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ABSTRACT

In this article, we give an analytic construction of ALF hyperkähler metrics on *smooth deformations* of the Kleinian singularity $\mathbb{C}^2/\mathcal{D}_k$, with \mathcal{D}_k the binary dihedral group of order $4k$, $k \geq 2$. More precisely, we start from the ALE hyperkähler metrics constructed on these spaces by Kronheimer, and use analytic methods, e.g. resolution of a Monge–Ampère equation, to produce ALF hyperkähler metrics with the same associated Kähler classes.

Introduction

This article deals with an analytic construction of a certain class of examples of four-dimensional *non-compact, complete, Ricci-flat manifolds*. One prominent feature of such spaces lies in their appearance as limit spaces, after rescaling, of families of compact Einstein 4-manifolds; this, among others, illustrates the interesting role played by non-compact complete Ricci-flat manifolds in Riemannian geometry in dimension 4.

Now, dimension 4 moreover allows one to specialise the question to *Ricci-flat Kähler*, and even to *hyperkähler*, non-compact, complete manifolds. If one adds furthermore a decay condition on the Riemannian curvature tensor, this leads to the following definition.

DEFINITION 0.1 (Gravitational instantons). Let (X, g, I, J, K) be a non-compact, complete, hyperkähler manifold of real dimension 4. Then X is called *gravitational instanton* if its Riemannian curvature tensor Rm^g satisfies the following L^2 condition:

$$\int_X |\text{Rm}^g|^2 \text{vol}^g. \quad (1)$$

Besides this differential-geometric definition, gravitational instantons also appear as fundamental objects in theoretical physics, where Condition (1) is thought of as a ‘finite type action’ assumption, in fields such as Quantum Gravity [Haw77] or String and M-Theories, see [CH05, CK99] and references therein.

Recall that hyperkähler metrics are Ricci-flat. The fundamental Bishop–Gromov theorem [Gro99] thus implies that on gravitational instantons, ball volume grows at most with Euclidean rate. In other words, introducing a ‘ball volume growth ratio function’

$$\nu : x \mapsto \frac{\rho_x^4}{\text{Vol}_g(B_g(o, \rho_x))}$$

(where ρ is on X the distance to some fixed point $o \in X$, the choice of which does not affect the asymptotic behaviour of ν), this function is at least bounded below by a positive constant c .

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If a bound $\nu \leq C$, $C > 0$, also holds, that is, if large balls on X do have Euclidean growth rate, one deals with *Asymptotically Locally Euclidean* (‘ALE’) instantons. These hyperkähler manifolds are very well understood: they are completely classified, after [BKN89] and [Kro89b] (with recent extension [Šuv12] and [Wri11] to the Kähler Ricci-flat case), and their classification corresponds to an exhaustive construction by Kronheimer [Kro89a]; we shall often refer to these spaces as *Kronheimer’s instantons* for this reason. In a nutshell, the hyperkähler structures of these spaces are asymptotic to that of a quotient \mathbb{R}^4/Γ , with Γ a finite subgroup of $SU(2) = Sp(1)$; when moreover Γ is fixed, these spaces are all diffeomorphic to the *minimal resolution of the Kleinian singularity* \mathbb{C}^2/Γ . The ALE class is moreover the standard one regarding the scaling limit process of Einstein 4-manifolds, since ALE spaces appear in the *non-collapsing* case, see e.g. [BKN89].

Now, a result by Minerbe [Min07] (see also [CC15a]) states the following quantisation on gravitational instantons: if¹ the volume form vol^g is replaced by the measure νvol^g in Condition (1), and if the asymptotic ball volume growth is less than Euclidean, i.e. quartic, then it is at most cubic; one jumps from a bound $\nu \geq c$ to a bound $\nu \geq c(\rho+1)$. If an analogous reverse upper bound $\nu \leq C(\rho+1)$ holds (the measure used in Condition (1) being vol^g in full generality, although one can push to νvol^g on many known examples), one then speaks about *Asymptotically Locally Flat*, or *ALF*, gravitational instantons. It is now known that such spaces indeed appear in their turn as scaled limits of (collapsing) Einstein 4-manifolds, see e.g. [Fos16] for a nice construction based on K3 surfaces. Moreover, under the additional condition $\text{Rm} = O(\rho^{-3})$ (actually, here, a consequence of $\text{Rm}^g \in L^2(\nu \text{vol}^g)$ [Min07]), Minerbe classifies (a rough) half of the ALF instantons [Min11]; their geometry at infinity is that of a circle fibration over \mathbb{R}^3 , and they are explicitly described by the so-called *Gibbons–Hawking ansatz*. This includes the prototypical Taub-NUT metric, living on \mathbb{R}^4 itself [EGH80].

Results. When $\text{Rm}^g \in L^2(\nu \text{vol}^g)$, the only possibility left for the asymptotic geometry of the ALF gravitational instantons is that of a circle fibration over \mathbb{R}^3/\pm [Min07]. This second family, for which a classification appeared only recently [CC15b, Theorem 1.2], includes: Atiyah–Hitchin’s ‘ D_0 -instanton’ [AH88] and a family of hyperkähler deformations of its double cover, the ‘ D_1 -instantons’; Page–Hitchin’s ‘ D_2 -instantons’ [Pag81, Hit83]; the ‘ \mathcal{D}_k (or D_{k+2})-families’, $k \geq 1$, produced by Cherkis and Kapustin [CK98, CK99] and made more precise by Cherkis and Hitchin [CH05]. To this regard, our main result consists in a construction of such spaces with independent methods (see ‘Comments’ below); it is aimed in particular at underlining the relation between ALE and ALF \mathcal{D}_k -instantons, and can be stated as follows.

THEOREM 0.2. *Let $(X, g, I_1^X, I_2^X, I_3^X)$ be an ALE gravitational instanton modelled on $\mathbb{R}^4/\mathcal{D}_k$, with \mathcal{D}_k the binary dihedral group of order $4k$, $k \geq 2$, in the sense that the infinities of X and $\mathbb{R}^4/\mathcal{D}_k$ are diffeomorphic, and that the hyperkähler structure of X is asymptotic to that of $\mathbb{R}^4/\mathcal{D}_k$ via the diffeomorphism in play. Then there exists on X a family of ALF hyperkähler structures $(g_m, J_{1,m}^X, J_{2,m}^X, J_{3,m}^X)_{m \in (0, \infty)}$, such that, for any fixed $m \in (0, \infty)$:*

- (i) *one can choose the diffeomorphism above so that g_m is asymptotic to the \mathcal{D}_k -quotient of \mathfrak{f}_m , the Taub-NUT metric with fibres of length $\pi(2/m)^{1/2}$ at infinity;*

¹ This strengthened decay curvature condition is indeed required in the quantisation results in [Min07], and [CC15a]. With the finite action condition $\text{Rm}^g \in L^2(\text{vol}^g)$ only, the integer quantisation of the ball volume growth rate is in general not quite granted, see [Hei10].

