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Rank two interval exchange transformations*

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Abstract. We consider interval exchange transformations T for which the lengths of the exchanged intervals have linear rank 2 over the field of rationals. We prove that, for such T, minimality implies unique ergodicity. We also provide an algorithm which tests T for aperiodicity and minimality.

1. *Introduction*

Two results of the paper are related to *i*nterval *e*xchange *t*ransformations (iets for briefness) of (rational) rank 2. We also introduce, for an arbitrary iet T, a finite set $C = C(T)$ which determines completely the minimal and periodic decomposition of T. An iet is said to be of (rational) rank k if the lengths of exchanged intervals span a k-dimensional space over Q (the field of rational numbers). The first result has been announced in [**1**, Theorem 4.1].

THEOREM 1.1. *A minimal rank* 2 *interval exchange transformation must be uniquely ergodic.*

We derive Theorem 1.1 as a simple corollary of Veech's recent result ([**15**, Theorem 1.2], see Theorem 3.5 below) which establishes a sufficient condition for a minimal iet to be uniquely ergodic. This condition is weaker than the condition 'Property P', used in [**1**] to show that most (in a sense) of the minimal iets of rank 2 are uniquely ergodic. The derivation of Theorem 1.1 (the end of § 5) from Veech's result leads to a significant shortening of my original proof (which will not be published). Besides, Veech's criterion provides one 'reason' for Theorem 1.1 to be true, while my first proof uses different arguments for the cases of irrational parameters with different diophantine properties. (Note that Veech's result has answered in the affirmative an important, but special case of my question [**1**, Question 1 in § 3] stated in the more general context of arbitrary symbolic flows. The answer in the general case is still unknown.)

The second result of the paper is an algorithm which decides, for a given iet T of rank 2, whether T is minimal (and hence also uniquely ergodic, by Theorem 1.1). We show how to establish the minimal decomposition of T if T is aperiodic, and how to find a periodic point of T if such points exist (§ 8). In fact, using our approach, one can effectively partition the domain of T onto minimal and periodic components.

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As we show, for an arbitrary iet, it suffices to establishes upper estimates for the possible periods and, more generally, for the lengths of possible connections (see the next paragraph and Remark 2.18). We can do it (see Theorem 7.9) for an iet T of rank 2 when using both ingredients of T: (a) discrete pattern, and (b) irrational parameter. (For precise definitions see § 5, in particular, (5.2) and (5.3).) An interesting open question is whether those upper estimates are possible in terms of (a) only. (For a consequence of the affirmative answer to the above question see the last two paragraphs of the paper.) As a corollary, we deduce that such properties as minimality (equivalently, unique ergodicity) and aperiodicity, as well as the numbers of periodic and minimal components, are stable under small perturbations of the irrational parameter of an iet of rank 2 (see Theorem 9.1).

In § 2 we set notation and discuss general properties of iets. We introduce the notions of a T-connection (a finite piece of a T-orbit, connecting the formal discontinuities of T, see Definition 2.7), associated with a given iet T. We show that the union of all T-connections $C(T)$ (which must be finite) determines a partition of the domain of T so that every subinterval of this partition belongs to one (periodic or minimal) component (Theorem 2.17). Although the fact that every iet can be decomposed into a finite number of minimal and periodic components is well known, Theorem 2.17 shows that this decomposition can easily be explicitly obtained provided that the set $C(T)$ is given. Note that Theorems 2.17 and 2.19 generalize Keane's sufficient condition ($C(T)=\varnothing$, stated as Theorem 2.6 below) for the minimality of an iet (see Keane **[6]** where the systematic study of iets has been initiated).

In § 3 we discuss the property of unique ergodicity, state Veech's sufficient condition for a minimal iet to be uniquely ergodic and state Proposition 3.8 that this condition is always fulfilled for iets of rank 2. This proposition implies Theorem 1.1. The proof of Proposition 3.8 is provided at the end of § 5. For completeness in § 4 we provide a short proof of Veech's criterion (which is close in spirit to Veech's original proof). Some of the intermediate results in the proof (Lemmas 4.4, 4.5 and the Identity (4.2)) are of independent interest and may prove to be useful in the further study of iets.

In §§ 6 and 7, for a rank 2 iet T, we bound the length of the minimal period and, in the case of aperiodicity of T, the length of possible T-connections (Theorem 7.9). Using these bounds, we show (§ 8) how to test T for aperiodicity and minimality. From the description of the algorithm it follows that the tested properties (aperiodicity, minimality) are stable under small perturbations of the irrational parameter (Theorem 9.1).

Some open questions are stated at the end.

2. *Interval exchange transformations and minimality*

Recall the definition of an interval exchange transformation. Let Λ_r denote the positive cone in $\mathbb{R}^r$:

$$\Lambda_r=\{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)|\lambda_i>0\},\qquad r\geq 1. \tag{2.1}$$

For $\lambda \in \Lambda_r$, set $\beta_0=0$, $\beta_i=\sum_{j=1}^{i}\lambda_j$, $X_i=[\beta_{i-1},\beta_i)(1\le i\le r)$ and $X=\bigcup X_i=[0,\beta_r)$. Let S_r be the set of permutations of the set $\{1,2,\ldots,r\}$. Given $\lambda\in|\Lambda_r$ and $\sigma\in S_r$, set $\lambda'=(\lambda'_2,\ldots,\lambda'_r)\in\Lambda_r$, where $\lambda'_{\sigma(i)}=\lambda_i$. Form corresponding β'_i and X'_i and consider the map $T:X\to X$ defined by

$$T(x)=x+\gamma_i, \qquad \text{for} \quad x\in X_i, \qquad 1\le i\le r, \tag{2.2}$$

where

$$\gamma_i=\beta'_{\sigma(i)-1}-\beta_{i-1}. \tag{2.3}$$

Thus T exchanges the semi-intervals X_i according to the permutation σ: $T(X_i)=X'_{\sigma(i)}$, is one-to-one and right-continuous at any point $x\in X$. T is called the (λ,σ)-iet. We sometimes write (λ,σ) to denote T. T is said to be of (rational) rank k if $\lambda_1,\lambda_2,\ldots,\lambda_r$ span a k-dimensional space over Q (the field of rational numbers).

$T=(\lambda,\sigma)$ is called minimal if the T-orbit

$$O_T(x)=\{T^n(x)\,|\,n>0\} \tag{2.4}$$

of any point $x\in X$ is dense in $X=[0,\beta_r)$. T is said to satisfy the infinite distinct orbit condition (idoc) if

(a) $O(\beta_i)$ are infinite sets for all $i=1,2,\ldots,r-1$,
and (2.5)
(b) $O(\beta_i)\cap O_T(\beta_j)=\varnothing$, for all $1\le i,j\le r-1$, $i\ne j$.

