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On Some Non-Riemannian Quantities in Finsler Geometry

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Abstract. In this paper we study several non-Riemannian quantities in Finsler geometry. These non-Riemannian quantities play an important role in understanding the geometric properties of Finsler metrics. In particular, we study a new non-Riemannian quantity defined by the S-curvature. We show some relationships among the flag curvature, the S-curvature, and the new non-Riemannian quantity.

1 Introduction

There are several non-Riemannian quantities in Finsler geometry, such as the distortion, the (mean) Cartan torsion, the S-curvature, the (mean) Berwald curvature, and the (mean) Landsberg curvature. We view the distortion and the (mean) Cartan torsion as non-Riemannian quantities of *order zero*, and the S-curvature, the (mean) Berwald curvature, and the (mean) Landsberg curvature as non-Riemannian quantities of *order one*. Differentiating these quantities along geodesics, we obtain some non-Riemannian quantities of *order two*.

Let F be a Finsler metric on an n-dimensional manifold M. In this paper we will consider two non-Riemannian quantities $\Xi = \Xi_i dx^i$ and $H = H_{ij} dx^i \otimes dx^j$ on the tangent bundle TM:

$$\Xi_i := \mathbf{S}_{\cdot i|m} y^m - \mathbf{S}_{|i}, \tag{1.1}$$

$$H_{ij} := \frac{1}{2} \mathbf{S}_{\cdot i \cdot j|m} y^m, \tag{1.2}$$

where $\mathbf{S}$ denotes the S-curvature of F, and "$\cdot$" and "$|$" denote the vertical and horizontal covariant derivatives, respectively, with respect to the Chern connection. We shall prove that H can be expressed by Ξ directly (Lemma 2.1).

One of the fundamental problems in Finsler geometry is to understand Finsler metrics of special curvature properties. We would like to investigate the following three classes of Finsler metrics with special non-Riemannian curvature properties:

(i) Almost isotropic S-curvature:

$$\mathbf{S} = (n+1)cF + \eta, \tag{1.3}$$

where $c = c(x)$ is a scalar function and η is a 1-form on M with $d\eta = 0$,

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(ii) Almost vanishing Ξ-curvature:

$$\Xi_i = -(n+1)F^2\Big(\frac{\theta}{F}\Big)_{y^i}, \tag{1.4}$$

where $\theta = a_i(x)y^i$ is a 1-form on M,

(iii) Almost vanishing H-curvature:

$$H_{ij} = \frac{n+1}{2}\theta F_{y^i y^j}, \tag{1.5}$$

where $\theta = a_i(x)y^i$ is a 1-form on M.

By (1.1) and (1.2), one can easily show that (1.3) implies (1.4) and (1.5) with $\theta = c_{x^m}(x)y^m$. However, the converse is not true. There are Finsler metrics with $\Xi = 0$ and $H = 0$, but the S-curvature is not almost isotropic. See Example 1.1. By Lemma 2.1, $H = \frac{1}{2}(\Xi_{i\cdot j} + \Xi_{j\cdot i})$, one can see that (1.4) implies (1.5), but the converse might not be true.

We also would like to investigate Finsler metrics with special Riemannian curvature properties. In particular, we consider Finsler metrics of almost isotropic flag curvature defined as follows,

$$\mathbf{K} = \frac{3\theta}{F} + \sigma, \tag{1.6}$$

where $\sigma = \sigma(x)$ is a scalar function and $\theta = a_i(x)y^m$ is a 1-form on M.

The non-Riemannian quantities $\mathbf{S}$, Ξ, and H are closely related to the flag curvature. First we have the following known results.

Theorem 1.1 ([4, 12]) *Let F be a Finsler metric of scalar flag curvature on an n-dimensional manifold M.*

(i) *If $\mathbf{S}$ is almost isotropic, given by* (1.3)*, then the flag curvature is almost isotropic, given by* (1.6) *with $\theta = c_{x^m}(x)y^m$.*
(ii) *For a* 1*-form θ, H almost vanishes, given by* (1.5)*, if and only if the flag curvature is almost isotropic, given by* (1.6)*. In particular, $H = 0$ if and only if $\mathbf{K} = \sigma$ (constant when $n \geq 3$).*

It is shown that every Randers metric of almost isotropic flag curvature must be of almost isotropic S-curvature ([16]). But this is not true for general Finsler metrics. See Example 1.1.

In this paper, we shall prove the following theorem.

