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ABSOLUTE RETRACTS AND VARIETIES OF REFLEXIVE GRAPHS

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Introduction. For a graph G, let $V(G)$ denote its vertex set and $E(G)$ its edge set. Here we shall only consider *reflexive* graphs, that is graphs in which every vertex is adjacent to itself. These adjacencies, i.e., the loops, will not be depicted in the figures, although we always assume them present. For graphs G and H, an *edge-preserving map* (or *homomorphism*) of G to H is a mapping of $V(G)$ to $V(H)$ such that $f(g)$ is adjacent to $f(g')$ in H whenever g is adjacent to g' in G. Because our graphs are reflexive, an edge-preserving map can identify adjacent vertices, i.e., possibly $f(g) = f(g')$ for some g adjacent to g', cf. Figure 1(a).

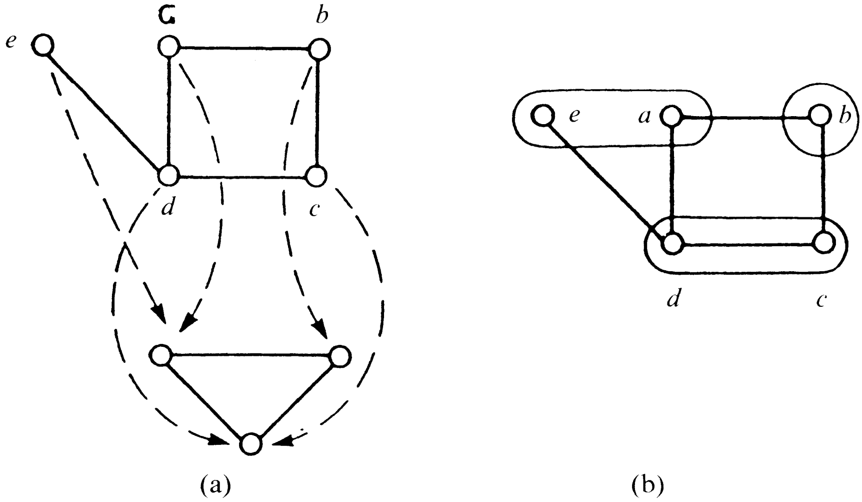

Figure 1. An example of an edge-preserving map and the equivalence relation it defines.

A retraction is a special kind of edge-preserving map. Before we turn to retractions though, we make two remarks about edge-preserving maps in general. Any edge-preserving map of G defines an equivalence relation on $V(G)$ in which two vertices are equivalent just if they are mapped to the same image. (In Figure1(b) we illustrate the classes of this equivalence for the edge-preserving map in (a).) Conversely, given any equivalence

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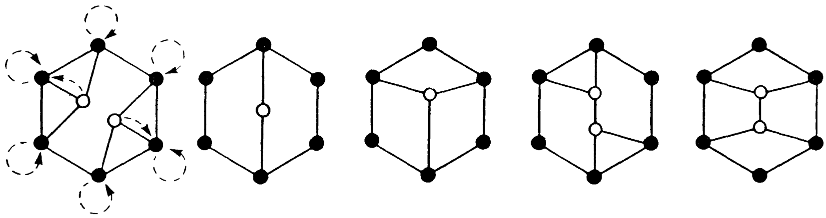
relation θ on $V(G)$, we would like to have an edge-preserving map which "shrinks" each equivalence class of θ to a vertex. This is easily done by defining the *quotient* G/θ as a graph whose vertices are the classes of θ, with two of them adjacent in G/θ just if, in G, there is an edge joining the two classes. For instance, the quotient of the equivalence depicted in Figure 1(b) is the triangle in Figure 1(a). The mapping which takes each vertex g of G to the class of θ containing g is then an edge-preserving map of G to G/θ. Thus we have another view of the edge-preserving map in Figure 1(a), as the map produced by the equivalence depicted in Figure 1(b). Our second remark concerns the notion of distance in a graph; we denote by $d_G(g, g')$ the length (number of edges) on a shortest path from g to g' in G, if one exists. It is easy to see that an edge-preserving map f of G to H takes a path of length l joining g and g' in G to a (possibly self-intersecting) path of length l joining $f(g)$ and $f(g')$ in H. Thus

$$d_H(f(g), f(g')) \leqq d_G(g, g')$$

for any edge-preserving map f of G to H, and any $g, g' \in V(G)$.

Now let H be a subgraph of G. A *retraction* of G to H is an edge-preserving map f of G to H such that $f(h) = h$ for all $h \in V(H)$. If there exists a retraction of G to H we say that H is a retract of G, cf. Figure 2(a)



(a) (b) (c) (d) (e)

Figure 2. The hexagon with shaded vertices is a retract of the graph in (a) but not of the graph in (b), (c), (d), or (e).

In Figure 2(b)-(e) we illustrate examples in which a retraction of G to H is impossible. What property of the way G lies in H makes it impossible? (We shall refer to such a property, on an intuitive level, as an "obstruction".) Given graphs G and H, an obvious obstruction to retracting G to H would be a "shortcut", i.e., a path of length $l < d_H(h, h')$ in G joining two vertices h, h' of H. (This is the case in Figure 2(b).) Indeed, according to our general remarks on edge-preserving maps, there could be in such a case no edge-preserving map of G to H fixing h and h', and hence no retraction of G to H. Let us say that H is an isometric subgraph of G if

$$d_H(h, h') = d_G(h, h') \quad \text{for any } h, h' \in V(H),$$

i.e., if there is no shortcut. Thus if a subgraph is a retract then it is necessarily an isometric subgraph. The class of graphs H which are retracts whenever this simple necessary condition is satisfied is characterized in Theorem 1. Figure 2(c) illustrates that the condition is not always sufficient. In fact, in Figure 2(c) there are three vertices having a vertex of distance 1 from each of them in G, but not in H. This is a prototype of another type of obstruction, which we call a "filled triple". A *triple* of H is a set of three (not necessarily distinct) vertices a, b, c of H together with three nonnegative integers i, j, k such that no $h \in V(H)$ satisfies

$$d_H(h, a) \leqq i, \ d_H(h, b) \leqq j, \text{ and } d_H(h, c) \leqq k.$$

If H is a subgraph of G, we say that the triple $(a, b, c; i, j, k)$ of H is *separated* in G if there is no $g \in V(G)$ satisfying

$$d_G(g, a) \leqq i, \ d_G(g, b) \leqq j, \ d_G(g, c) \leqq k.$$

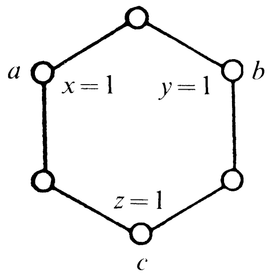


Figure 3. An example of a triple.

It follows from the earlier remarks on edge-preserving maps that if some triple of H is not separated (i.e., is "filled") in G, then there is no retraction of G to H. (For example the triple of Figure 3 which exists in the outer hexagon of Figure 2(c) is not separated in Figure 2(c) and hence there is no retraction onto the outer hexagon.) Hence having all triples separated is another necessary condition to be a retract, and Theorem 2 characterizes those graphs H which are retracts whenever this condition is satisfied. Note that having all triples separated is a stronger property than being an isometric subgraph: if

$$d_G(h, h') < d_H(h, h')$$

and if we set $a = h$, $b = c = h'$, $i = 0$, and $j = k = d_G(h, h')$ then $(a, b, c; i, j, k)$ is a triple in H which is not separated in G.

Naturally, this condition is still not sufficient for all graphs, as is illustrated in Figure 2(d). One would expect other types of obstructions, filled "quadruples", and larger "holes". In general, we define a *hole* of the graph H to be a pair (K, δ), where K is a nonempty set of vertices and δ a function from K to the nonnegative integers such that no $h \in V(H)$ has

$$d_H(h, k) \leqq \delta(k)$$

for all $k \in K$. For technical reasons we shall also require that if (K, δ) is a hole, then K has no subset K' with $|K'| < |K|$ such that $(K', \delta|K')$ is also a hole.