- (ii) the Kähler classes $[g_m(J_{j,m}^X \cdot, \cdot)]$, $j = 1, 2, 3$, are the same as those of the initial ALE hyperkähler structure, at least one of the parallel complex structures $aJ_{1,m}^X + bJ_{2,m}^X + cJ_{3,m}^X$, with $a^2 + b^2 + c^2 = 1$, of g_m equals one of those of g_X , and moreover $\text{vol}^{g_m} = \text{vol}^{g_X}$;
- (iii) the curvature tensor Rm^{g_m} has cubic decay.

As is understood here, the Taub-NUT metric \mathbf{f}_m is invariant under \mathcal{D}_k , and thus makes perfect sense on $\mathbb{R}^4/\mathcal{D}_k$. About the diffeomorphism chosen in point (i), given a \mathcal{D}_k ALE instanton X , one can conveniently extract a (n asymptotically tri-holomorphic) diffeomorphism F_X between the infinity of $\mathbb{R}^4/\mathcal{D}_k$ and that of the instanton from Kronheimer’s construction [Kro89a]; here, we simply correct such an F_X by right-composition with a rotation of \mathbb{R}^4 and with a map of the form $x \mapsto (1 + a/|x|^4)x$, $a \in \mathbb{R}$. The asymptotics between the ALF metric g_m and \mathbf{f}_m in point (i) are a direct and natural by-product of our construction, and are as follows: if $R = d_{\mathbf{f}_m}(0, \cdot)$ ($0 \in \mathbb{R}^4/\mathcal{D}_k$), then $(g_m - \mathbf{f}_m)$ and $\nabla^{\mathbf{f}_m}(g_m - \mathbf{f}_m)$ are $O(R^{-2+\epsilon})$ for all $\epsilon > 0$ (and we point out the unusual fact that our construction does give similar decays for both estimates of order 0 and 1, instead of establishing decays improving as the order of differentiation grows).

Before discussing in more detail how Theorem 0.2 is proved, we shall underline that our construction heavily relies on the computation of the asymptotics of the ALE instantons modelled on $\mathbb{R}^4/\mathcal{D}_k$. More precisely, the construction of these spaces by Kronheimer allows one to write down these asymptotics as power series, the main term of which is the Euclidean model $(\mathbf{e}, I_1, I_2, I_3)$, and this actually holds for any finite subgroup Γ of $\text{SU}(2)$ alluded to above. We describe in this article the first non-vanishing terms of those expansions.

THEOREM 0.3. *Let $(X, g_X, I_1^X, I_2^X, I_3^X)$ be an ALE gravitational instanton modelled on \mathbb{R}^4/Γ . Then one can choose a diffeomorphism Φ between X minus a compact subset and \mathbb{R}^4/Γ minus a ball such that:*

- (i) $\Phi_*g_X - \mathbf{e} = h_X + O(r^{-6})$, $\Phi_*I_1^X - I_1 = \iota_1^X + O(r^{-6})$ and if $\omega_1^X = g_X(I_1^X \cdot, \cdot)$ and $\omega_1^{\mathbf{e}} = \mathbf{e}(I_1 \cdot, \cdot)$, then $\Phi_*\omega_1^X - \omega_1^{\mathbf{e}} = \varpi_1^X + O(r^{-6})$, where h_X , ι_1^X and ϖ_1^X admit explicit formulas and are $O(r^{-4})$; for instance $\varpi_1^X = -\sum_{j=1}^3 c_j(X) dd_{J_j}^c(r^{-2})$ for some explicit constants $c_j(X)$.
- (ii) when Γ is not a cyclic subgroup of $\text{SU}(2)$, the $O(r^{-6})$ of the previous point can be replaced by $O(r^{-8})$.

Here the O are understood in an asymptotically Euclidean context: ε is $O(r^{-a})$ if for any $\ell \geq 0$, $|(\nabla^{\mathbf{e}})^\ell \varepsilon|_{\mathbf{e}} = O(r^{-a-\ell})$ near infinity.

Another crucial analytic tool in our construction is a Calabi–Yau type theorem, adapted to ALF geometry.

THEOREM 0.4. *Let $(\mathcal{Y}, g_{\mathcal{Y}}, J_{\mathcal{Y}}, \omega_{\mathcal{Y}})$ be an ALF Kähler 4-manifold of dihedral type. Let f be a smooth function such that $|(\nabla^{g_{\mathcal{Y}}})^\ell f|_{g_{\mathcal{Y}}} = O(\rho^{\beta-2-\ell})$ for some $\beta \in (0, 1)$ and for all $\ell \geq 0$. Then there exists a smooth function φ such that $\omega_{\mathcal{Y}} + dd_{J_{\mathcal{Y}}}^c \varphi$ is Kähler, such that $|(\nabla^{g_{\mathcal{Y}}})^\ell \varphi|_{g_{\mathcal{Y}}} = O(\rho^{-\beta-\ell})$ for all $\ell \geq 0$, and verifying the Monge–Ampère equation:*

$$(\omega_{\mathcal{Y}} + dd_{J_{\mathcal{Y}}}^c \varphi)^2 = e^f \omega_{\mathcal{Y}}^2. \tag{2}$$

Let us say at this stage that an ALF Kähler 4-manifold of dihedral type is a complete non-compact Kähler manifold of real dimension 4, agreeing at infinity with a dihedral quotient of $\mathbb{C}^2 = (\mathbb{R}^4, I_1)$ with Taub-NUT metric, ‘up to infinite order’. The precise meaning of this assertion is given below, when invoking Theorem 0.4.

