SHORT FORMULATIONS OF BOOLEAN ALGEBRA, USING RING OPERATIONS

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SPECIAL interest has recently attached to formulations of Boolean algebra in terms of ring operations [7], [1]. These axiomatizations have not been as brief as those reached through other modes of approach.[1]

The present note will show that the number of axioms when ring operations are used may be as small as in any present version that is not metamathematical,[2] that is, the number of axioms finally employed will be two.[3]

The first step will be to give a set of four postulates of more familiar appearance; these will mark a reduction by about two, as compared with similar versions of earlier date. Then, by employing a device due to Bernstein [2], we shall set up two postulates from which the previous four can be deduced.

FORMULATION IN FOUR TRANSFORMATION AXIOMS

Axioms

I. $(X + Y) + Z = X + (Y + Z)$. II. $(X + Y) + X = Y$.

III. $X(W + YZ) = XW + Y(ZX)$. IV. $X(X + 1) = Y + Y1$.

Theorems[4]

1. $X + Y = Y + X$.
Proof. $((X + Y) + (X + Y)) + (X + Y)$
$= ((X + Y) + X) + ((Y + X) + Y)$. From I, I.

The Theorem follows by II, II, II.

2. $(X + X) + Y = Y = Y + (X + X)$.
Proof. $Y = (X + Y) + X = (Y + X) + X$
$= Y + (X + X) = (X + X) + Y$. From II, 1, I, 1.

2a. $X + Y = Z + Z \rightarrow X = Y$.
Proof. $X + Y = Z + Z \rightarrow (X + Y) + Y$
$= X + (Y + Y) = X = (Z + Z) + Y = Y$. From I, 2, 2.

3. $X + X = Y + Y$.
Proof. $X + X = (X + X) + (Y + Y) = Y + Y$. From 2, 2.

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[1]References to several brief versions are given in [4]; see also [5].

[2]An interesting metamathematical version [6] uses only *one* transformation postulate.

[3]More precisely, there will be two "transformation" axioms, and these will not include the tacit assumptions as to closure with respect to operations or as to a minimal number of elements.

[4]An arrow $\rightarrow$ signifies (material) implication, as in Hilbert and Ackermann.

4. $X(Y + Y) = Z + Z$.
Proof. $X(XX + XX) = X(XX) + X(XX)$. From III.
The Theorem follows by 3.

5. $X(YZ) = Y(ZX) = Z(XY)$.
Proof. $X(YZ + YZ) = W + W = X(YZ) + Y(ZX)$. From 4, III.
The Theorem follows by 2a.

6. $X1 = X$.
Proof. $1(1 + 1) = X + X1$. From IV.
The Theorem follows by 4, 2a.

6a. $1X = X$.
Proof. $1(X1) = 1(X) = X(11) = X1 = X$. From 6, 5, 6, 6.

7. $XY = YX$.
Proof. $X(Y1) = XY = Y(1X) = YX$. From 6, 5, 6a.

7a. $X(YZ) = (XY)Z$.
Proof. $X(YZ) = Z(XY) = (XY)Z$. From 5, 7.

8. $X(Y + Z) = XY + XZ$.
Proof. $X(Y + 1Z) = XY + 1(ZX) = XY + 1(XZ)$. From III, 7.
The Theorem follows by 6a.

8a. $(X + Y)Z = XZ + ZY$.
Proof. $Z(X + Y) = ZX + ZY$. From 8.
The Theorem follows by 7.

9. $XX = X$.
Proof. $X(X + 1) = XX + X = Y + Y1 = Y + Y$. From IV, 8, 6.
The Theorem follows by 2a.

With the tacit assumptions (Note 3), I, 1, 2, 3, 5a, 7, 8, 8a show the system to be a ring, 9 Boolean, 6, 6a with unit, and accordingly a Boolean algebra [3, pp. 96, 97] and it is known that the axioms used are themselves valid in Boolean algebra. Note that $X + X$ is the zero, and Theorem 3 shows both unicity of zero and also that every element is its own negative. The complement of any X is $X + 1$.

Formulation in Two Transformation Axioms
(using a device of Bernstein[2])

Abbreviations:
$$P_1 = ((A + B) + C + (A + (B + C))$$
$$P_2 = C(F + DE) + (CF + D(EC))$$
$$P_3 = G(G + 1) + (H + H1)$$
$$0 = 1 + 1$$

Axioms

I′. $X + X = P_1 + (P_2 + P_3)$. II. $(X + Y) + X = Y$.

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Theorems[5]

3. $X + X = Y + Y = 1 + 1$. From I′.

2. $(X + X) + Y = Y = Y + (X + X)$.

Proof. $((Y + Y) + Y) + (Y + Y) = Y$
$= (Y + Y) + Y = Y + (Y + Y)$. From II, II, II.

The Theorem follows by 3.

2a. $X + Y = Z + Z \rightarrow X = Y$.

Proof. $X + Y = Z + Z \rightarrow (X + Y) + X = (Z + Z) + X$.

The Theorem follows by II, 2.

11. $P_1 = P_2 + P_3$. From I′ by 2a.

12. $P_2 = P_3$.

Proof. $((C + C) + C) + (C + (C + C)) = P_2 + P_3$. From 11.

The Theorem follows by 2, 2a.

I. $(X + Y) + Z = X + (Y + Z)$. From 11 by 12, 2a.

III. $X(W + YZ) = XW + Y(ZX)$.

Proof. $P_2 = 0(0 + 1) + (0 + 01) = 01 + 01$. From 12, 2.

The Theorem follows by 2a.

IV. $X(X + 1) = Y + Y1$. From 12 by III 2a.

Thus by means of two valid axioms we have derived the previous four[6].

References

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[5]Numbered as before, with 11 and 12 as new theorems.

[6]If it is desired to formalize also the tacit assumptions mentioned in Note 3, we may append A. $X \neq X + 1$, and the closure postulate B. $(X)\,(Y)\,(Z)\,(EV)\,(EW)\; V = X + Y \;\&\; W = VZ)$, where (X) is "for every X" and (EZ) "there exists a Z such that". Then (after obtaining 2) we can deduce as Theorems:

01. $(X)\,(Y)\,(EZ)\,(Z = X + Y)$; from B.

02. $(X)\,(Y)\,(EZ)\,(Z = XY)$: since $(X)\,(Y)\,(EW)\,(EZ)\,(W = X + (V + V) \;\&\; Z = WY)$, from B,B, whence Theorem by 2.

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