THEOREM 2.6. (Keane [6]). *If T satisfies the idoc and if r (the number of exchanged intervals) is at least two, then T is minimal.*

As a corollary of Theorem 2.6, Keane proves that, if all λ_i are linearly independent over Q (the field of rational numbers), if $r\ge 2$, and if σ is irreducible (no subset $\{1,2,\ldots,k\}$, $1\le k\le r-1$, is invariant under σ), then T is minimal.

DEFINITION 2.7. *For an iet $T=(\lambda,\sigma)$ denote by $D=D(T)$ the set $D=\{\beta_1,\beta_2,\ldots,\beta_{r-1}\}$ of formal discontinuities of T. A finite sequence $x_1,x_2,\ldots,x_k$ of points in $X=(0,\beta_r)$, $k>1$, is called a T-connection if the following two conditions are met:*

(a) $x_{i+1}=T(x_i)$, $1\le i\le k-1$,
(b) *both x_1 and x_k belong to $D=D(T)$.* (2.8)

Sometimes the set $\{x_1,\ldots,x_k\}$ itself is called a T-connection. By $C=C(T)$ we denote the union of all T-connections.

Clearly C is always finite, and $C=\varnothing$ if and only if T satisfies the idoc (see (2.5)). C is a distinct union of its maximal T-connections. Two points in C belong to the same maximal T-connection if and only if they belong to the same T-orbit. Note that C may contain or not contain 0.

PROPOSITION 2.9. *Let T be an iet which does not satisfy the idoc (so that $C=C(T)\ne\varnothing$ and therefore $C\backslash\{0\}\ne\varnothing$). Let $c_1,c_2,\ldots,c_t$ be the list of all points in $C\backslash\{0\}$ in the increasing order:*

$$0=c_0<c_1<c_2<\cdots<c_t<c_{t+1}=\beta_r, \qquad t\ge 1,$$

where $c_0 = 0$ and $c_{t+1} = \beta_r$ are added for convenience. Let

$$X = \bigcup Y_j, \qquad Y_j = [c_j, c_{j+1}), \quad 0 \le j \le t.$$

Then, for any $j = 0, 1, \ldots, t$, the induced by T (first return) map $T_j: Y_j \to Y_j$ is either the identity map or an iet which satisfies the idoc and is therefore minimal (by Theorem 2.6).

Proof. The induced map T_j is known to be an iet of s intervals, $1 \le s \le r+2$ (where r is the number of the intervals exchanged by T; see e.g. [**3**, Ch. 5.3]). We may assume that $s > 1$ because otherwise T_j is the identity map.

Denote by B_j the set of discontinuities of T_j:

$$B_j \subset (c_j, c_{j+1}) \subset Y_j, \qquad |B_j| \ge 1. \tag{2.10}$$

For $y \in Y_j$, denote by $n(y)$ the minimum positive integer such that $T^n(y) \in Y_j$. Thus $T_j(y) = T^{n(y)}(y)$. For $b \in B_j$, at least one of the following three conditions is satisfied:

$$\begin{aligned} &\text{(a)}\ \ T^k(b) \in D \text{ for some } k, \quad 0 \le k < n(b),\\ &\text{(b)}\ \ T^k(b) = c_{j+1} \text{ for some } k, \quad 0 < k < n(b),\\ &\text{(c)}\ \ T_j(b) = T^{n(b)}(b) = c_j \ne 0. \end{aligned} \tag{2.11}$$

For $b \in B_j$, let $u = u(b) = \min U$, where the set

$$U = U(b) = \{k \ge 0 \mid T^k(b) \in C \cup D\} \tag{2.12}$$

is easily seen to be non-empty in each of the cases (a), (b) and (c). Moreover, the inequalities

$$0 \le u = u(b) \le n(b), \tag{2.13}$$

clearly take place. (Note that (b) implies $j \le t-1$, $c_{j+1} \in C$, and that (c) implies $c_j \in C$, $0 < j < t$).

For $b \in B_j$, the relation

$$T^u(b) \in D \tag{2.14}$$

holds (for, if $u = 0$, then $b \in C \cup D$ but $b \notin C$ since $B_j \cap C = \varnothing$, and if $u \ge 1$, then $T^u(b) \in C \cup D$ but $T^{u-1}(y) \notin C \cup D$).

We have to show that T_j satisfies the idoc. Assume to the contrary that there exists a T_j-connection $y_1, y_2, \ldots, y_m$, where $m \ge 2$ and both y_1, y_m belong to B_j. Then $u(y_i)$, $n(y_i)$ are defined for both $i = 1, m$. It is clear that

$$y_m = T^k(y_1), \tag{2.15}$$

for some $k \ge n(y_1)$. From $v = k - u(y_1)$, $p = u(y_m) + v$, $x_1 = T^{u(y_1)}(y_1)$ and $x_2 = T^{u(y_m)}(y_m)$.

We claim that $v \ge 1$. Indeed, if $u(y_1) < n(y_1)$, we obtain $v = k - u(y_1) > k - n(y_1) \ge 0$. Otherwise (if $u(y_1) = n(y_1)$, see (13)), we observe that the only possibility in (11) for $b = y_1$ is (c), and therefore $y_2 = T_j(y_1) = c_j \notin B_j$, whence $m \ge 3$ and $k > n(y_1)$ (see (15)), and the required inequality follows: $v = k - u(y_1) = k - n(y_1) \ge 1$.

Thus we have $1 \le v \le p$. By (14), both x_1, x_2 belong to D. One easily verifies that $x_2 = T^p(x_1)$ and that $y_m = T^v(x_1)$. It follows that $y_m \in C$, which is impossible since $y_m \in B_j$ and $B_j \cap C = \varnothing$. The contradiction completes the proof. □

The following theorem follows directly from Proposition 2.9.

THEOREM 2.16. *Let T, $C = C(T)$, $t \geq 1$, Y_j and $T_j (0 \leq j \leq t)$ be as in Proposition* 2.9. *For any $x \in X = \bigcup Y_j$, let $A(x)$ be the set*

$$A(x) = \{j \mid Y_j \cap O_T(x) \neq \varnothing, \qquad 0 \leq j \leq t\}$$

(*see* 2.4) *and define the component of x to the set*

$$\text{comp}\,(x) = \bigcup_{j \in A(x)} Y_j \subset X.$$

The relation $z \in$ comp (x) is an equivalence relation on $X(x, z \in X)$. Thus comp (x) *is the equivalence class corresponding to x. For any fixed $x \in X$, one of the two possibilities takes place*:

(1) *x is periodic* (*relative to T*) *with a period $p \geq 1$. Then* comp (x) *is a distinct union of p intervals Y_j, and T cyclically exchanges those intervals.*

(2) *x is aperiodic. Then $O_T(x)$ is dense in* comp (x). *Moreover, $O_T(z)$ is also dense in* comp $(x) =$ comp (z), *for any $z \in$* comp (x).