Comments. We should start with some words on previous constructions of ALF dihedral gravitational instantons. As mentioned above, \mathcal{D}_k ALF instantons are known to exist since

the works [CK98, CK99], where such spaces are produced as *moduli spaces of solutions to Nahm's equations or of singular monopoles*; they have moreover been described in an explicit manner in [CH05], via *generalised Legendre transform and twistor theory*; see also [Dan94] for an alternative proposition using *hyperkähler quotient*. Despite this, and the fact that in these various cases the underlying spaces are Kronheimer's instantons, due to the difference in the methods of construction, we were not able to show directly that these previous examples and our examples coincide. However, according to the classification² given in [CC15b], *our construction and Cherkis–Kapustin's construction produce the same families of \mathcal{D}_k ALF hyperkähler metrics, $k \geq 2$, and are more precisely (almost) exhaustive, in the sense that any ALF dihedral gravitational instanton, except Atiyah–Hitchin's D_0 and D_1 -instantons, Page–Hitchin's D_2 -instantons (and the \mathcal{D}_1 family for our construction), fits into the produced examples (up to a tri-holomorphic isometry)*. Chen–Chen's classification seems nonetheless delicate to avoid in affirming that both constructions do coincide; it would thus be interesting to understand how the different constructions link up in this respect.

More closely to the statement of Theorem 0.2, notice that it is *not of a perturbative nature*: this corresponds to taking the parameter m in the whole range $(0, +\infty)$. The price to pay is somehow that so far, we do not control what happens when m goes to 0. We conjecture that the ALF hyperkähler structure converges back, in C_{loc}^∞ -topology, to the initial ALE one, as is the case on \mathbb{C}^2 ; this question will be handled in a future article.

Now in Theorem 0.3, the existence of the first order variation terms h_X , ι_1^X and ϖ_1^X is of course not new, as they already appear along Kronheimer's construction [Kro89a]. What is new though is their explicit determination, which we could only find in the literature for the simplest case of the Eguchi–Hanson space (see e.g. [Joy00, p. 153]), i.e. when $\Gamma = \mathcal{A}_2 = \{\pm \text{id}_{\mathbb{C}^2}\}$. Notice at this point that as suggested by the statement of Theorem 0.3, the shapes of h_X , ι_1^X and ϖ_1^X follow a general pattern which is only slightly affected by the order of the group Γ ; up to a multiplicative constant, we can indeed compute them on the explicit Eguchi–Hanson example. We think moreover that Theorem 0.3 is of further interest; for instance, the order of precision it brings could be useful in more general gluing constructions.

To conclude, we comment briefly on Theorem 0.4. This result comes within the general scope of generalising the celebrated Calabi–Yau theorem [Yau78] to non-compact manifolds, initiated by Tian and Yau [TY90, TY91]. A statement similar to ours can be found in [Hei10, Proposition 4.1], in a more general and abstract framework. One interest of our statement, nonetheless, simply lies in the fact that although we ask more precise asymptotics on our data than Hein does, we get in compensation sharper asymptotics on our solution φ , which echo in the asymptotics of Theorem 0.2.

Organisation of the article. This paper is divided into three parts, corresponding respectively to Theorems 0.2, 0.3 and 0.4, plus an appendix. Part 1 is devoted to the proof of Theorem 0.2. We first draw in § 1.1 a detailed program of construction of our hyperkähler ALF metrics, leading us to the expected result (Theorem 1.3). In § 1.2 are recalled essential facts on the Taub-NUT metric, seen as a Kähler metric on \mathbb{C}^2 . The construction itself occupies §§ 1.3 and 1.4; it consists of a gluing of the 'non-compact part' of the Taub-NUT metric with the 'compact part' of the ALE metric of some ALE instanton, which we subsequently correct into a Ricci-flat metric thanks to Theorem 0.4 (recalled as Theorem 1.16). The concluding § 1.5, mainly computational, deals with the proof of two technical lemmas useful to our construction.

² This classification answers a folklore conjecture and establishes a strong link between ALE and ALF instantons; it can be seen as an analogue of the classification of [Min11], including the specificities of the dihedral case.

In Part 2, which is mostly independent of Part 1, after recalling some basic facts about Kronheimer’s construction of ALE instantons, we state Theorem 2.1, which is a specified version of Theorem 0.3; in particular we give the promised explicit formulas (§ 2.1). We give further details on Kronheimer’s construction and classification in § 2.2, where we also fix the diffeomorphism of Theorem 0.3. Then we compute the tensors h_X , ι_1^X and ϖ_1^X in § 2.3; using similar techniques, we show in § 2.4 that the precision of the asymptotics is automatically improved when Γ is binary dihedral, tetrahedral, octahedral or icosahedral. We develop in § 2.5 a few informal digressive considerations on the approximation of complex structures of certain ALE instantons by the standard I_1 , relied on links observed in the construction of Part 2.

Part 3 is devoted to the proof of Theorem 0.4. As all the necessary elements are essentially already available in the literature, instead of running a continuity method as is classical for solving a Monge–Ampère equation, we proceed, for the sake of concision, by establishing *a posteriori* estimates. This is explained in the introduction of Part 3, and the required analysis is done in the following three sections.

Finally, the appendix gives a short account of a description of the Taub-NUT metric on \mathbb{C}^2 suggested by LeBrun [LeB91].

Throughout the article, \mathbb{C}^2 stands for (\mathbb{R}^4, I_1) with I_1 the standard complex structure given by the coordinates $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$; we denote by I_2 and I_3 the other two standard complex structures on $\mathbb{R}^4 \cong \mathbb{H}$, given respectively by the coordinates $(x_1 + ix_3, x_4 + ix_2)$ and $(x_1 + ix_4, x_2 + ix_3)$.

1. Construction of ALF hyperkähler metrics

1.1 Strategy of construction

Outline of the strategy. As described in [LeB91] and as we shall see in the next section, one can describe the Taub-NUT metric on \mathbb{R}^4 as a \mathcal{D}_k -invariant hyperkähler metric with volume form the standard Euclidean one $\Omega_{\mathbf{e}}$, Kähler for the standard complex structure I_1 , and compute a somehow explicit potential, φ say, for it.

Now, given one of Kronheimer’s ALE gravitational instantons $(X, g_X, I_1^X, I_2^X, I_3^X)$ modelled on $\mathbb{R}^4/\mathcal{D}_k$, we have a diffeomorphism Φ_X between infinities of X and $\mathbb{R}^4/\mathcal{D}_k$ such that $\Phi_{X*}g_X$ is asymptotic to the standard Euclidean metric \mathbf{e} , and $\Phi_{X*}I_1^X$ is asymptotic to I_1 . It is in this way quite natural to try and take, as an ALF metric on X , $dI_1^X d(\Phi_X^* \varphi)$ glued with $g_X(I_1^X \cdot, \cdot)$ written near the infinity of X as an I_1^X -complex hessian with sufficient precision, before we correct it into a hyperkähler metric. This naive idea works in a straightforward manner when (X, I_1^X) is a *minimal resolution of $(\mathbb{C}^2/\mathcal{D}_k, I_1)$ and Φ_X the associated map*. However this fails in the general case, where (X, I_1^X) is a *deformation of $(\mathbb{C}^2/\mathcal{D}_k, I_1)$* , without further precautions: the size of the Taub-NUT potential φ , roughly of order r^4 as well as its Euclidean derivatives, together with the error term $\Phi_{X*}I_1^X - I_1$ on the complex structure, even make wrong the assertion that the rough candidate $dI_1^X d(\Phi_X^* \varphi)$ is positive, in the sense that $dI_1^X d(\Phi_X^* \varphi)(\cdot, I_1^X \cdot)$ is a metric, near the infinity of X .