Theorem 1.2 *Let F be a Finsler metric of scalar flag curvature on an n-dimensional manifold M. Then for a* 1*-form θ on M, Ξ almost vanishes, given by* (1.4)*, if and only if $\mathbf{K}$ is almost isotropic, given by* (1.6)*. In particular, $\Xi = 0$ if and only if $\mathbf{K} = \sigma$ (constant when $n \geq 3$).*

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According to a theorem by Akbar-Zadeh, every Finsler metric of constant flag curvature on a *compact* manifold M must be Riemannian if $\mathbf{K} = \sigma < 0$ [1]. Thus for a Finsler metric of scalar flag curvature on a compact manifold with $\Xi = 0$, if $\mathbf{K} < 0$, then it must be Riemannian. In fact, this is true under a weaker condition on the flag curvature.

Theorem 1.3 *Let (M, F) be a compact Finsler manifold with $\Xi = 0$. If the flag curvature is negative, then it must be Riemannian.*

The condition $\Xi = 0$ cannot be dropped in Theorem 1.3. Take an arbitrary compact Riemannian manifold (M, α) of negative constant curvature and an arbitrary smooth function f on M. Consider a Randers metric $F = \alpha + \varepsilon df$ with sufficiently small number ε. F has negative flag curvature, but it is not Riemannian unless $\varepsilon = 0$. Thus the condition $\Xi = 0$ cannot be dropped in Theorem 1.3. There are Randers metrics on S^n with positive constant curvature and $\Xi = 0$ [13] [3]. Thus the condition $\mathbf{K} < 0$ cannot be dropped in Theorem 1.3. We do not know whether or not Theorem 1.3 is still true if the non-Riemannian condition $\Xi = 0$ is replaced by $H = 0$.

We go back to discussing the relationship between the S-curvature $\mathbf{S}$ and the Ξ-curvature. As we have shown above, if the S-curvature is almost isotropic, $\mathbf{S} = (n+1)cF + \eta$, then Ξ satisfies (1.4) with $\theta = c_{x^m}(x)y^m$. The converse might not be true in general (Example 1.1). However, for Randers metrics, they are equivalent. More generally, we have the following theorem.

Theorem 1.4 *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. The S-curvature is almost isotropic, given by* (1.3)*, if and only if the non-Riemannian quantity Ξ almost vanishes, given by* (1.4)*. In particular, $\mathbf{S} = (n+1)cF$ for some constant c if and only if $\Xi = 0$.*

We do not know whether or not Theorem 1.4 is still true for the non-Riemannian quantity H.

Example 1.1 Let $F = (\alpha + \beta)^2/\alpha$, where

$$\alpha := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{(1 - |x|^2)^2}, \quad \beta := \frac{\langle x, y\rangle}{(1 - |x|^2)^2}.$$

F is a projectively flat metric on the unit ball $B^n(1) \subset R^n$ with $\mathbf{K} = 0$. By Theorem 1.1(ii), we see that $H = 0$. By Theorem 1.2, we see that $\Xi = 0$. F is an (α, β)-metric. In [6], we classified (α, β)-metrics of isotropic S-curvature. By [6, Theorem 1.2], we can see that F is not of isotropic S-curvature. Actually, by a direct computation, we can verify that $\mathbf{S}$ is not almost isotropic.

2 Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M. It induces a spray G on TM. In local coordinates in TM, it is expressed by

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = \frac{1}{4}g^{il}\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\}$. Geodesics in M are just the projections of integral curves of G. Put

$$\Pi = \frac{\partial G^m}{\partial y^m}.$$

This is an important local quantity. Note that Π is a local scalar function that depends on the choice of a particular coordinate system.

When F is a Berwald metric, namely, $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in y, then $\Pi = \Gamma^m_{jm}y^m$ is a local exact 1-form. In fact, by a theorem of Szabo, there is a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ such that the spray coefficients G^i of F coincide the spray coefficients $\bar{G}^i = \frac{1}{2}\bar{\Gamma}^i_{jk}(x)y^j y^k$ of α. Then we have

$$\Pi = \Gamma^m_{jm}y^j = \bar{\Gamma}^m_{jm}y^j = y^m \frac{\partial}{\partial x^m}\Big(\ln\sqrt{\det(a_{ij})}\Big).$$

Therefore, Π is actually a local exact 1-form.