Remark. The results of this section clearly remain true if D (in Definition 2.7) is taken to be the set D' of actual discontinuities of T (rather than of all formal discontinuities), or any finite set containing D'.

A component comp (x) is called periodic (respectively, minimal) if x is periodic (respectively, aperiodic) relative to T. (Note that the periodicity property of x determines which one of the two possibilities in Theorem 2.16 takes place.)

Theorem 2.16 can be restated in the following way.

THEOREM 2.17. *The set $C(T)$ determines a partition of the interval $X = [0, \beta_r)$ so that every subinterval of this partition belongs to one* (*periodic or minimal*) *component.*

As we have noted in the Introduction, Theorem 2.17 allows us to determine the partition of X into minimal and periodic components, provided that the set $C = C(T)$ (which must be finite) is known. First we consider the partition $X = \bigcup Y_j$, $Y_j = [c_j, c_{j+1})$, $0 \leq j \leq t$, of X determined by C (see notation in Proposition 2.9). Next we define the ordered graph G with the set of vertices $G = \{0, 1, \ldots, t\}$. The ordered pair $\{u, v\} \in G \times G$ is said to form an arrow in G if $T(Y_u) \cap Y_v \neq \varnothing$.

By Theorem 2.17, each of the subintervals Y_j belongs as a whole to one (periodic or minimal) component. We thus obtain a one-to-one correspondence between the (minimal and periodic) components of X and the connected components of the graph G. Those components of G which form distinct cycles correspond to periodic components of X, the rest of the components of G correspond to the minimal components of X.

Remark 2.18. Let $T: X \to X$ be an iet and assume that some upper bound L on the lengths of all possible T-connections is known. Then one can find all T-connections (simply computing the T-orbits of the discontinuities β_j, $1 \leq j \leq r-1$, up to the length L). Following the procedure described in the preceding two paragraphs, one can effectively find the decomposition of X into minimal and periodic components.

Let $\Phi(T)$ be the family of all maximal T-connections and $\Phi_1(T) \subset \Phi(T)$ be the subfamily of non-periodic maximal T-connections (which do not form a T-orbit).

Every minimal component which is not exactly of the form $Y_0=[0, c_1)$ must contain a maximal non-periodic T-connection, and every such T-connection contains at least two discontinuities in D. Therefore the number of minimal components does not exceed $\operatorname{card}(\Phi_1(T))+1\le(r-1)/2+1=(r+1)/2$, where card $(\cdots)$ stands for the cardinality of the set in brackets. If, moreover, the permutation σ in $T=(\lambda, \sigma)$ is irreducible (no subset $\{1,2,\dots,k\}$, $1\le k\le r-1$, is invariant under σ), then $Y_0=[0, c_1)$ cannot be invariant under T, and the number of minimal components does not exceed $(r-1)/2$. (To make the estimate more accurate, r could be taken for the number of *actual* discontinuities only, rather than of all formal discontinuities β_i.)

In particular, every aperiodic iet (λ, σ) of 4 intervals with irreducible σ must be minimal. Similar arguments lead to the following result (which strengthens Keane's minimality criterion, Theorem 2.6).

THEOREM 2.18. *If* (λ, σ) *has at most one maximal T-connection, and if* σ *is irreducible, then either T is minimal, or all points of X are periodic with the same period and the T-orbit of* 0 *contains all (formal) discontinuities of T.*

3. *Minimality and unique ergodicity*

An iet $T:X\to X$ is said to be uniquely ergodic if there exists only one Borel probability measure on X: the Lebesgue measure. The unique ergodicity of T implies the minimality; the converse is true for $r=2,3$ (r is the number of exchanged intervals), but not for any $r>3$. The first counterexample (with $r=5$) is due to Keynes and Newton [**8**]; it is based on the work of Veech [**11**]. Keane [**7**] produced a counterexample with $r=4$, the smallest possible. Note that the minimality (even the aperiodicity) of T implies that the number of T-invariant normalized ergodic measures on X is finite. (The best estimates are given in [**5**] and [**13**]; see also [6], [10], [**3**, Section 4]. In [**2**] the finiteness of the set of ergodic measures is established in the more general setting of symbolic flows with a linear block growth.)

Masur [**9**] and Veech [**14**] independently proved that if $\sigma\in S_r$ is irreducible, then for Lebesgue almost all $\lambda\in\Lambda_r$ the (λ, σ)-iet is uniquely ergodic (Keane's conjecture).

Let $T:X\to X$ be an iet, $T=(\lambda, \sigma)$. Denote by D the set

$$D=\{\beta_j \mid 1\le j<r\} \tag{3.1}$$

of formal discontinuities T (see § 2 for notation). For all $n\ge1$ define

$$D_n=\left(\bigcup_{k=0}^{n-1} T^{-k}D\right)\cup\{0\}. \tag{3.2}$$

Each set D_n defines a partition of $X=[0, \beta_r)$:

$$0=\beta_0(n)<\beta_1(n)<\cdots<\beta_{r(n)-1}(n)<\beta_{r(n)}(n)=\beta_r, \tag{3.3}$$

where $\beta_i(n)$, $0\le i<r(n)$, are elements of D_n. We write $\varepsilon_n=\varepsilon_n(T)>0$ for the minimum separation of this partition:

$$\varepsilon_n=\min(\beta_j(n)-\beta_{j-1}(n) \mid 1\le j\le r(n)). \tag{3.4}$$

THEOREM 3.5. (Veech [15]). *Let T be a minimal iet which is not uniquely ergodic. Then*

$$\lim_{n\to\infty} n\varepsilon_n(T)=0. \tag{3.6}$$

Theorem 3.5 provides us with a sufficient condition for a minimal iet to be uniquely ergodic:

$$\limsup_{n\to\infty} n\varepsilon_n(T)>0. \tag{3.7}$$

Note that in [1] another sufficient condition, called 'Property P', has been introduced and applied

(a) to obtain a new proof of Keane's conjecture (see the second paragraph in this section),

(b) to show that 'most' (in a sense) of the minimal rank 2 iets satisfy Property P and therefore are uniquely ergodic,

(c) in the study of finite group extensions of irrational rotations,

(d) in the study of 'rational billiards' dynamical systems.

Veech's (sufficient) criterion for unique ergodicity (see (3.7)) turns out not only to be easier to formulate or verify than Property P, but it is also weaker: the condition (3.7) is always fulfilled for rank 2 iets (although not all rank 2 iets satisfy Property P).