Fortunately, up to choosing a different complex structure on X to work with, we can write down explicitly a sufficiently accurate potential for the first Kähler form $g_X(I_1^X \cdot, \cdot)$ ³ and make the appropriate corrections on φ so as to get a good enough ALF metric on X to start with, and

³ The problem of writing this form, or other representatives of its class, as *exact* I_1^X -complex Hessians at infinity is actually delicate (see the end of § 1.3.1), hence our choice of expressing $g_X(I_1^X \cdot, \cdot)$, which we happen to know very well, and which happens to be the only natural representative sufficiently well known to carry over the overall program, as an *approximate* I_1^X -complex hessian with enough precision for our construction.

then run the same machinery as in the minimal resolution case, up to minor but yet technical adjustments, so as to end up with Theorem 0.2.

Detailed strategy, and involvements of Kronheimer’s instantons asymptotics. We shall now be more specific about the different steps involved in the program we are following throughout this part.

- (i) Let $SO(3)$ act on the complex structures of X as follows: for $A = (a_{j\ell}) \in SO(3)$, define the triple (AI^X) . as

$$(AI^X)_j = ((AI^X)_j)_{j=1,2,3} = (a_{j1}I_1^X + a_{j2}I_2^X + a_{j3}I_3^X)_{j=1,2,3};$$

then $(X, g_X, (AI^X)_1, (AI^X)_2, (AI^X)_3)$ is again hyperkähler, and is therefore an ALE gravitational instanton modelled on $\mathbb{R}^4/\mathcal{D}_k$.

- (ii) With the model $\mathbb{R}^4/\mathcal{D}_k$ at infinity fixed, Kronheimer’s instantons are parametrised [Kro89a] by a triple $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, where \mathfrak{h} is a $(k + 2)$ -dimensional real vector space endowed with some scalar product $\langle \cdot, \cdot \rangle$, and D is a finite union of spaces $H \otimes \mathbb{R}^3$ with H a hyperplane in \mathfrak{h} (as notation suggests, \mathfrak{h} is a Lie algebra, the interpretation of which we will be more specific about in Part 2; for now, let us mention that there is a natural identification $\mathfrak{h} \simeq H^2_{\text{cpt}}(X, \mathbb{R})$, and, $H^2_{\text{cpt}}(X, \mathbb{R})$ being in turn isomorphic to $H^2(X, \mathbb{R})$, $(\zeta_1, \zeta_2, \zeta_3)$ corresponds to $([\omega_1^X], [\omega_2^X], [\omega_3^X])$ under this identification, which moreover sends Killing form products to cup products). Kronheimer’s ζ -parametrisation is compatible with the $SO(3)$ -action of point (i) in the sense that if ζ is the parameter associated to $(X, g_X, I_1^X, I_2^X, I_3^X)$, and if $(Y, g_Y, I_1^Y, I_2^Y, I_3^Y)$ is the instanton associated to $A\zeta$, defined by

$$A\zeta = ((A\zeta)_j)_{j=1,2,3} = (a_{j1}\zeta_1 + a_{j2}\zeta_2 + a_{j3}\zeta_3)_{j=1,2,3}, \tag{3}$$

then $(Y, g_Y, I_1^Y, I_2^Y, I_3^Y)$ and $(X, g_X, (AI^X)_1, (AI^X)_2, (AI^X)_3)$ are isometrically tri-holomorphic: this is Lemma 2.3, stated and proved in Part 2. Defined this way, $A\zeta$ is of course still in $\mathfrak{h} \otimes \mathbb{R}^3 - D$; otherwise $A\zeta \in H \otimes \mathbb{R}^3$ for one of the hyperplanes H mentioned above, and thus $\zeta = A^t(A\zeta) \in H \otimes \mathbb{R}^3$, which would be absurd.

- (iii) In general, one can take the diffeomorphism Φ_X between infinities of X and $\mathbb{R}^4/\mathcal{D}_k$ so that $\Phi_{X*}I_1^X - I_1 = O(r^{-4})$ with according decay on derivatives, which is not good enough for the construction we foresee. We can nonetheless improve the precision thanks to the following two lemmas.

LEMMA 1.1. *If $\xi \in \mathfrak{h} \otimes \mathbb{R}^3 - D$ is such that $|\xi_2|^2 - |\xi_3|^3 = \langle \xi_2, \xi_3 \rangle = 0$, and $(Y, g_Y, I_1^Y, I_2^Y, I_3^Y)$ is the associated ALE instanton, then one can choose Φ_Y such that there exists a diffeomorphism $\beth = \beth_\xi$ between infinities of \mathbb{R}^4 commuting with the action of \mathcal{D}_k , and such that*

$$|(\nabla^e)^\ell(\Phi_{Y*}I_1^Y - \beth^*I_1)|_e = O(r^{-8-\ell}) \quad \text{for all } \ell \geq 0.$$

Moreover, the shape of \beth is given by $\beth(z_1, z_2) = (1 + (a/r^4))(z_1, z_2)$, with $a = a_\xi \in \mathbb{R}$, and (z_1, z_2) the standard complex coordinates on (\mathbb{C}^2, I_1) , and moreover $|(\nabla^e)^\ell(\Omega_e - \beth^*\Omega_e)|_e = O(r^{-8-\ell})$ for all $\ell \geq 0$.

LEMMA 1.2. *For any $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$, there exists $A \in SO(3)$ such that $|(A\zeta)_2|^2 - |(A\zeta)_3|^2 = \langle (A\zeta)_2, (A\zeta)_3 \rangle = 0$.*

Lemma 1.1, which relies on our analysis of the asymptotics of Kronheimer’s instantons, is proved in § 1.3, assuming a general statement for these asymptotics that is seen in Part 2; the hypothesis on ξ_2 and ξ_3 is moreover commented on in § 2.5. Lemma 1.2, which is elementary, is proved at the end of this section.