Let $dV = f dx^1 \cdots dx^n$ be a volume form on M. The S-curvature of (F, dV) is given by

$$\mathbf{S} = \Pi - y^m \frac{\partial}{\partial x^m}\big(\ln f\big).$$

This is a well-defined geometric quantity ([14]). If $dV = dV_F$ is the Busemann–Hausdorff volume form, the corresponding S-curvature is called the S-curvature of F. Note that if the S-curvature is almost isotropic with respect to one volume form, then it is almost isotropic with respect to any volume form.

Lemma 2.1

$$H_{ij} = \frac{1}{4}\big\{\Xi_{i\cdot j} + \Xi_{j\cdot i}\big\}. \tag{2.1}$$

Proof By (1.1) and (1.2), we can express Ξ_i and H_{ij} by

$$\Xi_i = \Pi_{y^i x^m}y^m - \Pi_{x^i} - 2\Pi_{y^i y^m}G^m. \tag{2.2}$$

$$H_{ij} = \frac{1}{2}\Big\{\Pi_{y^i y^j x^m}y^m - 2\Pi_{y^i y^j y^m}G^m - \Pi_{y^j y^m}\frac{\partial G^m}{\partial y^i} - \Pi_{y^i y^m}\frac{\partial G^m}{\partial y^j}\Big\}. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$H_{ij} = \frac{1}{4}\Big\{\frac{\partial \Xi_i}{\partial y^j} + \frac{\partial \Xi_j}{\partial y^i}\Big\}. \tag{2.4}$$

Since $\Xi_{i\cdot j} = \frac{\partial \Xi_i}{\partial y^j}$, we get (2.1). ■

The quantity $\Xi = \Xi_i dx^i$ can be expressed in terms of the Riemannian curvature $R = R^i{}_k \frac{\delta}{\delta x^i} \otimes dx^k$ or the mean Cartan torsion $I = I_i dx^i$. The following lemma is well known.

Lemma 2.2 ([4, 8, 9, 11])

$$\Xi_i = -\tfrac{1}{3}\{2R^m_{\ i\cdot m} + R^m_{\ m\cdot i}\} = I_{i|p|q}y^p y^q + I_m R^m_{\ i}. \tag{2.5}$$

By Lemma 2.2, we immediately obtain two corollaries.

Corollary 2.3 ([10]) *For any R-quadratic Finsler metric, there is a two-form $\xi = \xi_{ij}(x)dx^i \wedge dx^j$ such that $\Xi_i = \xi_{ij}y^j$. Hence $H_{ij} = 0$.*

Proof Assume that F is R-quadratic, namely, $R^i_{\ k} = R_{j\ kl}^{\ i} y^j y^l$, where $R_{j\ kl}^{\ i} = R_{j\ kl}^{\ i}(x)$ denotes the hh-curvature of the Berwald connection, which depends only on the position $x \in M$.

We have

$$R^m_{\ i\cdot m} = R_{m\ il}^{\ m} y^l + R_{j\ im}^{\ m} y^j$$
$$R^m_{\ m\cdot i} = R_{i\ ml}^{\ m} y^l + R_{j\ mi}^{\ m} y^j.$$

Thus

$$\Xi_i = -\tfrac{1}{3}\{2R_{m\ il}^{\ m} + 2R_{l\ im}^{\ m} + R_{i\ ml}^{\ m} + R_{l\ mi}^{\ m}\}y^l$$

By the Bianchi identities, we get $\Xi_i = R_{m\ li}^{\ m} y^l$. Note that $\xi_{ij} := R_{m\ ij}^{\ m}(x)$ is anti-symmetric in i and j, *i.e.*, $\xi_{ij} + \xi_{ji} = 0$. Thus $\xi := \xi_{ij}dx^i \wedge dx^j$ is a two-form on M. By (2.4), we see that $H_{ij} = 0$. ■

The fact that $H = 0$ for all R-quadratic Finsler metrics is due to X. Mo [10].

Corollary 2.4 *Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold. Suppose that F is of scalar flag curvature $\mathbf{K} = K(x, y)$. Then*

$$\Xi_i = -\frac{n+1}{3}F^2 K_{\cdot i}. \tag{2.6}$$

Proof Suppose that F is of scalar flag curvature $\mathbf{K} = K(x, y)$. Then

$$R^m_{\ i} = K\{F^2\delta^m_i - g_{ip}y^p y^m\}.$$

Differentiating $R^m_{\ i}$, we get

$$R^m_{\ i\cdot m} = F^2 K_{\cdot i} - (n-1)Kg_{ip}y^p$$
$$R^m_{\ m\cdot i} = (n-1)F^2 K_{\cdot i} + 2(n-1)Kg_{ip}y^p.$$

