

A BESSEL FUNCTION INEQUALITY CONNECTED WITH STABILITY OF LEAST SQUARE SMOOTHING, II

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(Received 26 September, 1967)

1. Present Results. In accordance with customary notation, $J_\nu(t)$ denotes the Bessel function of the first kind and order ν , $j_{\nu,k}$ its k th positive zero and $j_{\nu,0} = 0$.

The object of this note is to prove that

$$\frac{\int_0^{j_{\nu,1}} t^{-\frac{1}{2}} J_\nu(t) dt}{\int_0^\infty t^{-\frac{1}{2}} J_\nu(t) dt} < 2 \quad \text{for all } \nu \geq \frac{3}{2}. \quad (1)$$

This inequality arose [7] in a problem of numerical analysis and was proved subsequently [3] for all sufficiently large ν . An equivalent inequality for Legendre polynomials has been established recently [4] (cf. also [5]) for $\nu = k + \frac{3}{2}$ ($k = 0, 1, 2, \dots$), the case relevant to the application with which the problem originated.

The proof here, valid for all $\nu \geq \frac{3}{2}$, is based on higher monotonicity properties of Bessel functions as discussed in [1] and [2]. As there, let

$$M_k = \int_{j_{p,k}}^{j_{p,k+1}} t^{\frac{1}{2}} |J_p(t)| dt \quad (k = 0, 1, 2, \dots). \quad (2)$$

First we demonstrate that

$$\int_0^{j_{p,1}} t^{\frac{1}{2}} J_p(t) dt < 2 \int_0^{j_{p,2k+1}} t^{\frac{1}{2}} J_p(t) dt - \int_{j_{p,2k}}^{j_{p,2k+1}} t^{\frac{1}{2}} J_p(t) dt \quad (3)$$

for all $p > \frac{1}{2}$ and $k = 1, 2, \dots$

Proof of (3). In [1, p. 63 (3.5), p. 70 (ii)] it has been shown that $\Delta^2 M_k > 0$ ($k = 0, 1, 2, \dots$), where $\Delta M_k = M_{k+1} - M_k$, $\Delta^2 M_k = \Delta M_{k+1} - \Delta M_k$. Hence

$$0 < [(M_0 - M_1) - (M_1 - M_2)] + [(M_2 - M_3) - (M_3 - M_4)] + \dots \\ + [(M_{2k-2} - M_{2k-1}) - (M_{2k-1} - M_{2k})].$$

Adding M_0 to both sides and rearranging gives

$$M_0 < 2(M_0 - M_1 + M_2 - \dots + M_{2k}) - M_{2k}.$$

Replacing each M_i ($i = 0, 1, \dots, 2k$) in this last inequality by its defining integral (2), we obtain (3).

Remark. If $p = \frac{1}{2}$, the two sides of (3) become equal.

The next inequality will apply to Bessel functions of order greater than $\frac{3}{2}$; the increase by one in the order arises from the use in the proof of a recursion formula which shifts the order by one. That is, it will be shown that

$$\int_0^{j_{\nu 1}} t^{-\frac{1}{2}} J_{\nu}(t) dt + A_{\nu} < 2 \int_0^{j_{\nu, 2k+1}} t^{-\frac{1}{2}} J_{\nu}(t) dt + B_{\nu k} \tag{4}$$

for $\nu > \frac{3}{2}$ and $k = 1, 2, \dots$, where A_{ν} and $B_{\nu k}$ are defined by

$$2\nu A_{\nu} = - \int_{j_{\nu-1, 1}}^{j_{\nu 1}} t^{\frac{1}{2}} J_{\nu-1}(t) dt + \int_{j_{\nu 1}}^{j_{\nu+1, 1}} t^{\frac{1}{2}} J_{\nu+1}(t) dt, \tag{5}$$

and

$$2\nu B_{\nu k} = \left. \begin{aligned} & -2 \int_{j_{\nu-1, 2k+1}}^{j_{\nu, 2k+1}} t^{\frac{1}{2}} J_{\nu-1}(t) dt + 2 \int_{j_{\nu, 2k+1}}^{j_{\nu+1, 2k+1}} t^{\frac{1}{2}} J_{\nu+1}(t) dt \\ & - \int_{j_{\nu-1, 2k}}^{j_{\nu-1, 2k+1}} t^{\frac{1}{2}} J_{\nu-1}(t) dt - \int_{j_{\nu+1, 2k}}^{j_{\nu+1, 2k+1}} t^{\frac{1}{2}} J_{\nu+1}(t) dt. \end{aligned} \right\} \tag{6}$$