PROPOSITION 3.8. *For every rank* 2 *iet the condition* (3.7) *holds.*

Thus the condition (3.7) seems to be the 'right' one to be used in the framework of the topics studied in [1]. The analog of Metric Theorem (see § 5 in [1]) becomes trivial if Collective Property P (Definition 5.1 in [1]) is replaced by the condition

$$\lim_{\varepsilon\to 0}\,[\liminf_{n\to\infty} u(n,\varepsilon)]=0. \tag{3.9}$$

The proof of Proposition 3.8 is supplied at the end of § 5. It is similar to that of Theorem 4.5 in [1] (which establishes Property P for 'most' of the iets of rank 2).

4. *Proof of Veech's Theorem* 3.5

Let $T=(\lambda,\sigma)$ be a fixed iet, let $D=D(T)=\{\beta_1,\beta_2,\ldots,\beta_{r-1}\}$ be the set of formal discontinuities of T (for notation see the beginning of § 2). For $n\geq 1$, define $D_n=D_n(T)$ and $\varepsilon_n=\varepsilon_n(T)$ as in (3.2) and (3.4). (The set D_n contains the formal discontinuities of T^n and the number 0, and $\varepsilon_n(T)$ is the minimum separation of the partition D_n. Finite subsets of the set $X=[0,\beta_r)$ are identified with the partitions of X they determine.)

For positive integers m, n define $D_{m,n}=D_m(T)\cup D_n(T^{-1})$ and let $\varepsilon_{m,n}$ be the minimum separation of the partition $D_{m,n}$. Observe the relations

$$\begin{aligned} &\text{(a)}\quad D_n(T^{-1})=T^n(D_n(T)) \\ &\text{(b)}\quad D_{m+n}(T)=T^{-n}(D_{m,n}). \end{aligned} \qquad m,n>0 \tag{4.1}$$

It is clear that T^{-n} is linear on each subinterval of $D_{m,n}$ and that T^n is linear on each subinterval of D_{m+n}, even on each subinterval of D_n (all intervals are assumed to be left closed and right open). Therefore T^n makes a one-to-one correspondence between the subintervals of D_{m+n} and $D_{m,n}$. In particular,

$$\varepsilon_{m,n}=\varepsilon_{m+n}, \qquad \text{for } m,n>0. \tag{4.2}$$

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By similar arguments, (4.1, a) implies

$$\varepsilon_n = \varepsilon_n(T) = \varepsilon_n(T^{-1}), \qquad \text{for } n > 0. \tag{4.3}$$

Recall (Definition 2.7) that $C = C(T)$ denotes the union of all T-connections. C is always a finite set.

LEMMA 4.4. *Let $n \geq 1$, let $Y = [u, v)$ be a subinterval of $X = [0, \beta_r)$ and assume that $|Y| \leq \varepsilon_n$ and that $Y \cap C = \varnothing$ ($|Y|$ stands for the length of the interval Y). Denote by p the maximal integer, $0 \leq p < n$, for which all the maps T^j, $0 \leq j \leq p$, are continuous on Y. Denote by q the maximal integer, $0 \leq q < n$, for which all the maps T^{-j}, $0 \leq j \leq q$, are continuous on Y. Then $p + q \geq n - 1$.*

Proof. By definition of p and q, there are points $\alpha \in D \cap T^p(Y)$ and $\beta \in D(T^{-1}) \cap T^{-q}(Y)$. Clearly, $\gamma = T^{-(p+q)}(\alpha)$ belongs to D_{p+q+1}. Since $\beta \in D(T^{-1})$, $T^{-1}(\beta) \in D_1$. So that $\delta = T^{-i}(\beta) \in D$ either for $i = 1$ or for $i = 2$.

Assume that $\beta = \gamma$. Then both α and δ belong to D and $T^{p+q+i}(\delta) = T^{p+q}(\beta) = T^{p+q}(\gamma) = \alpha$. Therefore $T^{q+i}(\delta) = T^q(\beta) \in Y$ belongs to a T-connection (connecting δ and α), contrary to the assumption $Y \cap C = \varnothing$. We conclude that β and γ are different points of $D_{p+q+1} \cup D(T^{-1}) \subset D_{p+q+1,i}$, so that, in view of (4.2), $|\beta - \gamma| \geq \varepsilon_{p+q+1,1} = \varepsilon_{p+q+2}$. On the other hand, β and γ belong to the interval $T^{-q}(Y)$ of the length $|Y|$, so that $\varepsilon_n \geq |Y| > |\beta - \gamma| = \varepsilon_{p+q+2}$. Therefore $n < p + q + 2$. □

LEMMA 4.5. *For any $x \in X$ and for any integer n either $T^n(x) = x$ or $|T^n(x) - x| \geq \varepsilon_{|n|+1}$.*

Proof. We may assume that $n > 0$ and that $d = T^n(x) - x \neq 0$. We have to show that $|T^n(x) - x| \geq \varepsilon_{n+1}$. Let $Y = [u, v)$ be the subinterval of D_n which contains the point x. Clearly T^n is a linear isometry on Y and therefore $T^n(u) - u = d$. Because $u \in D_n$, $u = T^{-k}(\alpha)$ for some non-negative integer $k \leq n - 1$ and some $\alpha \in D_1(T)$. Clearly then $T^n(u) \in D_{n-k}(T^{-1})$ and therefore $|d| = |T^n(u) - u| \geq \varepsilon_{k+1,n-k} = \varepsilon_{n+1}$, in view of (4.2). □

From now on we assume that $T = (\lambda, \sigma)$ is minimal. It follows from (3.7) that for some $\varepsilon > 0$ the set

$$S(\varepsilon) = \{n > 0 \mid n\varepsilon_n > \varepsilon\} \tag{4.6}$$

of integers is infinite. Fix such an ε and the corresponding set $S = S(\varepsilon)$. We have to prove that T is uniquely ergodic. Denote by μ Lebesgue measure on $\mathbb{R}$.

LEMMA 4.7. *T is ergodic relative to μ.*

Proof. Let $A \subset X$ be a measurable set such that $T(A) = A$. We have to prove that $\mu(A) = 0$ or 1. For an arbitrary δ such that $0 < \delta < \min(1/2, \varepsilon)$, there are $\gamma > 0$ and a measurable set $K \subset X$ with the following properties: $\mu(K) > \beta_r - \delta = \mu(X) - \delta$ and the relation

$$\mathrm{d}(J) = \mu(J \cap A)/\mu(J) \notin [\delta, 1 - \delta] \tag{4.8}$$

holds for every interval J satisfying $J \cap K \neq \varnothing$ and $\mu(J) < \gamma$.

Fix a subinterval I of the partition $C = C(T)$ (see Definition 2.7). Select an integer n so that $n + 1 \in S = S(\varepsilon)$ and $\varepsilon_{n+1} < \min(\gamma, |I|)$, where $|I| = \mu(I)$. (The minimality of T implies $\lim_{k\to\infty} \varepsilon_k = 0$.) Let $Y = [u, v)$ be a subinterval of I of the length ε_{n+1}.