Thus

$$2R^m_{\ i\cdot m} + R^m_{\ m\cdot i} = (n+1)F^2 K_{\cdot i}.$$

Plugging it into (2.5) we obtain $\Xi_i = -\frac{n+1}{3}F^2 K_{\cdot i}$. ■

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3 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2 This follows from Corollary 2.4 directly. We can rewrite (2.6) as follows:

$$\Xi_i + (n+1)F^2\Big(\frac{\theta}{F}\Big)_{\cdot i} = -\frac{n+1}{3}F^2\Big(K - \frac{3\theta}{F}\Big)_{\cdot i},$$

where θ is an arbitrary 1-form on M. Thus (1.4) holds if and only if (1.6) holds for some scalar function $\sigma = \sigma(x)$. ■

Proof of Theorem 1.3 The argument is similar to the proof of the main theorem in [15]. By Deicke's theorem, it suffices to prove that the mean Cartan torsion vanishes.

By assumption $\Xi = 0$. It follows from (2.5) that

$$I_{i|p|q}y^p y^q + I_m R^m_{\ i} = 0. \tag{3.1}$$

For a vector $y \in T_xM$, let $\mathbf{I}_y \in T_xM$ be defined by $g_y(\mathbf{I}_y, v) = I_i(x,y)v^i$. Let $\sigma = \sigma(t)$ be an arbitrary geodesic. Since F is complete, one may assume that σ is defined on $(-\infty, \infty)$. Let $\mathbf{I}(t) := \mathbf{I}_{\dot\sigma(t)}$. Equation (3.1) restricted to $\sigma(t)$ becomes

$$\mathrm{D}_{\dot\sigma}\mathrm{D}_{\dot\sigma}\mathbf{I}(t) + \mathbf{R}_{\dot\sigma(t)}(\mathbf{I}(t)) = 0. \tag{3.2}$$

Thus, the mean Cartan torsion is a Jacobi field along any geodesic. Let

$$\varphi(t) := g_{\dot\sigma(t)}\big(\mathbf{I}(t), \mathbf{I}(t)\big).$$

It follows from (3.2) that

$$\begin{aligned} \varphi''(t) &= 2g_{\dot\sigma(t)}\big(\mathrm{D}_{\dot\sigma}\mathrm{D}_{\dot\sigma}\mathbf{I}(t), \mathbf{I}(t)\big) + 2g_{\dot\sigma(t)}\big(\mathrm{D}_{\dot\sigma}\mathbf{I}(t), \mathrm{D}_{\dot\sigma}\mathbf{I}(t)\big) \\ &= -2g_{\dot\sigma(t)}\big(\mathbf{R}_{\dot\sigma(t)}(\mathbf{I}(t)), \mathbf{I}(t)\big) + 2g_{\dot\sigma(t)}\big(\mathrm{D}_{\dot\sigma}\mathbf{I}(t), \mathrm{D}_{\dot\sigma}\mathbf{I}(t)\big). \end{aligned} \tag{3.3}$$

By assumption, $\mathbf{K} < 0$. Thus $g_{\dot\sigma(t)}(\mathbf{R}_{\dot\sigma(t)}(\mathbf{I}(t)), \mathbf{I}(t)) \le 0$. It follows from (3.3) that $\varphi''(t) \ge 0$. Thus $\varphi(t)$ is convex and nonnegative. Suppose that $\varphi'(t_o) \ne 0$ for some t_o. If $\varphi'(t_o) < 0$, then

$$\varphi(t) \ge \varphi(t_o) - \varphi'(t_o)(t_o - t), \quad t < t_o.$$

If $\varphi'(t_o) > 0$, then

$$\varphi(t) \ge \varphi(t_o) + \varphi'(t_o)(t - t_o), \quad t > t_o.$$

One can see that $\lim_{t\to+\infty}\varphi(t) = \infty$ or $\lim_{t\to-\infty}\varphi(t) = \infty$. This implies that the mean Cartan torsion is unbounded, which contradicts the assumption. Therefore, $\varphi'(t) = 0$ and hence $\varphi''(t) = 0$. It follows from (3.3) that $\mathbf{R}_{\dot\sigma(t)}(\mathbf{I}(t)) = 0$. Since σ is arbitrary, one can conclude that $\mathbf{R}_y(\mathbf{I}_y) = 0$. Since $\mathbf{K} < 0$, we conclude that $\mathbf{I}_y = 0$. By Deicke's theorem [2], F is Riemannian. ■

