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A COMPACT IMBEDDING THEOREM FOR FUNCTIONS WITHOUT COMPACT SUPPORT

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The extension of the Rellich–Kondrachov theorem on the complete continuity of Sobolev space imbeddings of the sort

$$W_0^{m,p}(G) \to L^p(G) \tag{1}$$

to unbounded domains G has recently been under study [1–5] and this study has yielded [4] a condition on G which is necessary and sufficient for the compactness of (1). Similar compactness theorems for the imbeddings

$$W^{m,p}(G) \to L^p(G) \tag{2}$$

are well known for bounded domains G with suitably regular boundaries, and the question naturally arises whether any extensions to unbounded domains can be made in this case. Here G is an open domain in Euclidean n-space E_n, and, as usual, $W^{m,p}(G)$ [respectively $W_0^{m,p}(G)$] denotes for m a positive integer and $p \geq 1$ real the completion of the space of infinitely differentiable functions on G for which the norm below is finite [resp. the space of infinitely differentiable functions with compact support in G] with respect to the norm $\| \ \|_{m,p,G}$ defined by

$$\|u\|_{m,p,G}^p = \sum_{0 \leq |\alpha| \leq m} \int_G |D^\alpha u(x)|^p \, dx$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers; $|\alpha| = \sum \alpha_i$; $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$; $D_j = \partial/\partial x_j$.

The vanishing, in a generalized sense, on the boundary of G of elements of $W_0^{m,p}(G)$ plays a critical role in the establishment of the complete continuity of (1) for unbounded domains. For elements of $W^{m,p}(G)$ we no longer have this property and one might be led to expect that (2) cannot be compact for any unbounded G. For example, if G is the union of infinitely many balls B_j ($j = 1, 2, \ldots$) with pairwise disjoint closures then the sequence $\{u_j\}$ defined by

$$u_j(x) = \begin{cases} 0 & \text{if } x \notin \overline{B_j} \\ (\text{vol. } B_j)^{-1/p} & \text{if } x \in \overline{B_j} \end{cases}$$

is clearly bounded in $W^{m,p}(G)$ but not precompact in $L^p(G)$ no matter how rapidly

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the radius of B_j tends to zero as j tends to infinity. (As long as the radius of B_j tends to zero the imbedding (2) is compact by a theorem of [2].)

In the remainder of this paper we shall consider a very restricted class of unbounded domains G in E_2 for which a condition necessary and sufficient for the complete continuity of the imbedding

$$W^{1,2}(G) \to L^2(G) \tag{3}$$

can be given. In particular, therefore, there do exist extensions of the Rellich–Kondrachov theorem to unbounded domains for imbeddings of type (2).

DEFINITION. Hereafter f shall denote a positive, decreasing, continuously differentiable function on $[0, \infty)$ with bounded derivative f' and which satisfies $\int_0^\infty f(x)\,dx < \infty$. G shall denote the domain in E_2 bounded by the coordinate axes and the curve $y = f(x)$. For $R \geq 0$ we set $G_R = \{(x, y) \in G : x \geq R\}$ and $K_R = G - G_R$.

REMARK. C. Clark has shown in [5] that the imbedding (3) is not compact for a domain G of the above type but for which $\int_0^\infty f(x)\,dx = \infty$. The imbedding $W_0^{1,2}(G) \to L^2(G)$ is compact in this case provided only that $f(x) \to 0$ as $x \to \infty$.

Our principal result is the following

THEOREM 1. *The imbedding* $W^{1,2}(G) \to L^2(G)$ *is completely continuous if and only if*

$$\int_R^\infty f(x)\,dx = o(f(R)) \quad \text{as } R \to \infty. \tag{4}$$

EXAMPLE. $f(x) = e^{-x^2}$ satisfies (4) while $f(x) = e^{-x}$ does not. Condition (4) asserts that the half-life of f tends to zero as $x \to \infty$. In fact we require the following

LEMMA 1. *Condition* (4) *is satisfied if and only if for every* $\varepsilon > 0$, $f(R+\varepsilon) = o(f(R))$ as $R \to \infty$.

Proof. (a) Assume $f(R+\varepsilon) = o(f(R))$ as $R \to \infty$, for every positive ε. For such ε there exists R_0 such that if $x \geq R_0$ then $f(x+\varepsilon) \leq \frac{1}{2} f(x)$. Using the monotonicity of f we obtain for $R \geq R_0$

$$\begin{aligned} \int_R^\infty f(x)\,dx &= \sum_{m=0}^\infty \int_{R+m\varepsilon}^{R+(m+1)\varepsilon} f(x)\,dx \\ &\leq \varepsilon \sum_{m=0}^\infty f(R+m\varepsilon) \leq \varepsilon f(R) \sum_{m=0}^\infty 1/2^m = 2\varepsilon f(R) \end{aligned}$$

whence f satisfies condition (4).

