A is irreducible and non-negative then $(I + A)^{n-1}$ is positive. This result can be generalised in the following way also. Let A be non-negative and suppose A contains no r by s submatrix of zeros with $r + s = n$. There are at least two non zeros in every column of A and the non existence of the submatrix of zeros ensures that there are at least three non zeros in every column of A^2. In general, for $r \leq n - 1$, each column of A^r contains at least $r + 1$ non-zero entries, so that $A^{n-1} > 0$. Clearly, if A is irreducible then $I + A$ contains no such submatrix of zeros and hence $(I + A)^{n-1} > 0$.

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THEOREM 2. Let A be a primitive n by n matrix whose graph K_A contains a circuit of length d. Then the exponent of A is at most $d(2n-d-1)$.

Proof. Let the vertices which appear in the circuit be $i_1, i_2, \ldots, i_d$. Then A^d has a non-zero entry in the row i_k, column i_k position, for $k = 1, 2, \ldots, d$. By theorem 1, A^d has exponent at most $2n - d - 1$.

This result improves Wielandt's (4) in the case where $d < \frac{n}{2}$ for $n \geq 4$. More generally, if $d < kn$ where $0.5 < k < 1$, the result improves Wielandt's for all

$$n > \frac{2-k+\sqrt{(2-k)^2 - 8(1-k)^2}}{2(1-k)^2}$$

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