4 Randers Metrics

Consider a Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Let $\nabla\beta = b_{i;j}y^i dx^j$ denote the covariant derivative of β with respect to α. Let

$$r_{ij} := \tfrac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \tfrac{1}{2}(b_{i;j} - b_{j;i}), \quad s_j := b^i s_{ij},$$
$$e_{ij} := r_{ij} + b_i s_j + b_j s_i, \quad e_j := b^i e_{ij}.$$
$$q_{ij} := r_{im}s^m{}_j, \quad t_{ij} := s_{im}s^m{}_j,$$
$$q_j := b^i q_{ij} = r_m s^m{}_j, \quad t_j := b^i t_{ij} = s_m s^m{}_j$$
$$w_{ij} := q_{ij} + b_i t_j + s_i s_j.$$

Here and hereafter, we use a_{ij} to raise and lower the indices of tensors defined by b_i and $b_{i;j}$. We shall also denote $y_i := a_{ij}y^j$. The index "0" means the contraction with y^i. For example, $e_{k0} := e_{kl}y^l$, $e_{00} := e_{kl}y^k y^l$, and so on.

The following lemma is known.

Lemma 4.1 ([5]) *For a Randers metric $F = \alpha + \beta$, the following are equivalent:*

(i) $\mathbf{S} = (n+1)cF$;
(ii) $e_{00} = 2c(\alpha^2 - \beta^2)$.

To prove Theorem 1.4 it suffices to prove that $\Xi_i = -(n+1)\{\theta_{y^i}F - \theta F_{y^i}\}$ implies that $e_{00} = 2c(\alpha^2 - \beta^2)$.

The spray coefficients of F are given by

$$G^i = \bar{G}^i + H^i, \tag{4.1}$$

where

$$H^i := Py^i + \alpha s^i{}_0, \quad P = \frac{e_{00}}{2F} - s_0.$$

Let $H := [H^p]_{y^p}$. We have

$$H := [H^m]_{y^m} = [Py^m + \alpha s^m{}_0]_{y^m} = (n+1)P.$$

Plugging the formula (4.1) into (2.2), we obtain

$$\Xi_i = H_{\cdot i;m}y^m - H_{;i} - 2H_{\cdot i\cdot m}H^m = (n+1)\Big\{P_{\cdot i;m}y^m - P_{;i} - 2P_{\cdot i\cdot m}H^m\Big\},$$

where "$\cdot$ " and ";" denote the vertical and horizontal covariant differentiations with respect to α, respectively.

We have

$$
\begin{aligned}
P_{;i} &= \frac{1}{2(\alpha+\beta)^2}\Big\{ e_{00;i}\alpha + e_{00;i}\beta - \beta_{;i}e_{00}\Big\} - s_{0;i},\\
P_{\cdot i;m}y^m &= -\frac{1}{2\alpha(\alpha+\beta)^3}\Big\{ -2\alpha(\alpha+\beta)^2 e_{0i;0} + 2\alpha(\alpha+\beta)\beta_{;0}e_{0i} - 2\beta_{;0}e_{00}(y_i+\alpha b_i)\\
&\qquad + (\alpha+\beta)e_{00;0}y_i + \alpha(\alpha+\beta)e_{00;0}b_i + \alpha(\alpha+\beta)e_{00}b_{i;0}\Big\} - s_{i;0},\\
P_{\cdot i\cdot m}H^m &= \frac{1}{2(\alpha+\beta)^3}\Big\{ 2\alpha(\alpha+\beta)^2 w_{i0} - 2w_{00}(\alpha+\beta)(y_i+\alpha b_i)\\
&\qquad - 2\alpha(\alpha+\beta)s_0e_{0i} + 2e_{00}s_0(y_i+\alpha b_i) - (\alpha+\beta)e_{00}s_{i0}\Big\}.
\end{aligned}
$$

Note that

$$
b_{i;0} = e_{0i} + s_{i0} - s_i\beta - b_is_0, \quad \beta_{;i} = e_{0i} - s_{i0} - b_is_0 - s_{i0}.
$$