(b) Conversely, suppose f satisfies condition (4). Let $\varepsilon, \delta > 0$. There exists R_0 such that if $R \geq R_0$ then $\int_R^\infty f(x)\,dx < \varepsilon\delta f(R)$. But then by the monotonicity of f we have for $R \geq R_0$

$$\varepsilon f(R+\varepsilon) \leq \int_R^{R+\varepsilon} f(x)\,dx \leq \int_R^\infty f(x)\,dx < \varepsilon\delta f(R)$$

whence $f(R+\varepsilon) = o(f(R))$ as $R \to \infty$.

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LEMMA 2. *If* $R \geq 1$ *then for all* $\psi \in C^1([0, \infty))$ *we have*

$$\left|\int_R^\infty f(x)\psi(x)\,dx\right| \leq \delta(R)\left\{\int_0^\infty f(x)|\psi(x)|\,dx+\int_0^\infty f(x)|\psi'(x)|\,dx\right\} \tag{5}$$

where

$$\delta(R) = \sup_{S\geq R}\frac{1}{f(S)}\int_S^\infty f(x)\,dx.$$

Proof. For x, $\xi > 0$ we have

$$\psi(x) = \psi(\xi)+\int_\xi^x \psi'(t)\,dt.$$

Multiplication by $f(x)$ and integration first with respect to x over $[R, \infty)$ and then with respect to ξ over $[0, R]$ yields

$$\begin{aligned} R\int_R^\infty f(x)\psi(x)\,dx &= \int_R^\infty f(x)\,dx\int_0^R \psi(\xi)\,d\xi+\int_0^R d\xi\int_R^\infty f(x)\,dx\int_\xi^x \psi'(t)\,dt \\ &= \int_R^\infty f(x)\,dx\int_0^R \psi(\xi)\,d\xi+\int_0^R d\xi\left\{\int_\xi^R \psi'(t)\,dt\int_R^\infty f(x)\,dx\right. \\ &\quad \left.+\int_R^\infty \psi'(t)\,dt\int_t^\infty f(x)\,dx\right\}. \end{aligned}$$

Making use of the definition of δ and the monotonicity of f we now obtain

$$\begin{aligned} R\left|\int_R^\infty f(x)\psi(x)\,dx\right| &\leq \delta(R)f(R)\int_0^R |\psi(\xi)|\,d\xi \\ &\quad +\int_0^R d\xi\left\{\delta(R)f(R)\int_\xi^R |\psi'(t)|\,dt+\delta(R)\int_R^\infty f(t)|\psi'(t)|\,dt\right\} \\ &\leq \delta(R)\int_0^\infty f(\xi)|\psi(\xi)|\,d\xi+R\delta(R)\int_0^\infty f(t)|\psi'(t)|\,dt \end{aligned}$$

whence follows the Lemma.

LEMMA 3. *With* δ *defined as in Lemma* 2 *there exists a constant C such that for all* $u \in W^{1,2}(G)$ *we have*

$$\|u\|^2_{0,2,G_R} \leq C\delta(R)\|u\|^2_{1,2,G} \tag{6}$$

Proof. Without loss of generality we assume all functions are real-valued. Let $\phi \in C^1(G)$ have finite $W^{1,2}$-norm and let $\psi \in C^1([0, \infty))$ be defined by

$$\psi(x) = \frac{1}{f(x)}\int_0^{f(x)} (\phi(x, y))^2\,dy,$$

Then clearly

$$\left|\int_R^\infty f(x)\psi(x)\,dx\right| = \|\phi\|^2_{0,2,G_R}$$

$$\int_0^\infty f(x)|\psi(x)|\,dx = \|\phi\|^2_{0,2,G}$$

Moreover,

$$\psi'(t) = \frac{1}{[f(t)]^2}\left\{f(t)f'(t)[\phi(t,f(t))]^2+f(t)\int_0^{f(t)} 2\phi(t,y)\,D_1\phi(t,y)\,dy\right.$$
$$\left.-f'(t)\int_0^{f(t)}[\phi(t,y)]^2\,dy\right\}$$

whence we have, since f' is assumed bounded on $[0,\infty)$,

$$\begin{aligned} f(t)|\psi'(t)| &\le 2\int_0^{f(t)}|\phi(t,y)|\,|D_1\phi(t,y)|\,dy \\ &\quad +\text{const.}\left|[\phi(t,f(t))]^2-\frac{1}{f(t)}\int_0^{f(t)}[\phi(t,y)]^2\,dy\right| \\ &\le 2\int_0^{f(t)}|\phi(t,y)|\,|D_1\phi(t,y)|\,dy+\text{const.}\,|[\phi(t,f(t))]^2-[\phi(t,g(t))]^2| \end{aligned}$$

for some function g satisfying $0\le g(t)\le f(t)$. Hence

$$f(t)|\psi'(t)| \le \text{const.}\left\{\int_0^{f(t)}|\phi(t,y)|\,|D_1\phi(t,y)|\,dy+\int_0^{f(t)}|\phi(t,y)|\,|D_2\phi(t,y)|\,dy\right\}.$$