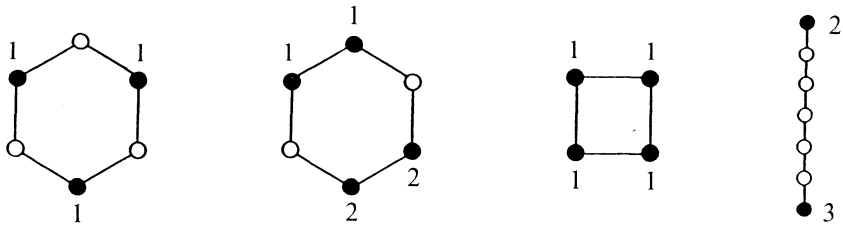


Figure 4. Examples of holes: each $k \in K$ is shaded and $\delta(k)$ is indicated.

An *m-hole* is a hole (K, δ) with $|K| = m$. Note that a triple is either a 3-hole, or contains a 2-hole. Also note that a 2-hole is a pair of vertices x, y with nonnegative integers $\delta(x)$, $\delta(y)$ so that

$$\delta(x) + \delta(y) < d(x, y),$$

cf. Figure 4(d). Another way to view holes is this. Denote by $D_H(h, r)$ the disc with centre h and radius r in H, i.e., let

$$D_H(h, r) = \{x \in V(H): d_H(h, x) \leqq r\}.$$

Then (K, δ) is a hole if and only if

$$\bigcap_{k \in K} D_H(k, \delta(k)) = \emptyset \quad \text{and}$$

$$\bigcap_{k \in K'} D_H(k, \delta(k)) \neq \emptyset \quad \text{if } K' \subset K, |K'| < |K|.$$

We have observed above that if H is a retract of G then for any hole (K, δ) of H we must have

$$\bigcap_{k \in K} D_G(k, \delta(k)) = \emptyset$$

also. Since each

$$\bigcap_{k \in K'} D_G(k, \delta(k)) \supseteq \bigcap_{k \in K'} D_H(k, \delta(k)) \neq \emptyset, \quad K' \subset K, |K'| < |K|,$$

we conclude that (K, δ) must also be a hole of G. Thus the obstruction described earlier as a "filled hole" is a hole of H which is not a hole of G. Just as for triples, we say that a hole (K, δ) of a subgraph H of G is *separated* in G if (K, δ) is also a hole of G. Having all holes separated is another necessary condition for being a retract, and Theorem 3 characterizes those graphs H which are retracts whenever this condition is satisfied. This condition is stronger than being an isometric subgraph (which is equivalent to having all 2-holes separated) or having all triples

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separated (which is equivalent to having all 2-holes and 3-holes separated), but it is not always sufficient, as illustrated in Figure 2(e). Since hole separation is our principal tool, we reserve the name *absolute retract* for those graphs for which this last condition is sufficient: H is an absolute retract if H is a retract of any G of which it is a subgraph and for which every hole of H is separated in G. Thus Theorem 3 characterizes absolute retracts. Graphs in Figures 6 and 7 are examples of absolute retracts; the hexagon is not an absolute retract, (cf. in Figure 2(d)).

For infinite graphs we consider the following "finitary" obstruction. When H is a subgraph of G, we can extend each equivalence relation θ on $V(H)$ to $V(G)$ by letting $g\,\theta\,g'$ just if $g = g'$ or $g\,\theta\,g'$ in H. (Cf. Figure 5.) Then an equivalence relation θ (and in particular, one with finitely many classes) on $V(H)$ such that H/θ is not a retract of G/θ represents an obstruction to retracting G to H. Indeed, any retraction f of G to H would define a retraction $\tilde{f}$ of G/θ to H/θ: simply let $\tilde{f}$ take each of the original classes of θ on $V(H)$ to itself, and each of the new classes on $V(G) - V(H)$, each of which must necessarily be a singleton, $\{g\}$, to the class containing $f(g)$.

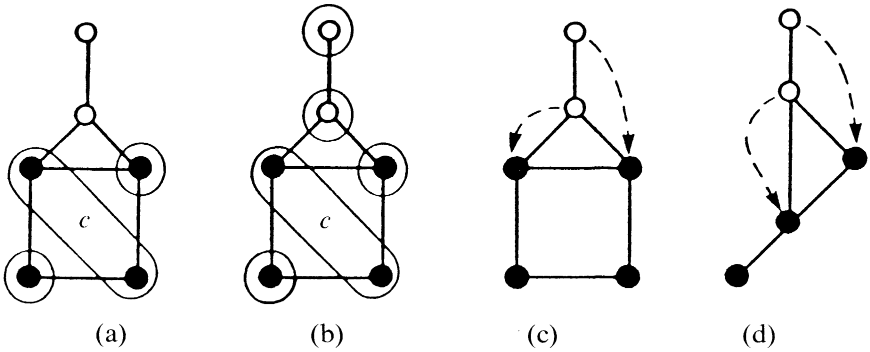

Figure 5. An equivalence θ on the four-cycle H (with shaded vertices), (a), is extended to the whole graph G, (b), and a retraction of G to H, (c), is modified to a retraction of G/θ to H/θ, (d).

In particular, we obtain the following necessary condition for the existence of a retraction of G to H: each finite quotient H/θ is a retract of the corresponding quotient G/θ. Graphs H for which this condition is also sufficient are said to satisfy the *finite separation property*, and are characterized in Theorem 4. (Formally, H has the finite separation property if H is a retract of any G of which it is a subgraph, provided that for each equivalence θ on $V(H)$ with finitely many classes, H/θ is a retract of G/θ.) Obviously, any finite graph H has the property (consider $\theta = \{ (h, h): h \in V(H) \}$).

Each of our characterizations is given in terms of varieties. A graph *variety*, (cf. [3] or [7]) is a class $\mathcal{V}$ of graphs which contains all products (defined below) and all retracts of members of $\mathscr{V}$. In symbols, $\mathbf{P}(\mathscr{V}) \subseteq \mathscr{V}$ and $\mathbf{R}(\mathscr{V}) \subseteq \mathscr{V}$. For a class $\mathscr{C}$ of graphs, let $\mathscr{C}^v$ stand for the smallest graph variety containing $\mathscr{C}$. We call $\mathscr{C}^v$ the *variety generated by* $\mathscr{C}$, or the variety of $\mathscr{C}$. In fact, $\mathscr{C}^v = \mathbf{RP}(\mathscr{C})$. The product we have in mind here is the product in the category of reflexive graphs and edge-preserving maps, known in graph theory as the strong (or normal) product [**1**], or the direct product [**14**]: The product of graphs G_i, $i \in I$, denoted by $\prod_{i \in I} G_i$ has as its vertex set the cartesian product $\prod_{i \in I} V(G_i)$ and two vertices $(g_i)_{i \in I}$ and $(h_i)_{i \in I}$ are adjacent just if g_i is adjacent to h_i in each G_i, $i \in I$. Note that because of reflexivity, one way g_i may be adjacent to h_i is $g_i = h_i$. In Figure 6 we illustrate the product of two graphs, as well as the easy fact that each map π_j of $\prod_{i \in I} G_i$ to G_j defined by

$$\pi_j((g_i)_{i \in I}) = g_j$$

(the j-th projection) is edge-preserving.

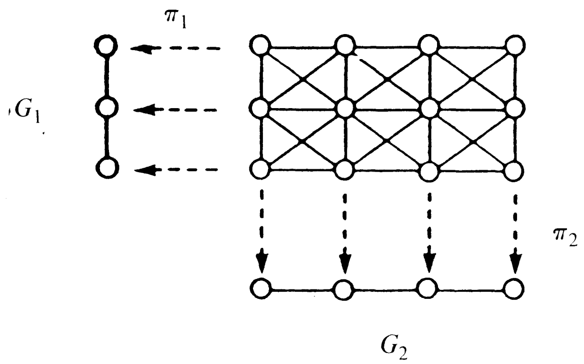

Figure 6. The product of two paths and the two projections.

Among the varieties playing a role here will be the variety of all finite paths (Theorem 1), of all finite graphs (Theorem 4), and of all **Y**-graphs, defined below, and illustrated in Figure 7 (Theorem 2).
In the last section we discuss some related problems.

The main results. Our point of departure is the following result.