We obtain the following formula for Ξ:

$$
\alpha(\alpha+\beta)^3\Xi_i = (n+1)\Big\{ S_{i\,\text{even}} + S_{i\,\text{odd}}\alpha\Big\},
$$

where

$$
\begin{aligned}
S_{i\,\text{even}} &:= (\alpha^2+3\beta^2)(e_{i;0}-e_{0;i})\alpha^2\\
&\quad + \Big\{ 2(w_{00}y_i - w_{i0}\alpha^2) + 2(w_{00}b_i - w_{i0}\beta)\beta\Big\}\alpha^2\\
&\quad + \Big\{ e_{0i;0}\alpha^2 - \tfrac{1}{2}e_{00;0}y_i - \tfrac{1}{2}e_{00;i}\alpha^2\Big\}\beta + \Big\{ e_{0i;0}\beta - \tfrac{1}{2}e_{00;0}b_i - \tfrac{1}{2}e_{00;i}\beta\Big\}\alpha^2\\
&\quad - 2s_0(e_{00}b_i - e_{0i}\beta)\alpha^2 + (e_{00}-2s_0\beta)(e_{00}y_i - e_{0i}\alpha^2),\\
S_{i\,\text{odd}} &:= (3\alpha^2+\beta^2)(e_{i;0}-e_{0;i})\beta\\
&\quad + 2(w_{00}y_i - w_{i0}\alpha^2)\beta + 2(w_{00}b_i - w_{i0}\beta)\alpha^2\\
&\quad + \Big\{ e_{0i;0}\alpha^2 - \tfrac{1}{2}e_{00;0}y_i - \tfrac{1}{2}e_{00;i}\alpha^2\Big\} + \Big\{ e_{0i;0}\beta - \tfrac{1}{2}e_{00;0}b_i - \tfrac{1}{2}e_{00;i}\beta\Big\}\beta\\
&\quad + (e_{00}-2s_0\beta)(e_{00}b_i - e_{0i}\beta) - 2s_0(e_{00}y_i - e_{0i}\alpha^2).
\end{aligned}
$$

Then

$$
\begin{aligned}
S_{i\,\text{odd}}\alpha^2 - S_{i\,\text{even}}\beta &= (\alpha^2-\beta^2)\Big\{ 2(e_{i;0}-e_{0;i})\alpha^2\beta + 2(w_{00}b_i - w_{i0}\beta)\alpha^2\\
&\qquad + (e_{0i;0}\alpha^2 - \tfrac{1}{2}e_{00;0}y_i - \tfrac{1}{2}e_{00;i}\alpha^2) - 2s_0(e_{00}y_i - e_{0i}\alpha^2)\Big\}\\
&\qquad + e_{00}^2(\alpha^2 b_i - \beta y_i).
\end{aligned}
$$

We assume that $\Xi_i = -(n+1)F^2\big(\frac{\theta}{F}\big)_{\cdot i}$. Then

$$
\alpha(\alpha+\beta)^3\Xi_i = (n+1)\Big\{ T_{i\,\text{even}} + T_{i\,\text{odd}}\alpha\Big\},
$$

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where

$$T_{i\,\mathrm{even}} = (3\alpha^2 + \beta^2)\beta(\theta y_i - \theta_i\alpha^2) + (\alpha^2 + 3\beta^2)\alpha^2(\theta b_i - \theta_i\beta)$$
$$T_{i\,\mathrm{odd}} = (\alpha^2 + 3\beta^2)(\theta y_i - \theta_i\alpha^2) + (3\alpha^2 + \beta^2)\beta(\theta b_i - \theta_i\beta).$$

Thus

$$T_{i\,\mathrm{odd}}\alpha^2 - T_{i\,\mathrm{even}}\beta = (\alpha^2 - \beta^2)\Big\{ (\alpha^2 + \beta^2)(\theta y_i - \theta_i\alpha^2) + 2\alpha^2\beta(\theta b_i - \theta_i\beta)\Big\}.$$

We conclude that

$$e_{00}^2(\alpha^2 b_i - \beta y_i) = (\alpha^2 - \beta^2)M_i, \tag{4.2}$$

where M_i is a homogeneous polynomial of degree four with $M_i y^i = 0$. Contracting (4.2) with $b^i = a^{ij}b_j$ yields

$$e_{00}^2(\alpha^2 b^2 - \beta^2) = (\alpha^2 - \beta^2)M_i b^i,$$

where $b := \|\beta_x\|_\alpha < 1$. There is no common factor in $(\alpha^2 b^2 - \beta^2)$ and $\alpha^2 - \beta^2$. Thus e_{00}^2 is divisible by $\alpha^2 - \beta^2$. Since $\alpha^2 - \beta^2$ is irreducible, e_{00} must be divisible by $\alpha^2 - \beta^2$. Therefore there is a scalar function $c = c(x)$ such that

$$e_{00} = 2c(\alpha^2 - \beta^2).$$

This proves Theorem 1.4.

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