- (iv) Recall that ζ is the parameter of our given instanton X . We choose A as in Lemma 1.2, consider the instanton Y associated to $\xi = A\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, and perform the gluing of the Kähler forms and correct a prototypical ALF metric into a hyperkähler metric, with the potential $\varphi^b := \beth^*\varphi$ instead of φ . Thanks to the better coincidence of the complex structures, the rough candidate $dI_1^Y d(\Phi_Y^* \varphi^b)$ is now positive at infinity, and actually also rather close to $\mathbf{f}^b := \beth^*\mathbf{f}$, with \mathbf{f} the Taub-NUT metric on \mathbb{R}^4 , which is Kähler for I_1 . We should moreover specify here that the gluing also requires a precise description of the Kähler form $\omega_1^Y := g_Y(I_1^Y \cdot, \cdot)$, which is again part of the analysis of the asymptotics of Kronheimer’s ALE instantons.
 We get this way, after a three-step correction process (making the metric Ricci-flat near infinity, putting it in Bianchi gauge, and concluding with a Calabi–Yau type theorem), a Ricci-flat, actually a hyperkähler, manifold $(Y, g'_Y, I_1^Y, J_2^Y, J_3^Y)$, with $\Phi_{Y*}g'_Y$ asymptotic to \mathbf{f}^b , and $[g'_Y(I_1^Y \cdot, \cdot)] = [g_Y(I_1^Y \cdot, \cdot)]$; the construction also gives $[g'_Y(J_j^Y \cdot, \cdot)] = [g_Y(I_j^Y \cdot, \cdot)]$, $j = 2, 3$.
- (v) We let $A^t = A^{-1}$ act back on the previous data to come back to X , and end up with a hyperkähler manifold $(X, g'_X, J_1^X, J_2^X, J_3^X)$, with $[g'_X(J_j^X \cdot, \cdot)] = [g_X(I_j^X \cdot, \cdot)]$, $j = 1, 2, 3$, and $\Phi_{X*}g'_X$ asymptotic to \mathbf{f}^b , provided that Φ_X is the composition of Φ_Y and the tri-holomorphic isometry in play in point (ii).

We shall also add that we can play on the metric \mathbf{f} in this construction. Indeed, \mathbf{f} is invariant under some fixed circle action on \mathbb{R}^4 , and the length for \mathbf{f} of the fibres of this action tends to some constant $L > 0$ at infinity. We can make this length vary in the whole $(0, \infty)$ and keep the same volume form for \mathbf{f} ; given $m \in (0, \infty)$ that we call the ‘mass parameter’, we then denote by \mathbf{f}_m the Taub-NUT metric giving length $L(m) = \pi\sqrt{2/m}$ to the fibres at infinity, and of volume form Ω_e (the choice of the parameter m instead of L will become clear in the next section).

We can then sum our construction up by the following statement, which is the main result of this part, and is a specified version of Theorem 0.2.

THEOREM 1.3. *Consider an ALE gravitational instanton $(X, g_X, I_1^X, I_2^X, I_3^X)$ modelled on $\mathbb{R}^4/\mathcal{D}_k$. Then there exists a one-parameter family $(g'_{X,m}, J_{1,m}^X, J_{2,m}^X, J_{3,m}^X)$ of smooth hyperkähler metrics on X such that, for any fixed $m \in (0, \infty)$:*

- the equality $[g'_{X,m}(J_{j,m}^X \cdot, \cdot)] = [g_X(I_j^X \cdot, \cdot)]$ of Kähler classes holds for $j = 1, 2, 3$;
- $g'_{X,m}$ and g_X have the same volume form;
- $g'_{X,m}$ is ALF in the sense that one has the asymptotics

$$|(\nabla^{\mathbf{f}_m}{}^\ell)(\Phi_{X*}g'_{X,m} - \mathbf{f}_m)|_{\mathbf{f}_m} = O(R^{-1-\delta}), \quad \ell = 0, 1,$$

for any $\delta \in (0, 1)$, and that $\text{Rm}^{g'_{X,m}}$ has cubic decay at infinity.

Here R is a distance function for \mathbf{f}_m , and Φ_X is an ALE diffeomorphism between infinities of X and $\mathbb{R}^4/\mathcal{D}_k$, in the sense that $|\Phi_{X*}g_X - e|_e$ and $|\Phi_{X*}I_j^X - I_j|_e$ are $O(r^{-4})$, with according decay on derivatives.

In this statement, $\mathbf{f}_m^b = \beth^*\mathbf{f}_m$, where $\beth = \beth_{A\zeta}$ is given by Lemma 1.1, $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$ is the parameter associated to $(X, g_X, I_1^X, I_2^X, I_3^X)$, and A is chosen as in Lemma 1.2. There might be

a slight ambiguity here, since different $A \in \text{SO}(3)$ could do, namely, given ζ as in Lemma 1.2, there may be many A satisfying its conclusions; we will see however in Remarks 1.4 and 1.11 that \square as we construct it is not affected by this choice.

Points (i) and (v) above do not need further development. We postpone the tri-holomorphic isometry of point (ii) to Part 2, §2.2.1, as it is easier to tackle with a few further notions on Kronheimer’s classification of ALE gravitational instantons. As for point (iii), as mentioned already, the proof of 1.1 is given in §1.3 assuming results from Part 2; apart from the proof of Lemma 1.2 which we shall settle now, our main task in the current part is thus the gluing and the subsequent corrections stated in point (iv), to which we devote §§1.3 and 1.4 below, after recalling a few useful facts on the Taub-NUT metric seen as a Kähler metric on (\mathbb{C}^2, I_1) in the next section.

Proof of Lemma 1.2. For $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$, define the matrix $Z(\zeta) = (\langle \zeta_j, \zeta_\ell \rangle)_{1 \leq j, \ell \leq 3}$ of its scalar products. It is elementary matrix calculus to check that the $\text{SO}(3)$ -action defined by (3), and referred to in the statement of the lemma, translates into $Z(A\zeta) = AZ(\zeta)A^t$.

Fixing ζ , we thus want to find $A \in \text{SO}(3)$ such that $AZ(\zeta)A^t$ has shape

$$\begin{pmatrix} \mu & * & * \\ * & \lambda & 0 \\ * & 0 & \lambda \end{pmatrix}. \tag{4}$$

Since $Z = Z(\zeta)$ is symmetric, there exists $O \in \text{SO}(3)$ such that $OZO^t = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, and we now look for $Q \in \text{SO}(3)$ such that $Q \text{diag}(\lambda_1, \lambda_2, \lambda_3)Q^t$ has shape (4); setting then $A = QO$ leads us to the conclusion. If two of the λ_j are the same then we are done, up to letting act one of the permutation matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Up to this action again, we can therefore assume $\lambda_1 > \lambda_2 > \lambda_3$.