THEOREM 1. *Let H be a graph. The following statements are equivalent*:
(1) *H is a retract of any graph G of which it is an isometric subgraph*;
(2) *H is in the variety of finite paths*;
(3) *H has no m-holes for $m \geqq 3$.*

For completeness we include a proof of Theorem 1, even though the equivalence of (2) and (3) was proved by R. J. Nowakowski and I. Rival [**9**], the equivalence of (1) and (3) (in different terminology and for finite graphs) is implicit in A. Quilliot [**12**], and the equivalence of (1) and

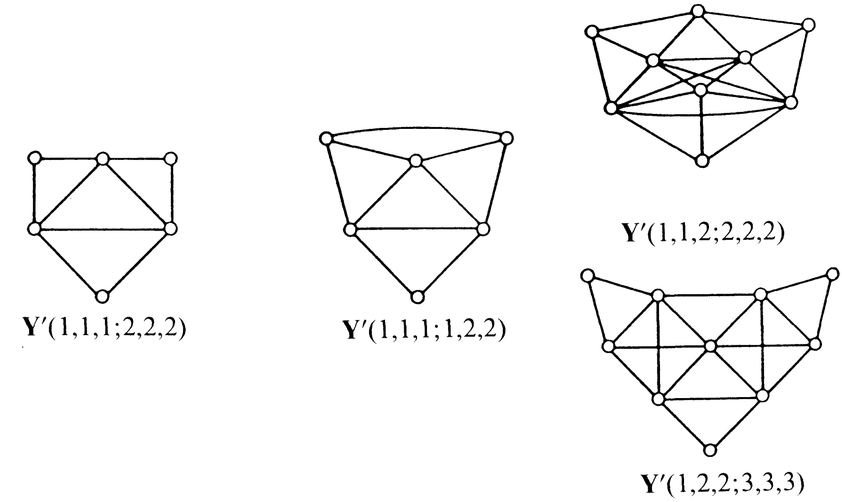

Figure 7. Example of **Y**-graphs.

(2) (in the essentially similar context of irreflexive bipartite graphs) was proved by P. Hell [**4**], cf. also [**5**, **6**]. (For topological spaces a similar result of this kind was published as early as 1931 in [**2**]. We shall also deduce this consequence. In this statement by a "cycle" we intend a closed path which may have chords. Such a "cycle" need not be an (induced) subgraph.)

COROLLARY. *Let H be a graph without cycles of length $l > 3$. If H is an isometric subgraph of a graph G, then H is a retract of G.*

This corollary is a generalization of the result of R. J. Nowakowski and I. Rival [**8**], and A. Quilliot [**12**], where it was proved for the case when H is a tree. It was also anticipated, in the context of irreflexive bipartite graphs by P. Hell [**4**, **5**], where it applied to all graphs H without cycles of length $l > 5$.

A similar theorem asserting the existence of a retraction onto a shortest cycle was proved for bipartite graphs by G. Sabidussi [**15**] (cf. also [**4**]) and for reflexive graphs by R. Nowakowski and I. Rival [**8**], and by A. Quilliot [**12**]. Sabidussi was apparently the first person to explicitly suggest studying retractions of graphs ([**15**]).

Theorem 1 says in effect that every (finite or infinite) graph which is a retract of any graph in which it is an isometric subgraph can be reconstructed, using the operations of retract and product, from finite paths alone. The simplicity of these "building blocks" is very appealing.

We now give a description of the building blocks for the class of graphs which are retracts whenever all their triples are separated. These graphs will have G parameters $(i, j, k; l, m, n)$ which are positive integers satisfying the following inequalities:

$$\text{(I)}\quad \begin{aligned} &l \leqq i + j,\ m \leqq j + k,\ n \leqq i + k \\ &l \leqq m + n,\ m \leqq l + n,\ n \leqq m + l \\ &l + n > j + k,\ l + m > i + k,\ m + n > i + j. \end{aligned}$$

The graph $\mathbf{Y}(i, j, k; l, m, n)$ is defined as follows. The vertices are triples

$$a = (0, \min(l, j + 1), \min(n, k + 1))$$

$$b = (\min(l, i + 1), 0, \min(m, k + 1))$$

$$c = (\min(n, i + 1), \min(m, j + 1), 0)$$

and all triples (x, y, z),

$$x = 1, \ldots, i, i + 1,$$

$$y = 1, \ldots, j, j + 1,$$

$$z = 1, \ldots, k, k + 1$$

such that $x = i + 1$, or $y = j + 1$, or $z = k + 1$, satisfying the following inequalities

$$\text{(II)}\quad \begin{aligned} &|x - y| \leqq l \text{ and } x + y \geqq l, && \text{if } x \leqq i \text{ and } y \leqq j \\ &|y - z| \leqq m \text{ and } y + z \geqq m, && \text{if } y \leqq j \text{ and } z \leqq k \\ &|z - x| \leqq n \text{ and } x + z \geqq n, && \text{if } x \leqq i \text{ and } z \leqq k. \end{aligned}$$

Two vertices (x, y, z), (x', y', z') are adjacent in $\mathbf{Y}(i, j, k; l, m, n)$ if

$$|x - x'| \leqq 1,\ |y - y'| \leqq 1, \text{ and } |z - z'| \leqq 1.$$

Figure 8 illustrates the general construction, as well as some concrete examples. Any finite face of these planar graphs is to be interpreted as a complete graph including all vertices inside and on the boundary of the face, cf. Figures 8c and 8d. It can be verified from (I) and (II) that if $x \leqq i$ (respectively $y \leqq j$, or $z \leqq k$) then the distance from (x, y, z) to a (respectively b, or c) is precisely x (respectively y, or z). If $x = i + 1$ then the distance to a is greater than i (and similarly for y, z). Finally, the distances between a and b, b and c, and a and c are respectively l, m, and n.

A **Y**-graph is any of the graphs $\mathbf{Y}(i, j, k; l, m, n)$ defined above. We have the following characterization:

THEOREM 2. *A graph H is a retract of each graph in which all triples of H are separated if and only if H is in the variety of* **Y**-*graphs.*

Remark. It is possible to simplify the **Y**-graphs a little bit, by removing the four central vertices

$$(i + 1, j + 1, k + 1),\ (i + 1, j + 1, k),\ (i + 1, j, k + 1), \text{ and}$$

$$(i, j + 1, k + 1).$$

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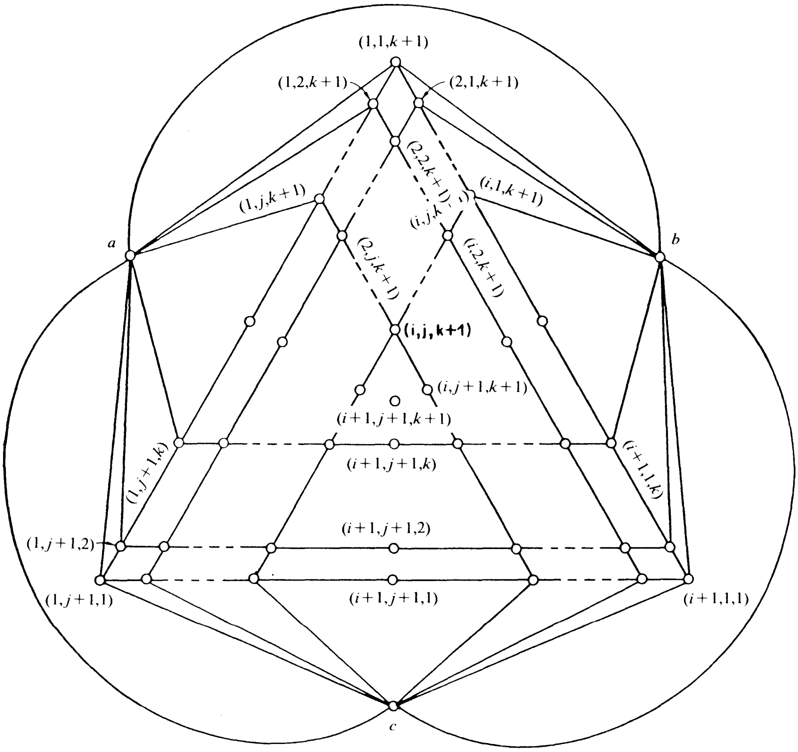

Each plane region of the above figure is to be interpreted as a complete graph consisting of all the vertices inside the region and on its boundary.