Proof of (4). In (3), we put $p = \nu + 1$ and $p = \nu - 1$; for this it must be assumed that $\nu > \frac{3}{2}$, since $p > \frac{1}{2}$. Adding the pair of inequalities obtained thus, we obtain

$$\begin{aligned} \int_0^{j_{\nu-1, 1}} t^{\frac{1}{2}} J_{\nu-1}(t) dt + \int_0^{j_{\nu+1, 1}} t^{\frac{1}{2}} J_{\nu+1}(t) dt &< 2 \int_0^{j_{\nu-1, 2k+1}} t^{\frac{1}{2}} J_{\nu-1}(t) dt + 2 \int_0^{j_{\nu+1, 2k+1}} t^{\frac{1}{2}} J_{\nu+1}(t) dt \\ &- \int_{j_{\nu-1, 2k}}^{j_{\nu-1, 2k+1}} t^{\frac{1}{2}} J_{\nu-1}(t) dt - \int_{j_{\nu+1, 2k}}^{j_{\nu+1, 2k+1}} t^{\frac{1}{2}} J_{\nu+1}(t) dt. \end{aligned}$$

When the indicated integrals are decomposed so as to isolate A_{ν} and $B_{\nu k}$, it follows immediately that, for $\nu > \frac{3}{2}$, $k = 1, 2, \dots$,

$$\int_0^{j_{\nu 1}} t^{\frac{1}{2}} [J_{\nu-1}(t) + J_{\nu+1}(t)] dt + 2\nu A_{\nu} < 2 \int_0^{j_{\nu, 2k+1}} t^{\frac{1}{2}} [J_{\nu-1}(t) + J_{\nu+1}(t)] dt + 2\nu B_{\nu k}.$$

This inequality yields (4) with, as noted, $\nu > \frac{3}{2}$, $k = 1, 2, \dots$, on using Bessel's recurrence formula, $J_{\nu-1}(t) + J_{\nu+1}(t) = 2\nu t^{-1} J_{\nu}(t)$ [6, p. 45 (1)].

To obtain (1), with the strict inequality stated there, from (4), it will be shown that $A_{\nu} > 0$ and that $B_{\nu k} \rightarrow 0$ as $k \rightarrow \infty$. A_{ν} is considered first:

$$A_{\nu} > 0 \quad \text{for all } \nu > \frac{3}{2}. \tag{7}$$

Proof of (7). The positive zeros of $J_{\nu}(t)$ and $J_{\nu+1}(t)$ are interlaced when $\nu > -1$ [6, p. 479], so that $0 < j_{\nu-1, 1} < j_{\nu 1} < j_{\nu-1, 2}$. Hence the integrand of the first integral in the definition (5) of A_{ν} is negative throughout the interval of integration. But the integral itself is preceded by a minus sign, so that the first term in the right member of (5) is positive. So is the second, since $0 < j_{\nu 1} < j_{\nu+1, 1}$.

This demonstrates (7). Now we require

$$\lim_{k \rightarrow \infty} B_{\nu k} = 0 \quad \text{for fixed } \nu. \tag{8}$$

Proof of (8). This is a consequence of the familiar asymptotic formulae [6, p. 199 (1) and p. 506 (with $\alpha = 0$)], valid for fixed ν (as here) and large t and k , respectively:

$$t^{\frac{1}{2}}J_{\nu}(t) = (2/\pi)^{\frac{1}{2}}[\cos(t - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)][1 + O(t^{-1})] \quad \text{as } t \rightarrow +\infty,$$

$$j_{\nu k} = (k + \frac{1}{2}\nu - \frac{1}{4})\pi + O(k^{-1}) \quad \text{as } k \rightarrow \infty.$$

Substituting these expressions in the defining integrals (6) for $B_{\nu k}$ and making the change of variable $\xi = t - \frac{1}{2}\nu\pi - \frac{1}{4}\pi - 2k\pi$, we obtain

$$\nu(2\pi)^{\frac{1}{2}}B_{\nu k} = 2 \int_0^{\pi} \sin \xi d\xi + \int_{-\pi}^0 \sin \xi d\xi - \int_0^{\pi} \sin \xi d\xi + O(k^{-1}) = O(k^{-1}) \quad \text{as } k \rightarrow \infty.$$

This verifies (8).

Proof of (1). With (8) established, we can now let $k \rightarrow \infty$ in (4), whose left member is independent of k , to obtain the inequality

$$\int_0^{j_{\nu 1}} t^{-\frac{1}{2}}J_{\nu}(t) dt + A_{\nu} \leq 2 \int_0^{\infty} t^{-\frac{1}{2}}J_{\nu}(t) dt$$

for $\nu > \frac{3}{2}$. But for such ν , $A_{\nu} > 0$, as shown in (7). This completes the proof of (1) for $\nu > \frac{3}{2}$.