Let p, q be defined as in Lemma 4.4. Then $p+q\geq n$. (We use the fact that $(u, v)\subset I$ is disjoint with C.) By definition of p and q, the sets $T^k(Y)$, $-q\leq k\leq p$, form (left closed and right open) intervals of equal lengths, $|T^k(Y)|=|Y|=\varepsilon_{n+1}$. Consider the set

$$M=\bigcup_{k=-q}^{-q+n} T^k(Y). \tag{4.9}$$

All the intervals in the union (4.9) are pairwise disjoint by Lemma 4.5. It follows that $\mu(M)=(n+1)\varepsilon_{n+1}>\varepsilon>\delta$, and hence $K\cap M\neq\varnothing$ since $\mu(K)>\mu(X)-\delta$. Therefore, for one of the intervals $Y'=T^k(Y)$ in the union (4.9), $Y'\cap K\neq\varnothing$. Since $|Y'|=|Y|=\varepsilon_{n+1}<\gamma$, by definition of K, the interval $J=Y'$ satisfies (4.8). Therefore, so does the interval $J=Y$, because A is invariant under T. Since $d(Y)$ depends continuously on the endpoint of Y, we may assume (replacing A by its complement $X\backslash A$ if needed) that $d(Y)\leq\delta$ (equivalently, $\mu(Y\cap A)\leq\delta\varepsilon_{n+1}$) holds for all subintervals Y of I with $|Y|=\varepsilon_{n+1}$.

Clearly the interval I can be covered by m subintervals, each of length ε_{n+1}, where $m\leq|I|/\varepsilon_{n+1}+1<2|I|/\varepsilon_{n+1}$. It follows that $\mu(I\cap A)\leq m\,\delta\varepsilon_{n+1}<2|I|\delta$. We conclude $\mu(I\cap A)=0$ because $\delta\in(0,\min(1/2,\varepsilon))$ is arbitrary. The minimality implies $\mu(A)=0$. □

Let η be any ergodic T-invariant Borel probability measure. For $n\geq 1$, $x\in X$ and a subinterval $Y\subset X$, denote by $N(Y,n,x)$ the number of $i=0,1,\ldots,n-1$ for which $T^i(x)\in Y$. There exists a point $x\in X$ (generic for η) such that

$$\lim_{n\to\infty} N(Y,n,x)/n=\eta(Y)$$

for every subinterval $Y\subset X$. On the other hand, by Lemma 4.5, $N(Y,n,x)<1+|Y|/\varepsilon_n$. For all $n\in S=S(\varepsilon)$ (see (4.6)) we obtain $N(Y,n,x)/n<|Y|/\varepsilon+1/n$ and therefore $\eta(Y)\leq|Y|/\varepsilon=\mu(Y)/\varepsilon$, so that η is absolutely continuous relative to μ. In view of Lemma 4.7, $\eta=\mu$. The proof of Theorem 3.5 is completed.

5. *Rank two interval exchanges*

We shall write rank $(T)=n$ to denote the fact that the rational rank of T is n (see the Introduction). Given an iet $T=(\lambda,\sigma)$, rank $(T)=2$, $\lambda\in\Lambda_r$ and $\sigma\in S_r$ (see § 2), the system of equations

$$\lambda_i=\bar\lambda_i\alpha+\tilde\lambda_i\beta,\qquad 1\leq i\leq r,$$

has a (non-unique) solution $(\bar\lambda_1,\tilde\lambda_1,\bar\lambda_2,\tilde\lambda_2,\ldots,\bar\lambda_r,\tilde\lambda_r,\alpha,\beta)$, where $\bar\lambda_i,\tilde\lambda_i\in Z$ (the set of integers) and $x, y\in\mathbb{R}$ (the set of reals). By change of scale (and the signs of $\bar\lambda_i$ if necessary) one can make $\beta=1$ and $\alpha>0$. Thus we obtain

$$\lambda_i=\bar\lambda_i\alpha+\tilde\lambda_i,\qquad 1\leq i\leq r, \tag{5.1}$$

where $\bar\lambda_i$, $\tilde\lambda_i\in Z$ and $\alpha>0$ is irrational since rank $(T)=2$. Clearly T is completely determined by its discrete pattern

$$(\bar\lambda_1,\tilde\lambda_1,\bar\lambda_2,\tilde\lambda_2,\ldots,\bar\lambda_r,\tilde\lambda_r,\sigma) \tag{5.2}$$

and by its irrational parameter $\alpha>0$:

$$T=(\bar\lambda_1,\tilde\lambda_1,\bar\lambda_2,\tilde\lambda_2,\ldots,\bar\lambda_r,\tilde\lambda_r,\sigma;\alpha). \tag{5.3}$$

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Consider the set

$$L(\alpha)=\{u=\bar{u}\alpha+\tilde{u}\mid \bar{u},\ \tilde{u}\in Z\}\subset\mathbb{R}. \tag{5.4}$$

Since α is irrational, the integers $\bar{u}=S_1(u)$ and $\tilde{u}=S_2(u)$ are uniquely determined by $u\in L(\alpha)$. For every T defined by (5.3), all the quantities λ_i, β_i, γ_i, λ'_i, β'_i, ε_n, as well as the points in the sets D_n and $D_{m,n}$ (see § 2, (3.2) and the beginning of § 4), belong to $L(\alpha)$. Denote

$$k_0=k_0(T)=\max\left(\max_{1\le i\le r}|\bar{\beta}_i|,\ \max_{1\le i\le r}|\bar{\gamma}_i|\right), \tag{5.5}$$

where $\bar{\beta}_i=S_1(\beta_i)$ and $\bar{\gamma}_i=S_1(\gamma_i)$.

By induction on n one easily verifies that $|\bar{u}|\le nk_0$ for any $u\in D_n$. This immediately implies the inequality

$$\varepsilon_n\ge\|\alpha,2nk_0\|, \tag{5.6}$$

where by definition

$$\|\alpha,m\|=\min_{1\le i\le m}\|i\alpha\|;\qquad m=1,2,\dots, \tag{5.7}$$

and $\|y\|$ denotes the distance from $y\in\mathbb{R}$ to the closest integer.

For every irrational $\alpha\in\mathbb{R}$ and $n\ge1$, it is well known from the theory of continued fraction representations that the relation

$$\|\alpha,q_n-1\|=\|q_{n-1}\alpha\|\ge1/(2q_n)$$

holds, where q_n stands for the nth denominator in the sequence of convergents p_n/q_n to α. Let $s(n)=[(q_n-1)/2k_0]$, where, for real y, $[y]$ denotes the largest integer less than or equal to y. Then the inequalities

$$\begin{aligned}\varepsilon_{s(n)}(T)&\ge\|\alpha,2k_0s(n)\|\ge\|\alpha,q_n-1\|\ge1/(2q_n)\\&\ge1/(4k_0(s(n)+1))\ge1/(8k_0s(n)),\end{aligned}$$

hold for all n large enough (such that $q_n\ge1+2k_0$).