The graph **Y** $(i, j, k; l, m, n)$ is a subgraph of the figure determined by the vertices satisfying the inequalities (II) and the adjacencies described just after (II).

Figure 8 (a)

In Figure 7 we illustrated these simplified graphs $\mathbf{Y}'$ $(i, j, k; l, m, n)$. It is not hard to modify the proof of Theorem 2 to work for the graphs $\mathbf{Y}'$.

In Figure 9 we illustrate the fact that in contrast to Theorem 1, (3), the absence of m-holes, $m \geqq 4$, does not assure that a graph is a retract of each supergraph in which all of its triples are separated.

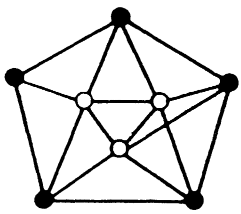

Figure 9. The pentagon with shaded vertices is not a retract of G, even though all of its triples are separated and it has no m-holes, $m \geqq 4$.

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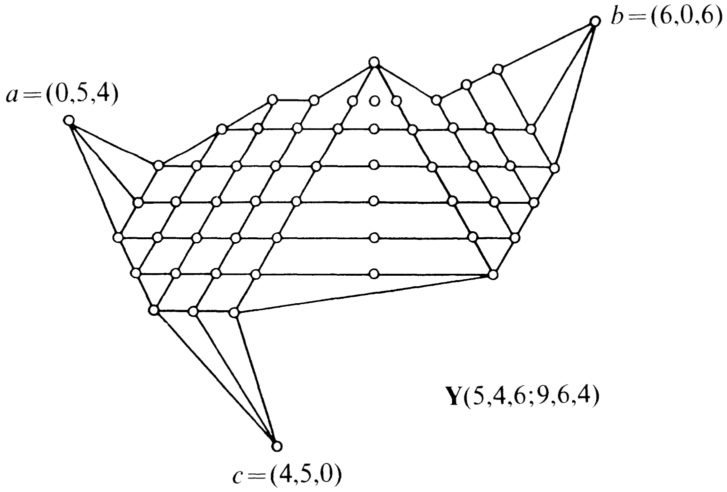
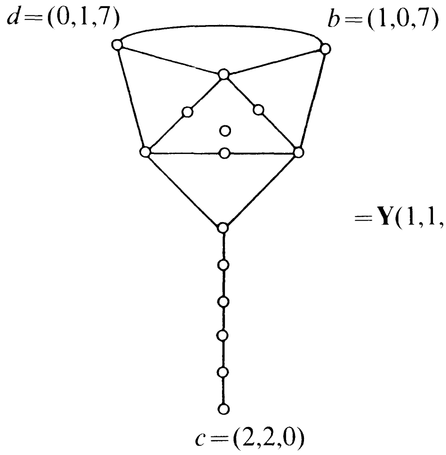


Figure 8 (b)



$=\mathbf{Y}(1,1,6;1,7,7)=$

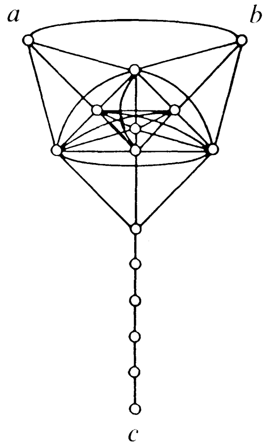


Figure 8 (c) Figure 8 (d)

There is some suggestion, in the definition of $\mathbf{Y}$ $(i, j, k; l, m, n)$, of how one may define the "building blocks" of the variety of absolute retracts. (The class of absolute retracts is a variety, cf. Lemma 5.) However, these building blocks seem to get too unwieldy to be of much current interest. One useful aspect of our "building block" characterizations, such as those given in Theorems 1 and 2, is that they render the problem of deciding whether a graph G has the property (of being a retract of any graph in which all appropriate holes are separated) a finite question. Indeed, it is easy to see that we may assume that G is connected (else investigate each component separately) and, that for connected graphs, only finitely many

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(2- and 3-) holes need be considered. Separating all of these holes gives rise to a finite product containing G (cf. the proofs of Theorems 1 and 2) and G will have the required property if and only if it is a retract of this product. To show that the problem of deciding whether G is an absolute retract is also a finite question, we give, instead of a variety ("building block") characterization, a related description.

For a graph H call a nonempty subset S of $V(H)$ *closed*, if for each $x \in V(H) - S$ there is an $h \in V(H)$ such that

$$d_H(x, h) > d_H(s, h),$$

for all $s \in S$. Equivalently, denoting

$$\mathrm{ecc}_S(h) = \sup_{s \in S} d_H(s, h)$$

(the S-eccentricity of h), $S \neq \emptyset$ is a closed subset if and only if

$$\bigcap_{h \in V(H)} D_H(h, \mathrm{ecc}_S(h)) = S.$$

The *coordinate graph* of H, denoted $\hat{H}$, has as its vertices the closed sets of H, and two vertices S, S' of $\hat{H}$ are adjacent just if

$$|\mathrm{ecc}_S(h) - \mathrm{ecc}_{S'}(h)| \leqq 1 \quad \text{for all } h \in V(H).$$

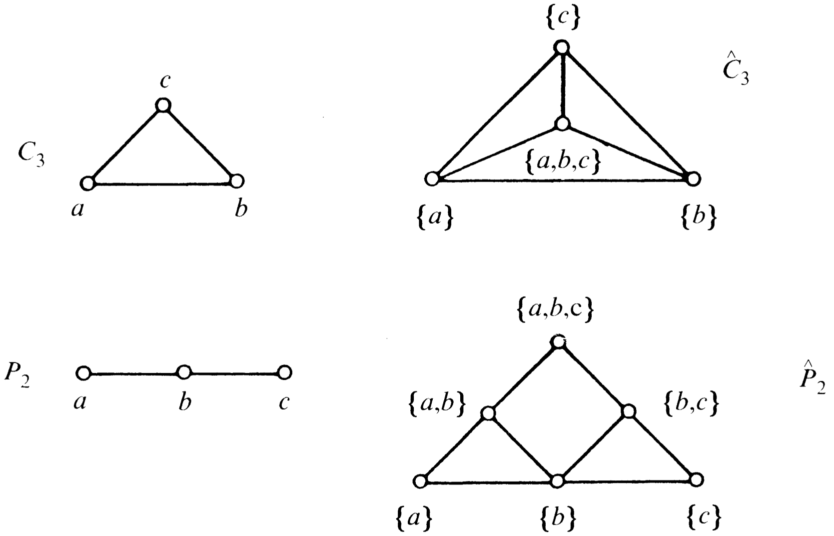

Figure 10. Two examples of coordinate graphs.

We observe that each singleton $S = \{s\}$ is a closed set; as

$$\mathrm{ecc}_S(h) = d_H(s, h) \quad \text{for all } h \in V(H)$$

(including $h = s$). It follows easily that H is (isomorphic to) a subgraph of $\hat{H}$.

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The graphs which are retracts whenever all their holes are separated, i.e., the absolute retracts, are characterized as follows:

THEOREM 3. *A graph H is an absolute retract if and only if it is a retract of its coordinate graph* $\hat{H}$.

Note that Theorems 1 and 2 give characterizations of absolute retracts without m-holes, $m \geqq 3$, $m \geqq 4$ respectively.

Finally, we characterize those graphs H which are retracts of any G for which all their finite quotients H/θ are retracts of G/θ:

THEOREM 4. *A graph H has the finite separation property if and only if H is in the variety of finite graphs.*

The proofs. In order to streamline the arguments we introduce additional terminology (which we avoided above to make the statements more transparent). Let P_j ($j = 1, 2, 3, 4$) be the following properties of a subgraph H of a graph G:

P_1: H is an isometric subgraph of G

P_2: All triples of H are separated in G

P_3: All holes of H are separated in G

P_4: H/θ is a retract of G/θ for all equivalences θ on $V(H)$ with finitely many classes.

Note that each property P_j is a necessary condition for the existence of a retraction of G to H. Let $\mathrm{AR}(P_j)$ denote the class of graphs H for which the condition P_j implies that H is a retract of G. Thus Theorem j characterizes $\mathrm{AR}(P_j)$, $j = 1, 2, 3, 4$; $\mathrm{AR}(P_3)$ is just the class of absolute retracts defined earlier; and $\mathrm{AR}(P_4)$ is the class of graphs with the finite separation property.

