Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Tuya Sa, SCH.1.17, Schofield Building, Loughborough University, Loughborough, LE11 3TU. Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most elegant solutions for either problem. It is not necessary to submit solutions to both. Solutions should arrive by 13th May 2024 and will be published in the July 2024 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the Mathematical Association website

https:www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others***.

Problem 2024.1 **(Nick Lord)**

For positive real numbers α, β, γ, consider the inequality

$$\alpha p^2 + \beta q^2 > \gamma r^2. \qquad (*)$$

(a) Show that, if $\frac{1}{\gamma} \geqslant \frac{1}{\alpha} + \frac{1}{\beta}$, then (*) holds whenever p, q, r are the sides of a non-degenerate triangle.

(b) Is the converse of (a) true?

Problem 2024.2 **(Himadri Lal Das)**

For a positive integer n, $d(n)$ denotes the number of digits in n. Prove that

$$\frac{20}{19} + \frac{30}{29} + \dots + \frac{100}{99} \leqslant \sum_{n=1}^{\infty} \frac{10(d(n))^2 - 4d(n) + 3}{n^{d(n)-1}} \leqslant \frac{10}{9} + \frac{20}{19} + \dots + \frac{90}{89}.$$

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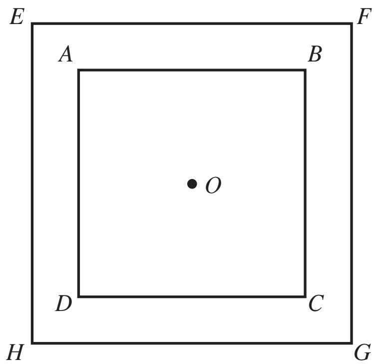
Solutions to 2023.5 and 2023.6

Both problems were solved by Emily Warren, Ella Bradbury, Geya Wang, and Pediredla Suhaas.

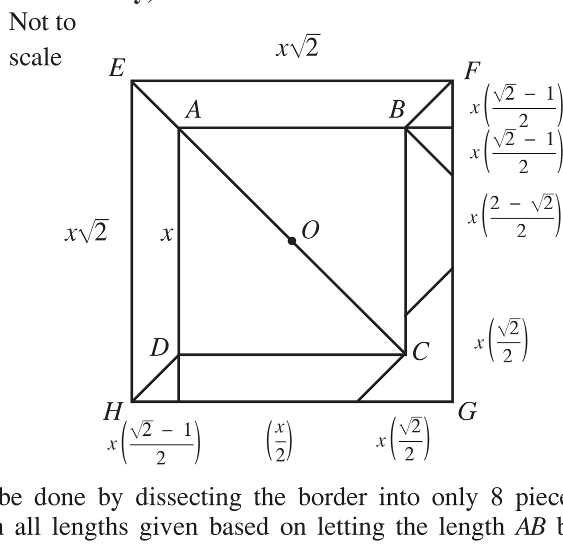
Problem 2023.5 **(Geoffrey Strickland)**



ABCD is a square with centre O. *EFGH* is an enlargement of *ABCD* with centre O, such that its area is twice that of *ABCD*.

Show how the border between the two squares may be dissected into no more than ten pieces which will fit into *ABCD*.

Solution (**Ella Bradbury**)



It can be done by dissecting the border into only 8 pieces as shown above, with all lengths given based on letting the length *AB* be x, and all angle cuts made such that the magnitude of gradients of angle sides created from them are always equal to 1.

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Problem 2023.6 **(Paul Stephenson)**

The n^{th} triangle number, $T_n = \frac{1}{2}n(n+1)$ for $n \geqslant 1$. Determine which triangle numbers cannot be represented as the difference of two squares.

Solution (**Emily Warren**)

For T_n to be represented as the difference of two squares, we have

$$T_n = (x + y)(x - y).$$

$(x + y)$ and $(x - y)$ must be two factors that multiply to give T_n. Their difference is $2y$, so they are either both even or both odd.

In the case where they are both odd, T_n has no even factors so $\frac{1}{2}n(n + 1) \neq 2p$ so $n(n + 1) \neq 4p$.

Neither n nor $(n + 1)$ are multiples of 4. (They are consecutive numbers, and one has no factors of 2.) This is only possible if $n = 4m + 1$ or $n = 4m + 2$. (This can also be written as $8m + 1$, $8m + 5$, $8m + 2$, $8m + 6$).

In the case where they are both even, T_n factorises into two even factors so T_n is a multiple of 4. Hence

$$\tfrac{1}{2}n(n + 1) = 4p \text{ and } n(n + 1) = 8p.$$

So n must be a multiple of 8. (These are consecutive numbers, so one has no factor of 2.)

Therefore we can say that T_n can be expressed as the difference of 2 squares if n is of the form:

$$8m,\ 8m + 1,\ 8m + 2,\ 8m + 5,\ 8m + 6 \text{ or } 8m + 7.$$

Therefore T_n can *not* be expressed as the difference of 2 squares when $n = 8m + 3$ or $n = 8m + 4$.

Prize Winners

The first prize of £25 is awarded to Emily Warren. The second prize of £20 is awarded to Ella Bradbury.

TUYA SA

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