Since $\lim_{n\to\infty}s(n)=+\infty$, we conclude that

$$\limsup_{n\to\infty}n\varepsilon_n(T)\ge1/(8k_0).$$

This completes the proofs of Proposition 3.8. Now, Theorem 1.1 in the introduction follows immediately from Veech's criterion of unique ergodicity (3.7).

6. *Upper estimates on the length of minimal period*

Let $T=(\lambda,\sigma)$ be a fixed iet. For every $n\ge1$, T^n is also an iet and we shall always assume that the exchanged intervals

$$X_i(n)=[\beta_{i-1}(n),\beta_i(n)),\quad 1\le i\le r(n), \tag{6.1}$$

are determined by the partition $D_n=D_n(T)$ of the interval $X=[0,\beta_r)$ (see (3.2)), even if T^n is continuous at some of the points in D_n. We employ notation similar to that in the beginning of § 2.:

(a) $T^n=(\lambda(n),\sigma(n)),\quad \lambda(n)=(\lambda_1(n),\lambda_2(n),\dots,\lambda_{r(n)}(n))\in\Lambda_{r(n)}$,
and (6.2)

(b) $\beta_0(n)=0,\quad \beta_i(n)=\sum_{j=1}^{i}\lambda_j(n),\quad 1\le i\le r(n).$

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Thus $\lambda_i(n)=|X_i(n)|$, $\beta_r=\beta_{r(n)}(n)=|X|$, and $r(n)$ denotes the number of the intervals (6.1) exchanged by T^n.

For every $n\geq 1$, denote by $\gamma(n)$ the function

$$(\gamma(n))(x)=\gamma(n,x)=T^n(x)-x, \tag{6.3}$$

and by $\gamma_i(n)$ the constant value of this function on the interval $X_i(n)$:

$$T^n(x)=x+\gamma_i(n), \qquad \text{for} \quad x\in X_i(n), \quad 1\leq i\leq r(n), \tag{6.4}$$

where $\gamma_i(n)=\beta'_{(\sigma(n))(i)-1}(n)-\beta_{i-1}(n)$, by analogy with (2.3). The following identities are self-evident:

(a) $\gamma(m+n,x)=\gamma(m,T^nx)+\gamma(n,x), \qquad m,n\geq 1, x\in X$

and (6.5)

(b) $\gamma_i(1)=\gamma_i, \qquad 1\leq i\leq r.$

Recall that $\varepsilon_n=\varepsilon_n(T)$ stands for the minimum separation of the partition D_n:

$$\varepsilon_n=\min(\lambda_i(n)\,|\,1\leq i\leq r(n)) \tag{6.6}$$

(see (3.4)).

Now assume that $T=(\lambda,\sigma)=(\bar\lambda_1,\tilde\lambda_1,\bar\lambda_2,\tilde\lambda_2,\ldots,\bar\lambda_r,\tilde\lambda_r,\sigma;\,\alpha)$ is of rank two (see § 5). Then all the values $\lambda_i(n)$, $\beta_i(n)$, $\gamma_i(n)$, $\gamma(n,x)$, $\varepsilon_n,\ldots$ belong to $L(\alpha)$ (see (5.4)). Therefore, for any of these values u, the integers $\bar u=S_1(u)$ and $\tilde u=S_2(u)$ are defined.

LEMMA 6.7. *Assume that, for some $m\geq 1$, either $\bar\gamma(m,x)>0$ for all $x\in X$, or $\bar\gamma(m,x)<0$ for all $x\in X$. Then T is aperiodic (does not have periodic points; $\bar\gamma(m,x)$ stands for $S_1(\gamma(m,x))$).*

Proof. Assume to the contrary that $T^p(x)=x$, for some $x\in X$ and $p\geq 1$. It follows from (6.5, a) that

$$\bar\gamma(mp,x)=\sum_{i=1}^{p}\bar\gamma(m,T^{m(i-1)}(x))\neq 0,$$

and hence $T_{mp}(x)-x=\gamma(mp,x)\neq 0$, a contradiction with the assumption $T^p(x)=x$. □

Let $k_0=k_0(T)$ be defined by (5.5). Clearly $k_0\geq 1$.

LEMMA 6.8. *Under the assumptions of Lemma 6.7, the length of any T-connection does not exceed $(m^2+2m)k_0$.*

Proof. Let $x_1,x_2,\ldots,x_n$ be a T-connection (see Definition 2.7) and assume that $n\geq(m^2+2m)k_0+1$. The contradiction is obtained as follows.

Denote $u=[(n-1)/m]$ and $v=n-um-1$ ($[y]$ denotes the maximal integer $\leq y\in\mathbb{R}$). Clearly $u\geq(m+2)k_0$ and $0\leq v\leq m-1$. It follows from (6.5, a) that

$$|\bar x_{um+1}-\bar x_1|=|\bar\gamma(um,x_1)|=\left|\sum_{i=1}^{u}\bar\gamma(m,T^{m(i-1)}(x_1))\right|\geq u.$$

Since $|\bar\gamma(1,x)|\leq k_0$ for all $x\in X$, it follows that

$$|\bar x_n-\bar x_{um+1}|=\left|\sum_{i=um+1}^{n-1}\bar\gamma(1,x_i)\right|\leq vk_0\leq(m-1)k_0.$$

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Therefore

$$|\bar{x}_n - \bar{x}_1| \geq |\bar{x}_{um+1} - \bar{x}_1| - |\bar{x}_n - \bar{x}_{um+1}| \geq u - (m-1)k_0 \geq 3k_0 > 2k_0,$$

which is impossible since $|\bar{x}_1|$, $|\bar{x}_n|$ do not exceed k_0 by definition of k_0 (see (5.5) and (2.8, b)). □

For every $n \geq 1$, define the partition of $X = [0, \beta_r)$:

$$X = X_+(n) \cup X_-(n) \cup X_*(n), \tag{6.9}$$

where $X_+(n) = \{x \in X \mid \bar{\gamma}(n, x) \geq 2k_0\}$, $X_-(n) = \{x \in X \mid \bar{\gamma}(n, x) < 0\}$, $X_*(n) = \{x \in X \mid 0 \leq \bar{\gamma}(n, x) \leq 2k_0 - 1\}$ and k_0 is as above (see (5.5)).