The following lemma will allow us to deduce $\mathscr{C}^v \subseteq \mathrm{AR}(P_j)$ from knowing just $\mathscr{C} \subseteq \mathrm{AR}(P_j)$:

LEMMA 5. *Let j be* 1, 2, 3, *or* 4.

(a) *If* $H' \in \mathrm{AR}(P_j)$ *and if H is a retract of H', then also* $H \in \mathrm{AR}(P_j)$.

(b) *If* $H_i \in \mathrm{AR}(P_j)$, *for all* $i \in I$, *then the product*

$$H = \prod_{i \in I} H_i$$

also belongs to $\mathrm{AR}(P_j)$.

Proof of Lemma 5. (a) Let H be a subgraph of G and assume that P_j holds ($j = 1, 2, 3$, or 4). We will modify G so that it contains all of H'. This is formally done by using the notion of amalgams. Let X and Y be graphs with disjoint vertex sets and suppose we have a fixed isomorphism of a subgraph X' of X to a subgraph Y' of Y. The *amalgam* of X and Y over X' (or Y') is the quotient of $X \cup Y$ under the equivalence whose only nontrivial classes are the pairs of corresponding vertices of X', Y'. In other words, the amalgam is constructed from X and Y by identifying the corresponding vertices (and edges) of X' and Y'. (Note that both X and Y are subgraphs of the amalgam.)

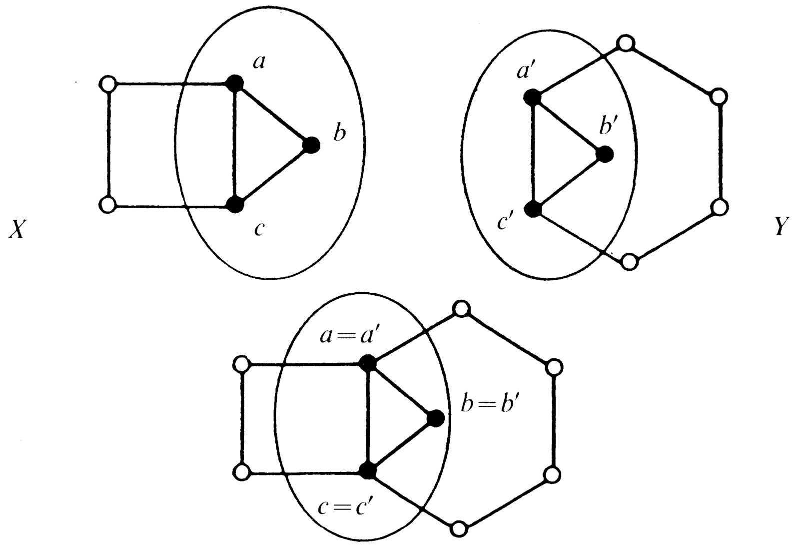

Figure 11. The amalgam of X and Y over a triangle.

Now in our situation we have H as a subgraph of G and (an isomorphic copy of) H as a subgraph of H'. Thus we can construct the amalgam G' of G and H' over H. If we can argue that property P_j holds for H' in G' then by assumption H' is a retract of G', and by composing retractions H is a retract of G'. Since G is a subgraph of G' we would also have H a retract of G.

It remains to show that H' in G' has property P_j. For $j = 1, 2$, and 3 the proof is fairly evident. (Recall that H has property P_j in G.) It suffices to notice that any path connecting a vertex of G to a vertex of H' must contain vertices from H. Thus any m-hole in H' not separated in G' would give rise to an m-hole in H not separated in G. For $j = 4$ we need to prove that each finite H'/θ is a retract of G'/θ. The equivalence θ on $V(H')$ can be restricted to $V(H)$, and has, in $V(H)$, also only finitely many classes. By assumption, there is a retraction r of G/θ to H/θ. The retraction r can now be extended to a retraction of G'/θ to H'/θ: the vertices of G'/θ are the vertices of G/θ, for which we keep the existing value of $r(v)$, and the vertices w of H'/θ, for which we let $r(w) = w$. It is a routine exercise to verify that the extended map r is well-defined (i.e., the two definitions agree where they both apply), and edge-preserving.

(b) Assume that

$$H = \prod_{i \in I} H_i$$

is a subgraph of G such that P_j holds ($j = 1, 2, 3$, or 4).

Let θ_k be the equivalence on $V(H)$ in which

$$(h_i)_{i \in I} \theta_k (h'_i)_{i \in I}$$

just if $h_k = h'_k$. Suppose we can argue that H/θ_k also has property P_j in G/θ_k. Thus there would be, for each $i \in I$, a retraction r_i of G/θ_i to $H/\theta_i \cong H_i$. A retraction r of G to $H = \prod_{i \in I} H_i$ may then be defined by

$$r(g) = (r_i(\{g\}))_{i \in I} \quad \text{for } g \notin V(H)$$

$$r(g) = g \quad \text{for } g \in V(H).$$

Indeed, if $g \notin V(H)$, then the class of θ_i containing g is the singleton $\{g\}$. If $g \notin V(H)$ is adjacent to $g' \in V(H)$ in G, then each $r_i(\{g\})$ is adjacent to the θ_i-class containing g' in $H_i \cong H/\theta_i$. Hence, in $H = \prod_{i \in I} H_i$, $(r(g))_i$ is adjacent to $(g')_i$ for each $i \in I$, and so $r(g)$ is adjacent to g'. It easily follows that r is an edge-preserving map.

It remains to demonstrate that each H/θ_i ($\cong H_i$) has property P_j in G/θ_i. Let $j = 1, 2$, or 3, and suppose that some m-hole of H/θ_i is not separated in G/θ_i. Then there is an m-hole in H which is not separated in G. For $j = 4$, if some $H/\theta_k = H_k$ has a finite quotient $(H/\theta_k)/\theta$ which is not a retract of $(G/\theta_k)/\theta$, then define $\tilde{\theta}$ on $V(H)$ by

$$h \, \tilde{\theta} \, h' \text{ just if}$$

$$h = (h_i)_{i \in I}, \ h' = (h'_i)_{i \in I} \text{ and } h_k \theta h'_k,$$

and observe that $(H/\theta_k)/\theta \cong H/\tilde{\theta}$ is not a retract of $(G/\theta_k)/\theta \cong G/\theta$ while $\tilde{\theta}$ has only finitely many classes on $V(H)$. In each case we conclude that the fact that H has property P_j in G implies that $H/\theta_i = H_i$ has property P_j in G/θ_i.

In the next lemma we shall show how to construct, for a given graph H, a graph G in which H satisfies property P_j ($j = 1, 2, 3$).

Let (K, δ) be a hole of the graph H. We say that an arbitrary graph G *separates* the hole (K, δ) of H if there is an edge-preserving map f of H to G such that

$$\bigcap_{k \in K} D_G(f(k), \delta(k)) = \emptyset.$$

In this case, we also call f a *separating map* of (K, δ) in G. Note that if H is a subgraph of G then the inclusion map i of H to G,

$$i(h) = h, \quad h \in V(H),$$

is a separating map of (K, δ) if and only if (K, δ) is separated in G according to an earlier definition.

LEMMA 6. *Let H be a fixed graph. Let M be a set of integers $m \geqq 2$, with $2 \in M$. If for each $m \in M$ and each m-hole (K, δ) of H there is a separating map $f_{k,\delta}$ of (K, δ) in some $G_{K,\delta}$, then H is (isomorphic to) a subgraph of*

$$G = \Pi\{G_{K,\delta}:(K, \delta) \text{ is an } m\text{-hole}, m \in M\}$$

such that all m-holes, $m \in M$, of H are separated in G.

We shall use Lemma 6 to conclude, for a graph $H \in \mathrm{AR}(P_j)$, that it is a subgraph of some product in which it has property P_j, and hence that H is a retract of the product giving us a variety characterization of $\mathrm{AR}(P_j)$. (Here $j = 1, 2, 3$; a similar argument will also be used for $j = 4$, cf. the proof of Theorem 4.)