Setting

$$Q = \begin{pmatrix} \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3}\right)^{1/2} & 0 & \left(\frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}\right)^{1/2} \\ 0 & 1 & 0 \\ -\left(\frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}\right)^{1/2} & 0 & \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3}\right)^{1/2} \end{pmatrix},$$

a direct computation gives $Q \text{diag}(\lambda_1, \lambda_2, \lambda_3)Q^t = \begin{pmatrix} \lambda_1 + \lambda_3 - \lambda_2 & 0 & -\Lambda \\ 0 & \lambda_2 & 0 \\ -\Lambda & 0 & \lambda_2 \end{pmatrix}$, where $\Lambda = (\lambda_1 - \lambda_2)^{1/2}(\lambda_2 - \lambda_3)^{1/2}$. □

Remark 1.4. Our choice for Q is a little arbitrary; however, one can show that the only possibilities for writing $Q \text{diag}(\lambda_1, \lambda_2, \lambda_3)Q^t$, that is, AZA^t , under shape (4) are the $\begin{pmatrix} \lambda_1 + \lambda_3 - \lambda_2 & \Lambda \cos \phi & \Lambda \sin \phi \\ \Lambda \cos \phi & \lambda_2 & 0 \\ \Lambda \sin \phi & 0 & \lambda_2 \end{pmatrix}$, $\phi \in \mathbb{R}$, and again $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

1.2 The Taub-NUT metric as a Kähler metric on (\mathbb{C}^2, I_1)

Before we proceed to the gluing of the Taub-NUT metric with the ALE metric of one of Kronheimer’s instantons, we recall a few facts about this very Taub-NUT metric on \mathbb{C}^2 , that will be used in the analytic upcoming §§1.3 and 1.4. Our main references here are [GH76, LeB91].

1.2.1 *Gibbons–Hawking versus LeBrun ansätze.*

Gibbons–Hawking ansatz. As recalled in the Introduction, the Taub-NUT metric on \mathbb{R}^4 is often described via the Gibbons–Hawking ansatz as follows: given $m \in (0, \infty)$, set

$$\mathbf{f}_m = V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta^2,$$

where (y_1, y_2, y_3) is a circle fibration of $\mathbb{R}^4 \setminus \{0\}$ over $\mathbb{R}^3 \setminus \{0\}$, V is the function $(1 + 4mR)/2R$ (harmonic in the y_j coordinates) with $R^2 = y_1^2 + y_2^2 + y_3^2$, and where η is a connection 1-form for this fibration such that $d\eta = *\mathbb{R}^3 dV$. Thus defined, the metric \mathbf{f}_m confers length $\pi\sqrt{2/m}$ to the fibres at infinity, and is hermitian for the almost-complex structures

$$J_a : \begin{cases} Vdy_a \mapsto \eta, \\ dy_b \mapsto dy_c, \end{cases}$$

with $(a, b, c) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. These are in fact complex structures, verifying the quaternionic relations $J_a J_b J_c = -1$, for which \mathbf{f}_m is Kähler, thanks to the harmonicity of V : \mathbf{f}_m is thus hyperkähler. One checks moreover that this way, the metric \mathbf{f}_m and the complex structures extend as such through $0 \in \mathbb{R}^4$.

We now switch point of view to a description better adapted to our construction.

LeBrun’s potential. As depicted in [LeB91] and reviewed in detail in Appendix A, one can give a somehow more concrete support of the description of \mathbf{f}_m , through which the complex structure J_1 mentioned above is the standard I_1 on \mathbb{C}^2 , and $\text{vol}^{\mathbf{f}_m} = \Omega_{\mathbf{e}}$, the standard Euclidean volume form. One starts with the following implicit formulas:

$$\begin{aligned} |z_1| &= e^{m(u^2 - v^2)}u, \\ |z_2| &= e^{m(v^2 - u^2)}v, \end{aligned} \tag{5}$$

defining functions $u, v : \mathbb{C}^2 \rightarrow \mathbb{R}$, invariant under the circle action $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta}z_1, e^{-i\theta}z_2)$ which makes \mathbb{S}^1 as a subgroup of $\text{SU}(2)$; notice the role of m in these formulas, which enlightens our choice of taking it as the parameter of the upcoming construction. One then sets $y_1 = \frac{1}{2}(u^2 - v^2)$, $y_2 + iy_3 = -iz_1z_2$, $R = \frac{1}{2}(u^2 + v^2) = (y_1^2 + y_2^2 + y_3^2)^{1/2}$; these are \mathbb{S}^1 -invariant functions, making (y_1, y_2, y_3) as a principal- \mathbb{S}^1 fibration $\mathbb{C}^2 \rightarrow \mathbb{R}^3$ away from the origins. One finally defines

$$\varphi_m := \frac{1}{4}(u^2 + v^2 + m(u^4 + v^4)) = \frac{1}{2}(R + m(R^2 + y_1^2)). \tag{6}$$

One can then check (see Appendix A) that $dd_{I_1}^c \varphi_m$ is positive in the sense of I_1 -hermitian 2-forms, and that $(dd_{I_1}^c \varphi_m)^2 = 2\Omega_{\mathbf{e}}$. If one sets moreover $V = (1 + 4mR)/2R$, and $\eta = I_1 V dy_1$, noticing by passing that η is then a connection 1-form for the fibration with $d\eta = *\mathbb{R}^3 dV$, one has $\mathbf{f}_m := V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta^2 = (dd_{I_1}^c \varphi_m)(\cdot, I_1 \cdot)$. This metric is well defined at $0 \in \mathbb{C}^2$, as $(dd_{I_1}^c \varphi_m)(\cdot, I_1 \cdot) = \mathbf{e}$ at that point.

The metric \mathbf{f}_m is therefore Kähler for I_1 with volume form $\text{vol}^{\mathbf{f}_m} = \Omega_{\mathbf{e}}$ on the whole \mathbb{C}^2 ; by the standard properties of Kähler metrics, it is thus Ricci-flat. One recovers a complete hyperkähler data after checking that the defining equations

$$\mathbf{f}_m(J_j \cdot, \cdot) = \omega_j^{\mathbf{e}} \quad \text{where } \omega_j^{\mathbf{e}} = \mathbf{e}(I_j \cdot, \cdot), \quad j = 2, 3, \tag{7}$$

with I_2, I_3 the other two standard complex structures on $\mathbb{R}^4 \cong \mathbb{H}$, give rise to integrable complex structures, verifying respectively $J_j : Vdy_j \mapsto \eta, dy_k \mapsto dy_\ell$ for $(j, k, \ell) = (2, 3, 1), (3, 1, 2)$, as well as the quaternionic relations together with I_1 .