LEMMA 6.10. *Assume that for some $m \geq 1$ $X_*(m) = \varnothing$. Then each of the sets $X_+(m)$, $X_-(m)$ is invariant under T.*

Proof. Denote $x_n = S_1(T^n(x))$, $x \in X$, $n \geq 0$, and observe that $\bar{\gamma}(m, T(x)) = x_{m+1} - x_1$, that $\bar{\gamma}(m, x) = x_m - x_0$, and that $|x_{n+1} - x_n| \leq k_0$ for all $n \geq 0$. The lemma follows immediately from the inequality

$$|\bar{\gamma}(m, T(x)) - \bar{\gamma}(m, x)| \leq |x_{m+1} - x_m| + |x_1 - x_0| \leq 2k_0. \qquad \square$$

LEMMA 6.11. *Assume that for some $m \geq 1$ $X_*(m) = \varnothing$. Then T is aperiodic, and the length of any T-connection cannot exceed $(m^2 + 2m)k_0$.*

Proof. The arguments used in the proofs of Lemmas 6.8 and 6.10 work under the present conditions. □

In the following two lemmas we bound the lengths of minimal periods and the lengths of T-connections, in terms of the growth of the sum $\sum_{1 \leq i \leq H} \varepsilon_i(T)$.

LEMMA 6.12. *With the notations as above, assume that*

$$\sum_{1 \leq i \leq H} \varepsilon_i \geq 2k_0 |X| (|X| + 1), \tag{6.13}$$

for some $H \geq 1$ ($|Y|$ stands for the Lebesgue measure of the set $Y \subset \mathbb{R}$). Then either T possesses a periodic point $x \in X$ of period $< H$ or there exists a positive integer $m \leq H$ such that $X_*(m) = \varnothing$.

Proof. Assume that $X_*(i) \neq \varnothing$ for all $i = 1, 2, \ldots, H$. Since every set $X_*(i)$ is measurable relative to the partition D_i, it follows from the definition of ε_i (see (3.4) or (6.6)) that $|X_*(i)| \geq \varepsilon_i = \varepsilon_i(T)$. Denote

$$Y(i, j) = \{x \in X_*(i) \mid \bar{\gamma}(i, x) = j\}.$$

It is clear that

$$X_*(i) = \bigcup_{1 \leq j \leq 2k_0 - 1} Y(i, j).$$

Therefore the inequality

$$\sum_{i,j} \| Y(i, j) \| = \sum_i X_*(i) \geq \sum_i \varepsilon_i(T) \geq 2k_0 |X| (|X| + 1),$$

holds, where the indices i and j run over the sets $\{1, 2, \ldots, H\}$ and $\{0, 1, 2, \ldots, 2k_0 - 1\}$, respectively. For some $j = j_0$, the inequality

$$\sum_i \| Y(i, j_0) \| \geq |X| (|X| + 1)$$

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is fulfilled. It follows that, for some $x_0 \in X$, the cardinality of the set

$$S = \{i \in \{1, 2, \ldots, H\} \mid x_0 \in Y(i, j_0)\}$$

is at least $|X|+1$. For every $i \in S$, $\bar{\gamma}(i, x_0) = S_1(\gamma(i, x_0)) = j_0$, whence

$$\gamma(i, x_0) = j_0 \alpha + \tilde{\gamma}(i, x_0),$$

where $\tilde{\gamma}(i, x_0) = S_2(\gamma(i, x_0))$ (see (5.4)). Therefore, for every pair $u, v \in S$, the inequality

$$|\tilde{\gamma}(u, x_0) - \tilde{\gamma}(v, x_0)| = |\gamma(u, x_0) - \gamma(v, x_0)| = |T^u(x_0) - T^v(x_0)| < |X|$$

takes place. Since the cardinality of S is at least $|X|+1$, and since all $\tilde{\gamma}(i, x_0)$, $i \in S$, are integers, there exists a pair of different $u, v \in S$ such that $\tilde{\gamma}(u, x_0) = \tilde{\gamma}(v, x_0)$. Then $T^u(x_0) = T^v(x_0)$, whence $T^p(x_0) = x_0$ for $p = |u - v|$. Clearly $1 \le p \le H - 1$. This completes the proof of the lemma. □

LEMMA 6.14. *Under the conditions of Lemma* 6.12, *either T is aperiodic, and the length of every T-connections does not exceed* $(H^2+2H)k_0$, *or there exists a T-periodic point* $x \in X$ *of period* $< H$.

Proof. Assume that there are no T-periodic points of period $< H$. Then, by Lemma 6.12, for some positive integer $m \le H$, the relation $X_*(m) = \varnothing$ holds. Therefore, by Lemma 6.11, T must be aperiodic and $(m^2+2m)k_0 \le (H^2+2H)k_0$ is an upper bound on the lengths of possible T-connections. □

7. *Asymptotic growth of the sum* $\sum_{1\le i\le H} \varepsilon_i$.

We shall see that $\sum_{1\le i\le H} \varepsilon_i(T) \to \infty$ as $H \to \infty$. Moreover, we shall establish a lower estimate on the growth of this sum, in terms of arithmetical data on the irrational parameter α (Lemma 7.8). This allows us to find H satisfying (6.13), thus providing (see Lemma 6.14) an upper estimate on the lengths of possible periods and connections (Theorem 7.9).

It follows from (5.6) (see also (5.7) for notation) that

$$\sum_{1\le i\le H} \varepsilon_i \ge \sum_{1\le i\le H} \|\alpha, 2ik_0\|. \tag{7.1}$$

Let $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction representation of α (recall that α is an irrational positive number, and thus the integers $a_n = a_n(\alpha) \ge 1$ and the sequence of convergents p_n/q_n are defined). The denominators q_n of the convergents satisfy the recurrence

$$q_{n+1} = a_{n+1} q_n + q_{n-1}, \qquad n \ge 0, \tag{7.2}$$

where by definition $q_{-1} = 0$ and $q_0 = 1$. The sequence $\|\alpha, k\|$, $k \ge 1$, is non-increasing (see (5.7) for notation), and, for $k \ge 1$, $\|\alpha, k\| > \|\alpha, k+1\|$ if and only if $k = q_n - 1$ for some $n \ge 1$. For every $n \ge 1$ the following inequalities are fulfilled:

$$1/(2q_{n+1}) < \|\alpha, q_n\| = \|\alpha, q_{n+1} - 1\| = \|\alpha q_n\| = |q_n \alpha - p_n| < 1/q_{n+1} \tag{7.3}$$

The facts quoted in the preceding paragraph are basic in the theory of continued fraction representations of real numbers.