Proof of Lemma 6. The isomorphism φ taking H onto a subgraph $G = \Pi G_{K,\delta}$ is defined by

$$\varphi(h) = (f_{K,\delta}(h))_{(K,\delta)}.$$

It is easy to check that φ is indeed an isomorphism; for instance $\varphi(h) \neq \varphi(h')$ for $h \neq h'$ is verified by considering $f_{K,\delta}$ for the hole $K = \{h, h'\}$ with

$$\delta(h) = 0, \quad \delta(h') = d_H(h, h') - 1.$$

(Recall that $2 \in M$.) Identifying H with its isomorphic image $\varphi(H)$ we note that each projection $\pi_{K,\delta}$, when restricted to $V(H)$, becomes just $f_{K,\delta}$. Now suppose that some m-hole (K_o, δ_o) $m \in M$, of H were not separated in $G = \Pi G_{K,\delta}$, i.e., that there is a $g \in V(G)$ such that

$$d_G(g, k) \leqq \delta(k), \quad \text{for each } k \in K_o.$$

Since each projection $\pi_{K,\delta}$ is an edge-preserving map of G to $G_{K,\delta}$, we have the following inequality for the distance in $G_{K,\delta}$:

$$d(\pi_{K,\delta}(g), f_{K,\delta}(k)) = d(\pi_{K,\delta}(g), \pi_{K,\delta}(k)) \leqq d_G(g, k) \leqq \delta(k),$$

for each $k \in K_o$. In other words,

$$\pi_{K,\delta}(g) \in \bigcap_{k \in K_o} D_{G_{K,\delta}}(f_{K,\delta}(k), \delta(k)),$$

for each m-hole (K, δ), $m \in M$. In particular, for $K = K_o$, $\delta = \delta_o$, this contradicts the definition of $f_{K,\delta}$. Hence each m-hole of H, $m \in M$, is separated in G.

The proof of Theorem 1. We first prove the equivalence of (1) ($H \in \mathrm{AR}(P_1)$) and (2) (H is in the variety of finite paths). It is easy to see that each finite path is in $\mathrm{AR}(P_1)$. Indeed, if the path P_n with vertices $a_0, a_1, \ldots, a_n$ is an isometric subgraph of G, then the map r defined by

$$r(g) = a_i \text{ if } d_G(g, a_0) = i, i = 0, 1, \ldots, n - 1$$

$$r(g) = a_n \text{ otherwise,}$$

is a retraction of G to P_n. (If G were not connected, then all vertices of the components not containing P_n could be mapped to any vertex of P_n.) According to Lemma 5 (with $j = 1$), it follows that each graph in the variety of finite paths also belongs to $\mathrm{AR}(P_1)$. Conversely, assume that $H \in \mathrm{AR}(P_1)$. Let (K, δ) be a 2-hole in H, i.e., $K = \{k, k'\}$ and

$$\delta(k) + \delta(k') < d_H(k, k').$$

Let n be a positive integer,

$$\delta(k) + \delta(k') < n \leqq d_H(k, k').$$

(Note that such an integer exists even if $d_H(k, k') = \infty$.) We define a map $f_{K,\delta}$ from H to P_n (with vertices $a_0, a_1, \ldots, a_n$) by

$$f_{K,\delta}(h) = a_i \text{ if } d_H(h, k) = i \leqq n - 1$$

$$f_{K,\delta}(h) = a_n \text{ otherwise.}$$

Then $f_{K,\delta}$ is an edge-preserving map of H to P_n which separates the hole (K, δ) in P_n. Since each 2-hole is separated by a finite path, Lemma 6 (with $M = \{2\}$) assures that H is an isometric subgraph of a product of finite paths, and because $H \in \mathrm{AR}(P_1)$, it is a retract of it. Therefore H is in the variety of finite paths.

Next we prove the equivalence of (1) ($H \in \mathrm{AR}(P_1)$) and (3) (H has no m-hole, $m \geqq 3$). The easier half is to see that (1) implies (3). Any m-hole (K, δ) with $m = |K| \geqq 3$ allows one to define a graph G containing H as well as a new vertex v of degree m with m disjoint paths (of new vertices) joined to the elements of K in such a way that the length of the added path from v to $k \in K$ is precisely $\delta(k)$ (cf. Figure 12). (Note that it follows from the definition of a hole that if $|K| > 2$ then $\delta(k) > 0$ for all $k \in K$.)

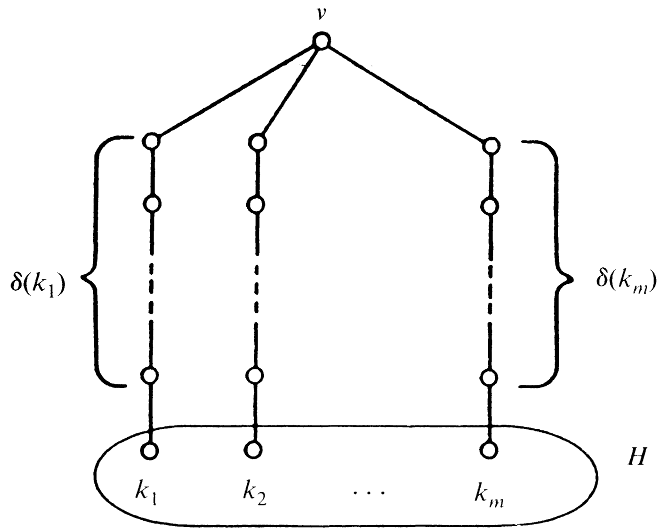

Figure 12. The construction of G for a hole (K, δ) in H.

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Because (K, δ) is a hole in H but not in G, H is not a retract of G. Because $(\{k, k'\}, \delta)$ is not a hole for any $k, k' \in K$, H is an isometric subgraph of G. Thus $H \notin \mathrm{AR}(P_1)$. Conversely, we shall now show that (3) implies (1). Suppose that H has no m-holes, $m \geqq 3$, and that H is an isometric subgraph of some G. It follows that every hole of H is separated in G, and that for each vertex $g \in V(G)$ there exists a vertex $h_g \in V(H)$ such that

$$d_H(h_g, x) \leqq d_G(g, x)$$

for all $x \in V(H)$. The idea of showing that H is a retract of G will be to map one vertex g to its h_g and note that in the resulting situation H remains an isometric subgraph so that we can continue. To do this formally, we define a *partial retraction* to be an equivalence θ on $V(G)$ such that

(i) no two vertices of H are equivalent
(ii) H is an isometric subgraph of G/θ.

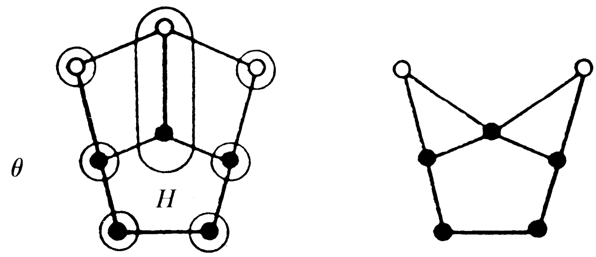

Figure 13. A partial retraction.

Note that (i) implies that H is (can be viewed as) a subgraph of G/θ. Clearly, there are partial retractions, e.g., $\{(g, g):g \in V(G)\}$ is one. Partial retractions are ordered by inclusion and in this order there are maximal elements. (This is obvious when G is finite and follows, for infinite G, by a routine application of Zorn's lemma.) We claim that a maximal partial retraction G corresponds to a retraction r of G to H, i.e., that G/θ is H. Indeed if G/θ had a vertex v outside of H, then we would have a situation where H is isometric in $G' = G/\theta$ and hence there is a vertex $h_v \in V(H)$ with

$$d_H(h_v, x) \leqq d_{G'}(v, x)$$

for all $x \in V(H)$. Let θ' be the equivalence on $V(G')$ in which each class is a singleton except for the class $\{v, h_v\}$. Let $G'' = G'/\theta'$ and note that H is isometric in G''; otherwise

$$d_{G''}(h_1, h_2) < d_H(h_1, h_2) \leqq d_{G'}(h_1, h_2),$$

for some vertices h_1, h_2 of H. Therefore we have a situation depicted in Figure 14.