It follows that

$$\begin{aligned} \int_0^{\infty} f(t)|\psi'(t)|\,dt &\le \text{const.}\left\{\|\phi.D_1\phi\|_{0,1,G}+\|\phi.D_2\phi\|_{0,1,G}\right\} \\ &\le \text{const.}\left\{\|\phi\|_{0,2,G}\|D_1\phi\|_{0,2,G}+\|\phi\|_{0,2,G}\|D_2\phi\|_{0,2,G}\right\} \\ &\le \text{const.}\left\{\|\phi\|^2_{0,2,G}+\|D_1\phi\|^2_{0,2,G}+\|D_2\phi\|^2_{0,2,G}\right\}. \end{aligned}$$

Substitution in (5) now yields (6) for ϕ and the lemma follows by completion.

Proof of Theorem 1. (a) Sufficiency. We assume (4) and hence that $\delta(R)\to 0$ as $R\to\infty$. Let $\{u_i\}_{i=1}^{\infty}$ be a bounded sequence in $W^{1,2}(G)$. We must show that $\{u_i\}$ is precompact in $L^2(G)$ and for this it suffices by a standard diagonalization argument to show that

(i) for every $\varepsilon>0$ there exists R such that for all i, $\|u_i\|_{0,2,G_R}<\varepsilon$, and
(ii) for every bounded subdomain $G'\subset G$ the sequence $\{u_i \mid G'\}$ is precompact in $L^2(G')$.

We note that (i) is an immediate consequence of Lemma 3. If $G'\subset G$ is bounded then $G'\subset K_R=G-G_R$ for some R. The set K_R is bounded and sufficiently regular that Rellich's compactness theorem is known to hold for it and so $\{u_i \mid K_R\}$ being bounded in $W^{1,2}(K_R)$ is precompact in $L^2(K_R)$. Hence $\{u_i \mid G'\}$ is precompact in $L^2(G')$ and so the imbedding (3) is completely continuous.

(b) Necessity. We assume that (4) does not hold and so by Lemma 1 there exist $\delta,\varepsilon>0$ and a positive sequence $\{R_j\}$ with $R_j\to\infty$ as $j\to\infty$ such that $f(R_j+3\varepsilon)$

$\geq \delta f(R_j)$. Without loss of generality we may assume that $R_{j+1} \geq R_j + 3\varepsilon$ for each j. Define functions u_j as follows

$$u_j(x, y) = c_j \begin{cases} x - R_j & \text{for } R_j \leq x \leq R_j + \varepsilon \\ \varepsilon & \text{for } R_j + \varepsilon \leq x \leq R_j + 2\varepsilon \\ R_j + 3 - x & \text{for } R_j + 2\varepsilon \leq x \leq R_j + 3\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

where c_j is chosen so that

$$\|u_j\|^2_{0,2,G} \geq \varepsilon^2 c_j^2 \int_{R_j+\varepsilon}^{R_j+2\varepsilon} f(x)\,dx \geq \varepsilon^3 \delta c_j^2 f(R_j) = 1.$$

But then

$$\begin{aligned} \|D_1 u_j\|^2_{0,2,G} &= c_j^2 \left(\int_{R_j}^{R_j+\varepsilon} + \int_{R_j+2\varepsilon}^{R_j+3\varepsilon} \right) f(x)\,dx \\ &\leq 2\varepsilon c_j^2 f(R_j) = 2\varepsilon^{-2}\delta^{-1} \end{aligned}$$

so that $\{u_j\}_{j=1}^{\infty}$ is bounded in $W^{1,2}(G)$ and bounded away from zero in $L^2(G)$. Since the functions u_j have mutually disjoint supports $\{u_j\}$ is not precompact in $L^2(G)$ and (3) is not completely continuous.

REMARK. Condition (4) is not merely a restriction on the magnitude of $f(x)$ as $x \to \infty$ but is also concerned with the steadiness of its decay. In fact it is easy to construct $f(x) \leq e^{-x^2}$ for which, however, $f(x+1) \neq o(f(x))$ as $x \to \infty$.

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