Let us now take a look at the length of the S^1 -fibres at infinity. Consider the vector field $\xi := i(z_1(\partial/\partial z_1) - \bar{z}_1(\partial/\partial \bar{z}_1) - z_2(\partial/\partial z_2) + \bar{z}_2(\partial/\partial \bar{z}_2))$ giving the infinitesimal action of S^1 . One has $dy_1(-I_1\xi) = V^{-1}$, thus $\eta(\xi) = 1$, and $dy_j(\xi) = 0$, $j = 1, 2, 3$; since R is S^1 -invariant, the length of the fibres is just $2\pi V^{-1/2}$, which tends to $\pi\sqrt{2/m}$.

Remark 1.5. Even if we can let m vary, this description actually leads to essentially one metric; indeed, if κ_s is the dilation of factor $s > 0$ of \mathbb{R}^4 , one gets with help of (5) and (6) the following homogeneity property: $\kappa_s^* \mathbf{f}_m = s^2 \mathbf{f}_{ms^2}$, which is of course coherent with the length of the fibres at infinity and the fact that $\text{vol}^{\mathbf{f}_m} = \text{vol}^{\mathbf{f}_{ms^2}} = \Omega_{\mathbf{e}}$.

From now on, we see the mass parameter m as fixed, and we drop the indices m when there is no risk of confusion.

The Taub-NUT metric and the action of the binary dihedral group on \mathbb{C}^2 . For $k \geq 2$, which we fix until the end of this part, the action of the binary dihedral group \mathcal{D}_k of order $4k$ seen as a subgroup of $SU(2) = Sp(1)$ is generated by the matrices $\zeta_k := \begin{pmatrix} e^{i\pi/k} & 0 \\ 0 & e^{-i\pi/k} \end{pmatrix}$ and $\tau := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. One has $\zeta_k^* y_j = y_j$, $j = 1, 2, 3$, and thus $\zeta_k^* R = R$, and $\zeta_k^* \eta = \eta$, whereas $\tau^* y_j = -y_j$, $j = 1, 2, 3$, thus $\tau^* R = R$, and $\tau^* \eta = -\eta$. The Taub-NUT metric \mathbf{f} is therefore \mathcal{D}_k -invariant, and descends smoothly to $(\mathbb{R}^4 \setminus \{0\})/\mathcal{D}_k$: this is the metric we are going to glue at infinity of \mathcal{D}_k -ALE instantons in the next section. Before though, we need a few more analytical tools for the Taub-NUT metric as we describe it here.

1.2.2 Orthonormal frames, covariant derivatives and curvature. In addition to the above relations between the vector field ξ , and the 1-forms η and dy_j , $j = 1, 2, 3$, one has that the data

$$(e_0, e_1, e_2, e_3) := (V^{1/2}\xi, -I_1 V^{1/2}\xi, V^{-1/2}\zeta, V^{-1/2}I_1\zeta), \tag{8}$$

is the dual frame of the orthonormal frame of 1-forms

$$(e_0^*, e_1^*, e_2^*, e_3^*) := (V^{-1/2}\eta, V^{1/2}dy_1, V^{1/2}dy_2, V^{1/2}dy_3) \tag{9}$$

on $\mathbb{C}^2 \setminus \{0\}$, provided that the vector field ζ is defined by

$$\zeta := \frac{1}{2iR} \left(e^{4my_1} \left(z_2 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial z_1} \right) + e^{-4my_1} \left(z_1 \frac{\partial}{\partial \bar{z}_2} - \bar{z}_1 \frac{\partial}{\partial z_2} \right) \right), \tag{10}$$

see Appendix A; we keep the notation $(e_j)_{j=0,\dots,3}$ and $(e_j^*)_{j=0,\dots,3}$ throughout this part. An explicit computation made in Appendix A then gives the estimates

$$|(\nabla^{\mathbf{f}})^\ell e_j|_{\mathbf{f}} = O(R^{-1-\ell}) \text{ near infinity for all } \ell \geq 1 \text{ and } j = 0, \dots, 3.$$

Consequently, for all $\ell \geq 0$, $|(\nabla^{\mathbf{f}})^\ell \text{Rm}^{\mathbf{f}}|_{\mathbf{f}} = O(R^{-3-\ell})$, this justifies the terminology ‘Asymptotically Locally Flat’ for \mathbf{f} ; this estimate, done using the Gibbons–Hawking ansatz, can also be found e.g. in [Min07, § 1.0.3].

We close this section with two further useful estimates, giving an idea of the geometric gap between \mathbf{e} and \mathbf{f} : first, at the level of distance functions, rearranging (5) gives $R \leq 2r^2$, which is sharp in general; second, there exists $C = C(m) > 0$ such that outside the unit ball of \mathbb{C}^2 , $C^{-1}r^{-2}\mathbf{e} \leq \mathbf{f} \leq Cr^2\mathbf{e}$, which, again, is sharp in general. Details are given in § A.2, in Appendix A.

1.3 Gluing the Taub-NUT metric to an ALE metric

As is usual when gluing Kähler metrics, we shall work on potentials to glue the ALF model-metric to an ALE one. The previous section gives us the potential φ for the ALF metric (6); the following paragraph provides us a sharp enough potential for the ALE metric.