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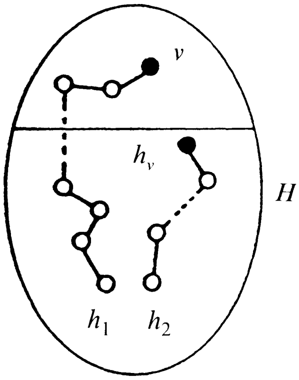

Figure 14. How H could fail to be isometric in G'' even though it is isometric in G'.

In other words

$$d_{G''}(h_1, h_2) = d_{G'}(h_1, v) + d_H(h_v, h_2)$$

(or the similar inequality with h_1, h_2 interchanged). But

$$d_H(h_v, h_1) \leqq d_{G'}(v, h_1)$$

and so

$$d_H(h_1, h_2) \leqq d_H(h_v, h_1) + d_H(h_v, h_2) \leqq d_{G'}(v, h_1) + d_H(h_v, h_2)$$
$$= d_{G''}(h_1, h_2)$$

contradicting the choice of h_1, h_2. In conclusion, H is isometric in $G'' = G'/\theta'$ and hence there is a partial retraction θ^* on $V(G)$ properly containing θ, contrary to the maximality of θ; therefore the edge-preserving map associated with a maximal partial retraction is in fact a retraction.

The proof of the corollary to Theorem 1. By Theorem 1, it will suffice to show that a graph with an m-hole, $m \geqq 3$, must have a cycle of length greater than 3. Therefore, let (K, δ) be a hole in H, $K = \{k_1, k_2, \ldots, k_m\}$ and $m \geqq 3$. By the definition of a hole, there exist distinct vertices $l_1, l_2, \ldots, l_m$ such that

$$l_i \in \bigcap_{j \neq i} D_H(k_j, \delta(k_j)),$$

$i = 1, 2, \ldots, m$. Let W_{ij} be a shortest path from l_i to k_j in H. Let z be the first vertex on W_{21} common to W_{31}. (Possibly $z = l_2, l_3$, or k_1.) If H has no cycle of length at least 4, then $W_{21} \cup W_{31}$ has the general form depicted in Figure 15.

Note that $l_1 \notin W_{21} \cup W_{31}$ because

$$d_H(l_1, k_1) > \delta(k_1) \geqq d_H(l_i, k_1)$$

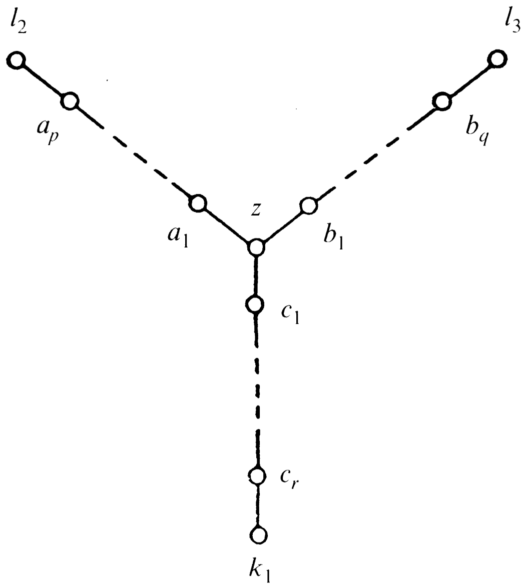

Figure 15. The union $W_{21} \cup W_{31}$.

for $i = 2, 3$. Let

$$A = \{a_1, a_2, \ldots, a_p\} \cup W_{23} \quad \text{and} \quad B = \{b_1, b_2, \ldots, b_q\} \cup W_{32}.$$

Now assume that W_{12} intersects A. By the above inequality (with $i = 2$) as well as by

$$d_H(l_2, k_2) > \delta(k_2) \geqq d_H(l_1, k_2)$$

we easily conclude that H has a cycle of length at least 4. A similar argument applies if W_{13} intersects B. Hence we may assume that W_{12} intersects B but not A and W_{13} intersects A but not B. Then l_1, z, the first vertex of W_{12} on B, and the first vertex of W_{13} on A lie on a cycle of length at least 4.

The proof of Theorem 2. To prove that each member of the variety of **Y**-graphs belongs to AR(P_2) it is enough, in view of Lemma 5, (with $j = 2$) to show that each **Y** $(i, j, k; l, m, n)$ is itself in AR(P_2). To this end, let **Y** $(i, j, k; l, m, n)$ be a subgraph of a graph G in which all triples of **Y** $(i, j, k; l, m, n)$ are separated. Since **Y** $(i, j, k; l, m, n)$ is connected, we may assume that G is connected as well. Therefore we can define, for each $g \in V(G)$,

$$s(g) = \min(i + 1, d_G(g, a)), \min(j + 1, d_G(g, b)), \min(k + 1, d_G(g, c)).$$

The mapping s is a retraction of G onto **Y** $(i, j, k; l, m, n)$. In particular, each vertex (x, y, z) of $\mathbf{Y}(i, j, k; l, m, n)$ is mapped to itself by s. Each vertex $g \in V(G)$ has at least one component of $s(g)$ attains its highest value, or else g would fill the triple $(a, b, c; i, j, k)$. Moreover, no $s(g) = (x, y, z)$ can have $x \leqq i, y \leqq j$ and $x + y < l$, because then the distance of x and y in G would be less than l (= distance in $\mathbf{Y}(i, j, k; l, m, n)$). Similarly, no $s(g) = (x, y, z)$ can have $|x - y| > l$ because of the triangle inequality for the distance in G. The other inequalities of II follow analogously. Hence s maps G onto $\mathbf{Y}(i, j, k; l, m, n)$. If g and g' are adjacent in G, then in $s(g) = (x, y, z)$, $s(g') = (x', y', z')$

$$|x - x'| \leqq 1, |y - y'| \leqq 1, \text{ and } |z - z'| \leqq 1,$$

and s is edge-preserving by the definition of $\Delta(i, j, k)$.

Conversely, assume that $H \in \mathrm{AR}(P_2)$. According to our remark on triples and holes, H is a retract of any G in which all 2-holes and 3-holes of H are separated. With Lemma 6 (for $M = \{2, 3\}$) in mind, we now find separating maps for all holes (K, δ), $|K| = 2$ or 3.

Consider the case of 3-holes first. A triple $(a, b, c; i, j, k)$ is a 3-hole of H if a, b, c are distinct and

$$d_H(a, b) = l \leqq i + j,$$
$$d_H(b, c) = m \leqq j + k,$$
$$d_H(a, c) = n \leqq i + k.$$

We shall define the separating map f of the 3-hole $(a, b, c; i, j, k)$, mapping H to $\mathbf{Y}(i, j, k; l, m, n)$ as follows: for $h \in V(H)$, let

$$f(h) = (\min(i + 1, d_H(h, a)), \min(j + 1, d_H(h, b)), \min(k + 1, d_H(h, c))).$$

Since $\mathbf{Y}(i, j, k; l, m, n)$ has no vertex (x, y, z) with $x \leqq i, y \leqq j, z \leqq k$, the triple $(a, b, c; i, j, k)$ of H remains separated in $\mathbf{Y}(i, j, k; l, m, n)$. It is also clear that f is edge-preserving. It remains to verify that each $f(h)$ is a vertex of $\mathbf{Y}(i, j, k; l, m, n)$. Because of the distances in H among a, b, and c (cited above), no $f(h) = (x, y, z)$ can have $x \leqq i, y \leqq j$, and $x + y < l$, or $y \leqq j, z \leqq k$, and $y + z < m$, or $x \leqq i, z \leqq k$, and $x + z < n$. Similarly, no $f(h) = (x, y, z)$ can satisfy $|x - y| > l$ and $x \leqq i$ or $x = i + 1 > y$, etc. because of the triangle inequality in H.

For the 2-holes we have already seen in the proof of Theorem 1 that each can be separated by a finite path, and $\mathbf{Y}(i, j, k; l, m, n)$ certainly contains an isometric path of length i. Therefore for each 2-hole of H there is a separating map in some $\mathbf{Y}(i, j, k; l, m, n)$. By Lemma 6 then H is a subgraph of a product G of $\mathbf{Y}$-graphs, and all triples of H are separated in G. Since $H \in \mathrm{AR}(P_2)$, H is a retract of G, i.e., H is in the variety of $\mathbf{Y}$-graphs.