LEMMA 7.4. *For every $n\geq 0$, the following inequality holds*:

$$\sum_{i=q_n}^{q_{n+2}-1} \|\alpha, i\| > 1/4. \tag{7.5}$$

Proof

$$\sum_{i=q_n}^{q_{n+2}-1} \|\alpha, i\| \geq (q_{n+2}-q_n)\|\alpha, (q_{n+2}-1)\| > (q_{n+2}/2)(1/(2q_{n+2})) = 1/4.$$

We used the inequality (7.3) and the fact that $q_{n+2}\geq 2q_n$. □

LEMMA 7.6. *For every $k\geq 1$, the following inequality holds*:

$$\sum_{i=1}^{q_{2k}-1} \|\alpha, i\| > k/4.$$

Proof. Sum up (7.5) for $n=0, 2, \ldots, (2k-2)$. □

LEMMA 7.7. *Let $b_1, b_2, \ldots$ be a non-increasing sequence of positive numbers. Let n and u be positive integers. Then*

$$\sum_{i=1}^{n} b_{iu} \geq 1/u\left(\sum_{i=1}^{u(n+1)} b_i\right) - b_1.$$

Proof.

$$\sum_{i=1}^{n} b_{iu} \geq 1/u \sum_{i=u+1}^{u(n+1)} b_i = 1/u\left(\sum_{i=1}^{u(n+1)} b_i - \sum_{i=1}^{u} b_i\right) \geq 1/u\left(\sum_{i=1}^{u(n+1)} b_i\right) - b_1.$$

LEMMA 7.8. *Let $R>0$ be real, let $t\geq 8k_0L+4k_0$ be an integer, denote $H=[(q_{2t}-1)/(2k_0)]$ (where $[y]$ denotes the largest integer $\leq y\in\mathbb{R}$). Then the inequality*

$$\sum_{1\leq i\leq H} \varepsilon_i > R$$

takes place.

Proof. We apply in succession the inequality (7.1), Lemma 7.7 (with $u=2k_0$ and $b_i=\|\alpha, i\|$) and Lemma 7.6:

$$\sum_{1\leq i\leq H} \varepsilon_i \geq \sum_{1\leq i\leq H} \|\alpha, 2ik_0\| \geq 1/(2k_0) \sum_{i=1}^{q_{2t}-1} \|\alpha, i\| - 1/2$$

$$> t/(8k_0) - 1/2 \geq R. \quad □$$

THEOREM 7.9. *Let $t\geq 16(k_0)^2|X|(|X|+1)+4k_0$ be an integer, denote $H=[(q_{2t}-1)/(2k_0)]$. Then either T is aperiodic and the lengths of all T-connections are $\leq(H^2+2H)k_0$ or there exists a T-periodic point $x\in X$ of period $<H$.*

Proof. Taking $R=2k_0|X|(|X|+1)$ in Lemma 7.8, we obtain inequality (6.13). It remains to apply Lemma 6.14. □

8. *Tests for aperiodicity and minimality*

Through this section we assume that a rank 2 interval exchange transformation T is given in the form (5.3). We shall describe an algorithm to test T for aperiodicity and minimality.

One computes the integers $k_0=k_0(T)\geq 1$ (see (5.5)), t (the minimal integer which is $\geq 16(k_0)^2|X|(|X|+1)+4k_0$) and $H=[(q_{2t}-1)/(2k_0)]$ (see Theorem 7.9; q_n stands

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for the n-th denominator in the sequence of convergents p_n/q_n to α.). Note that inequality (6.13) holds (see the first sentence in the proof of Theorem 7.9).

For $m = 1, 2, \ldots$ in succession, one tests the following two conditions:
(c1) $T^m(x) = x$, for (at least) one of the discontinuities $x = \beta_j (1 \le j \le r-1)$ of T.
(c2) $X_*(m) = \varnothing$ (see (6.9)).

One stops testing when one of the conditions, either (c1) or (c2), is fulfilled. Lemmas 6.12 guarantees that this happens when $m = m_0 \le H$. If (c1) is fulfilled, then T has a periodic point. If (c2) is fulfilled, then, by Lemma 6.11, T is aperiodic.

One may test only the condition (c1) for $m = 1, 2, \ldots, H$. If the condition fails for all m, one concludes (in view of Lemma 6.14) that T is aperiodic. This method is easier to describe, but it is normally less effective if T is aperiodic since H could be very large.

The condition (c2) means that $\bar{\gamma}(m, x) \notin \{0, 1, \ldots, 2k_0 - 1\}$, for all $x \in X$. This condition in the above described algorithm can be replaced by the weaker one:
(c3) There is a set S of $2k_0$ successive integers containing 0 and such that $\bar{\gamma}(m, x) \notin S$, for all $x \in X$.

If T is found to be aperiodic, one has an upper bound $((m_0)^2 + 2m_0)k_0$ one the lengths of possible T-connections (see Lemma 6.8). (If the value of m_0 is not known, the *a priori* upper estimate $(H^2 + 2H)k_0 \ge ((m_0)^2 + 2m_0)k_0$ on the lengths of possible T-connections can be used). Given an upper bound on the lengths of possible T-connections, the decomposition of X onto minimal and periodic components can be found (see Remark 2.18).

We have proved slightly more. If the rank two iet T is found to be aperiodic, the problem of minimal decomposition of T can be effectively solved. In particular, one can decide whether T is minimal (equivalently, uniquely ergodic).

Note without proof that in the remaining case (T has periodic points) the problem of finding an upper bound L on the lengths of possible T-connections can also be effectively solved using our approach. From the above discussion it is only clear that the length of the minimal period does not exceed H.

9. *Concluding remarks*

Assume that a rank 2 interval exchange transformation T is given in the form (5.3). T is completely determined by its discrete pattern (5.2) and its irrational parameter $\alpha > 0$. The following theorem follows from the algorithm presented in the preceding section.

THEOREM 9.1. *The following properties of T are not affected by small perturbations of the irrational parameter α:*
(1) *aperiodicity*;
(2) *minimality (equivalently, unique ergodicity)*;
(3) *the numbers of minimal and periodic components.*

Proof. We can test T for the Properties (1) and (2) in the way described in the preceding section. Clearly the results of the tests will not be affected by small changes of α. The above argument works also for (3) assuming that T is aperiodic.

In the remaining case (T has periodic points) similar arguments can be used to prove that the numbers of minimal and periodic components are invariant under small changes of α (we omit the details; see the last paragraph in §8). □

One can try to generalize Theorem 9 for the case rank$(T) = n > 2$ (so that T is determined by its discrete pattern and by $(n-1)$ linearly independent over Q parameters). However, the obvious version of such generalization turns out to be wrong already for $n = 3$.

We have established an upper bound on the length of minimal period and (in aperiodic case) on the lengths of possible T-connections (Theorem 7.9). These bounds depend only on k_0, $|X|$ and the irrational parameter α. We conjecture that such bounds are possible in terms of the discrete pattern of T only (that is, that the bounds may be independent of α). The conjecture would imply, for a given discrete pattern of a rank 2 iet, that with a monotonic change of α, the validity of the Properties (1) and (2) in Theorem 9.1 changes only a finite number of times. Our results imply that the changes are possible only at rationals.

Another interesting problem is whether spectral properties of a rank 2 iet T (like weak mixing) are also stable under small perturbations of the irrational parameter. More generally, one would like to have a systematic way to examine spectral properties of T (assuming that T is minimal).

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