For $\nu = \frac{3}{2}$ a direct numerical check suffices. Now [6, p. 391 (1)],

$$\int_0^{\infty} \frac{J_{\nu}(t) dt}{t^{\nu-\mu+1}} = \frac{\Gamma(\frac{1}{2}\mu)}{2^{\nu-\mu+1}\Gamma(\nu-\frac{1}{2}\mu+1)} \tag{9}$$

for $0 < \mu < \nu + \frac{3}{2}$. (In [6] there is a misprint, $\frac{1}{2}$ replacing $\frac{3}{2}$.)

Here $\nu = \frac{3}{2}$, $\mu = 2$, so that the denominator in (1) becomes $(2/\pi)^{\frac{1}{2}}$. Moreover, $t^{\frac{1}{2}}J_{\frac{3}{2}}(t) = (2/\pi)^{\frac{1}{2}}(t^{-1} \sin t - \cos t)$, so that we need to show that

$$\int_0^{j_{3/2,1}} \frac{\sin t - t \cos t}{t^2} dt < 2.$$

The integrand is $D_i(-t^{-1} \sin t)$ and so, keeping in mind that $\tan(j_{\frac{3}{2},1}) = j_{\frac{3}{2},1}$, we obtain

$$\int_0^{j_{3/2,1}} \frac{\sin t - t \cos t}{t^2} dt = 1 - \frac{\sin(j_{3/2,1})}{j_{3/2,1}} = 1 - \cos(j_{3/2,1}) \doteq 1.2172336 < 2,$$

since $j_{\frac{3}{2},1} \doteq 4.4934095$. Thus (1) holds for $\nu = \frac{3}{2}$ and is established now for all $\nu \geq \frac{3}{2}$.

Remarks. (a) It is interesting to note how close the numerical value of the left member of (1) for $\nu = \frac{3}{2}$, namely 1.2172336, is to the limit (10) which this left member approaches as $\nu \rightarrow \infty$, namely 1.2743521.

(b) Inequality (1) can be verified for $\nu = \frac{3}{2}$ without the use of numerical tables. As shown above, this case is equivalent to showing that $1 - \cos(j_{\frac{3}{2},1}) < 2$, i.e., $\cos(j_{\frac{3}{2},1}) > -1$. But from the interlacing property for zeros [6, p. 479], used already in the proof of (7), we have $j_{\frac{3}{2},1} < j_{\frac{3}{2},1} < j_{\frac{3}{2},2}$, i.e., $\pi < j_{\frac{3}{2},1} < 2\pi$. Thus, $\cos(j_{\frac{3}{2},1}) > -1$, as required.

2. Further Problems. The question arises as to whether the inequality (1) can be extended to additional values of ν or improved in other ways. That it is valid not only for $\nu \geq \frac{3}{2}$, as shown here, but also for $\nu \geq \frac{1}{2}$ can be established, but to do so requires a substantial extension of certain results of [1] used here. This has been done and will be published elsewhere.

We have not been able to improve the inequality (1) by reducing its right member. The left member has been shown [3] to have the limit, as $\nu \rightarrow \infty$,

$$\frac{1}{3} + \frac{1}{3} \int_0^c [J_{1/3}(t) + J_{-1/3}(t)] dt \doteq 1.2743521, \quad (10)$$

where c is the first positive zero of the integrand. Numerical calculations (for which we thank the University of Alberta Department of Computing Science) suggest that the left member of (1) increases to this limit as ν increases. If so, then, of course, the inequality (1) could be sharpened by replacing 2 by the constant (10), the "best possible", and the inequality would remain strict. The rate of increase of the left member would have to be slow, starting as it does with the value 1.2172336 when $\nu = \frac{3}{2}$, and approaching 1.2743521 as $\nu \rightarrow \infty$. When $\nu = \frac{1}{2}$ it is the classic Gibbs ratio in Fourier series,

$$\frac{2}{\pi} \int_0^\pi t^{-1} \sin t dt \doteq 1.1789797.$$

These calculations suggest still another possible property of the ratio which is the left member of (1). For $\nu = k + \frac{1}{2}$ ($k = 0, 1, 2, \dots$), let this ratio be called r_k . The conjecture is that $(-1)^n \Delta^{n+1} r_k > 0$ ($n, k = 0, 1, \dots$), where, as usual $\Delta^{n+1} r_k$ is defined inductively as $\Delta^n r_{k+1} - \Delta^n r_k$.

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