The proof of Theorem 3. If H is an absolute retract, then H is a retract of $\hat{H}$ provided that each hole of H is separated in $\hat{H}$. Therefore, let (K, δ)

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be a hole of H and assume that

$$\bigcap_{k \in K} D_{\hat{H}}(K, \delta(k)) \neq \emptyset,$$

i.e., that some closed set $S \subseteq V(H)$ satisfies

$$d_{\hat{H}}(S, \{k\}) \leqq \delta(k) \quad \text{for all } k \in K.$$

It is easily seen by induction on $d_{\hat{H}}(T, T')$ that the distance in $\hat{H}$ satisfies

$$\sup\{ |\mathrm{ecc}_T(h) - \mathrm{ecc}_{T'}(h)| : h \in V(H)\} \leqq d_{\hat{H}}(T, T')$$

for any $T, T' \in V(\hat{H})$. Taking $h = k$, $T = S$, and $T' = \{k\}$, we have

$$\mathrm{ecc}_S(k) = |\mathrm{ecc}_S(k) - \mathrm{ecc}_{\{k\}}(k)| \leqq d_{\hat{H}}(S, \{k\}) \leqq \delta(k)$$

for all $k \in K$. Thus any $s \in S$ satisfies

$$d_H(s, k) \leqq \delta(k) \quad \text{for all } k \in K,$$

contrary to the assumption that (K, δ) is a hole of H.

To prove the second half of Theorem 3, assume that r is a retraction of $\hat{H}$ onto H, and that H is a subgraph of a graph G such that all holes of H are separated in G.

To each vertex g of G we associate the set

$$S(g) = \bigcap_{h \in V(H)} D_H(h, d_G(g, h)).$$

We shall now show that each $S(g)$ is a closed set in $V(H)$. Firstly, note that $S(g) \neq \emptyset$, or else a subset K of $V(H)$ of minimum cardinality, with

$$\bigcap_{k \in K} D_H(k, d_G(g, k)) = \emptyset,$$

would define a hole in H which is not separated in G (g fills it there). Next note that for each $k \in V(H)$ and each $s \in S(g)$

$$d_H(s, h) \leqq d_G(g, h)$$

and so

$$\mathrm{ecc}_{S(g)}(h) \leqq d_G(g, h)$$

for all $h \in V(H)$. Therefore

$$\bigcap_{h \in V(H)} D_H(h, \mathrm{ecc}_{S(g)}(h)) \subseteq \bigcap_{h \in V(H)} D_H(h, d_G(g, h))$$

$$= S(g) \subseteq \bigcap_{h \in V(H)} D_H(h, \mathrm{ecc}_{S(g)}(h)),$$

implying that $S(g)$ is a closed set of H, i.e., $S(g) \in V(\hat{H})$. It is now easy to verify that S is an edge-preserving map of G to $\hat{H}$ and that the composition $r \circ S$ is a retraction of G onto H.

The proof of Theorem 4. As we have already observed, each finite graph has the finite separation property (f.s.p.), i.e., belongs to $AR(P_4)$. Hence by Lemma 5 (with $j = 4$), each member of the variety of finite graphs also has the f.s.p.

Conversely, assume that $H \in AR(P_4)$, i.e., that H has the f.s.p. Let I enumerate all finite quotients $H_i = H/\theta_i$ of H, and let f_i be the associated edge-preserving map of H onto H_i. Let

$$G = \prod_{i \in I} H_i$$

and define a map f of $V(H)$ to $V(G)$ by

$$f(h) = (f_i(h))_{i \in I}.$$

It is easy to see that f is an injective edge-preserving map and that if h, h' are not adjacent in H then $f(h)$, $f(h')$ are not adjacent in G. Hence H is isomorphic to the subgraph of G induced by $f(V(H))$, and we shall simply assume that H is a subgraph of G by identifying $V(H)$ with $f(V(H))$. Since G is a product of finite graphs, we will be done if we can show that H is a retract of G. Therefore we assume that H is not a retract of G. Since H has the f.s.p., there is an equivalence relation θ on $V(H)$ such that $H/\theta = H'$ is finite and H/θ is not a retract of $G/\theta = G'$. Notice that $H' = H_i$ for some $i \in I$. We can now in fact define a retraction r of G' onto H' as follows: let $r(v) = v$ if $v \in V(H')$; if $v \in V(G') - V(H')$, then by the definition of G', v is a class $\{g\}$ for some $g \in V(G) - V(H)$ and we let $r(v) = \pi_i(g)$. Since $\pi_i \circ f = f_i$, the map r is edge-preserving, contrary to the fact that H' is not a retract of G'. Thus H is a retract of G, and hence a member of the variety of finite graphs.

Remarks and observations. Let H be a graph. We say that H has the *finite intersection property* (f.i.p.), if for any $K \subseteq V(H)$ and any map δ of K to the non-negative integers,

$$\bigcap_{k \in K} D_H(k, \delta(k)) = \emptyset$$

implies

$$\bigcap_{k \in K'} D_H(k, \delta(k)) = \emptyset$$

for some finite subset K' of K. In our terminology, f.i.p. amounts to requiring that no hole of H be infinite.

It is easy to verify, along the lines of Lemma 5, that the class of graphs satisfying the f.i.p. is a variety. It is also not hard to see, using a proof by contradiction and the trick illustrated in Figure 14 that the f.s.p. implies the f.i.p. It is tempting to conjecture that the two properties are equivalent, but this turns out not to be the case:

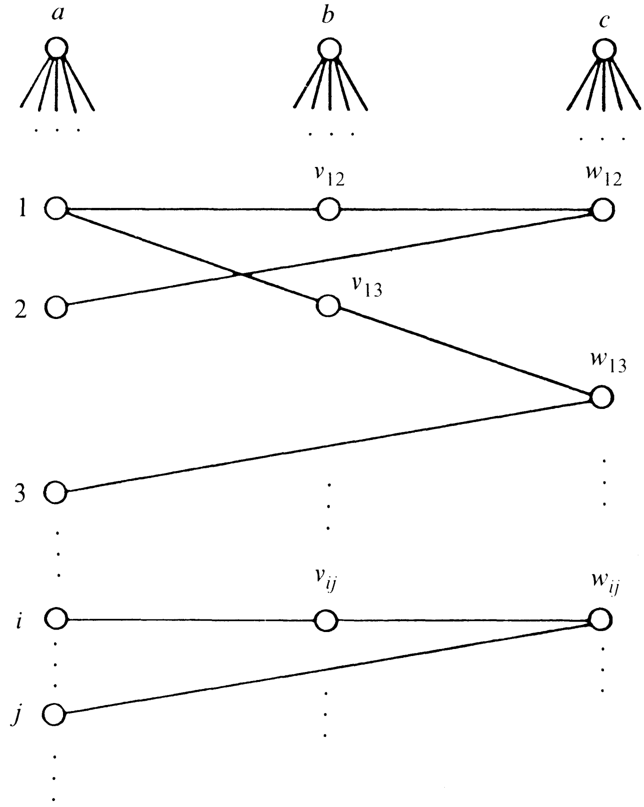

Figure 16. A graph H with the f.i.p. but without the f.s.p.

In H, the vertex a is adjacent to all vertices $1, 2, \ldots$, the vertex b is adjacent to all vertices v_{ij} $(i < j)$, and the vertex c to all vertices w_{ij} $(i < j)$. Moreover v_{ij} is adjacent to i, w_{ij} to j, and v_{ij} to w_{ij}, for each $i < j$. It is easy to check that H does not have the f.s.p. Indeed, form G by adjoining to H the triangle $a'b'c'$ with a' adjacent to a, b' to b, and c' to c. There is no retraction of G onto H because a' would have to map somewhere in the left column, b' in the centre column, and c' in the right column of H. But there is no triangle in H joining one vertex of each column. However, for any finite quotient H/θ of H such a triangle is formed and a retraction of G/θ onto H/θ becomes possible. The verification that H of Figure 16 satisfies the f.i.p. is somewhat more tedious, and we shall omit it.

As a last remark concerning infinite retracts, we note that a compactness argument can be used to prove the following: A finite graph H is a retract of a graph G if and only if H is a retract of each finite subgraph of G which contains H. Similar statements about retractions of graphs may be found in [**4**, **6**, **10**].

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