1.3.1 *Approximation of the ALE Kähler form as a complex hessian.*

Asymptotics of the Kähler form and the complex structure. In view of steps (iii) and (iv) of the program developed in § 1.1, since we are performing our gluing on some specific ALE instantons, we fix for the rest of this part

$$\xi \in \mathfrak{h} - D, \quad \text{such that: } |\xi_2|^2 - |\xi_3|^2 = \langle \xi_2, \xi_3 \rangle = 0, \tag{11}$$

and consider the associated ALE instanton $(Y, g_Y, I_1^Y, I_2^Y, I_3^Y)$. Lemma 1.1 gives an ALE diffeomorphism $\Phi_Y : Y \setminus K \rightarrow (\mathbb{R}^4 \setminus B) / \mathcal{D}_k$, where K is some compact subset of Y and B a ball in \mathbb{R}^4 centred at the origin; recall that by ‘ALE diffeomorphism’ we mean that for all $\ell \geq 0$,

$$|(\nabla^e)^\ell(\Phi_{Y*}g_Y - e)|_e = O(r^{-4-\ell}),$$

and likewise on the complex structures. Before using the more specific properties of Φ_Y at the level of complex structures, let us mention the following: we want to proceed to a gluing of Kähler metrics, and the convenient way of doing so is to glue *the Kähler forms, via their potentials*. We already have a candidate for the potential of an ALF metric at infinity at hand: as evoked, this would be $\Phi_{Y*}\varphi^b$ (see point (iv) in § 1.1). Conversely, we need to kill the ALE metric near infinity, and for this we want a sharp enough potential, in a sense that we make clear below, see Proposition 1.12. We thus need for this a sharp knowledge of *the Kähler form* $\omega_1^Y := g_Y(I_1^Y \cdot, \cdot)$, and since we are about to compute I_1^Y -complex Hessians as well, we also need a precise description of the complex structure I_1^Y . These are given by the following, from which Lemma 1.1 actually follows as we shall see at the end of this section, with the same Φ_Y .

LEMMA 1.6. *One can choose the ALE diffeomorphism Φ_Y such that*

$$\Phi_{Y*}\omega_1^Y = \omega_1^e - c(|\xi_1|^2\theta_1 + 2\langle \xi_1, \xi_2 \rangle\theta_2 + 2\langle \xi_1, \xi_3 \rangle\theta_3) + O(r^{-8}) \tag{12}$$

where $c > 0$ is some universal constant, $\theta_j = \frac{1}{4}dd^c_{I_j}(r^{-2})$, $j = 1, 2, 3$, on the one hand, and if ι_1^Y denotes $\Phi_{Y*}I_1^Y - I_1$, then it is given by:

$$e(\iota_1^Y \cdot, \cdot) = -c(|\xi_2|^2 + |\xi_3|^2)\frac{rdr \cdot \alpha_1}{r^6} + O(r^{-8})$$

where c is the same constant as above and $\alpha_1 = I_1 r dr$, on the other hand.

We can moreover assume that $\Phi_{Y*}\Omega_Y = \Omega_e$, where $\Omega_Y = \text{vol}^{g_Y}$.

In this statement the error terms $O(r^{-8})$ are understood in the ‘Euclidean way’, namely for any $\ell \geq 0$, the ℓ th ∇^e -derivatives of these tensors are $O(r^{-8-\ell})$. This lemma requires further notions on Kronheimer’s construction, and is more precisely a direct application of Theorem 2.1 of Part 2 to $Y = X_\xi$ with ξ verifying (11). Notice however the error term order -8 , whereas one would expect -6 , if one thinks for instance about the Eguchi–Hanson metric [Joy00, Example 7.2.2]; this estimate is crucial in proving Lemma 1.1, and is specific to (groups containing) dihedral binary groups. Besides, the assertion on the volume forms is only needed in the next paragraph.

Approximating ω_1^Y as an I_1^Y -complex hessian. We shall see for now how Lemma 1.6 allows us to approximate the Kähler form ω_1^Y as an I_1^Y -complex hessian, with respect to the Taub-NUT metric pushed-forward to Y .

PROPOSITION 1.7. Take Φ_Y as in Lemmas 1.1 and 1.6, and denote by $\tilde{\mathbf{f}}$ a smooth extension of $\Phi_Y^*\mathbf{f}$ on Y . Then there exists a function Ψ on Y such that near infinity,

$$|(\nabla^{\tilde{\mathbf{f}}})^\ell(\omega_1^Y - dd_{I_1^Y}^c \Psi)|_{\tilde{\mathbf{f}}} = O(R^{-2}), \quad \ell = 0, 1, 2. \tag{13}$$

More precisely, Ψ can be decomposed as a sum $\Phi_Y^*\Psi_{\text{euc}} + \Phi_Y^*\Psi_{\text{mxd}}$, where on the one hand, $\Psi_{\text{euc}} = O(r^2)$, $|d\Psi_{\text{euc}}|_{\mathbf{e}} = O(r)$, and

$$|(\nabla^{\mathbf{e}})^\ell(\omega_1^{\mathbf{e}} - c|\xi_1|^2\theta_1 - dd_{\Phi_{Y^*}I_1^Y}^c \Psi_{\text{euc}})|_{\mathbf{e}} = O(r^{-8-\ell}) \quad \text{for all } \ell \geq 0, \tag{14}$$

and on the other hand, $\Psi_{\text{mxd}} = O(R^{-1})$, $|d\Psi_{\text{mxd}}|_{\mathbf{f}} = O(R^{-1})$, and

$$|(\nabla^{\mathbf{f}})^\ell(-2c(\langle \xi_1, \xi_2 \rangle \theta_2 + \langle \xi_1, \xi_3 \rangle \theta_3) - dd_{\Phi_{Y^*}I_1^Y}^c \Psi_{\text{mxd}})|_{\mathbf{f}} = O(R^{-2}), \quad \ell = 0, 1, 2. \tag{15}$$

Proof. Notice that once the statement on Ψ_{euc} (the ‘Euclidean component’ of Ψ) and Ψ_{mxd} (the ‘mixed component’) are known, estimates (13) follow at once by transposition to Y of estimates (14) and (15) and of the expansion of ω_1^Y stated in Lemma 1.6, keeping the following fact in mind.

LEMMA 1.8. If $\ell \geq 0$, and α is a tensor of type $(2, 0)$, $(1, 1)$ or $(0, 2)$ such that $|(\nabla^{\mathbf{e}})^k \alpha|_{\mathbf{e}} = O(r^{-2a-k})$, $a \geq 1$, for $k = 0, \dots, \ell$ on \mathbb{R}^4 , then $|(\nabla^{\mathbf{f}})^k \alpha|_{\mathbf{f}} = O(R^{1-a})$, $k = 0, \dots, \ell$.

This lemma takes into account estimates such as $R = O(r^2)$ and $C^{-1}r^{-2}\mathbf{e} \leq \mathbf{f} \leq Cr^2\mathbf{e}$ of Proposition A.9 at level $\ell = 0$, and follows for positive ℓ from explicit computations using the material given in Appendix A, where the proof of Lemma 1.8 is thus postponed.

Remark 1.9. This lemma moreover gives an essential hint about why we push to an $O(r^{-8})$ Euclidean precision in Lemma 1.6 and in (14), that is, why we do need the analysis of Part 2. In the forthcoming gluing (§ 1.3.2), we are concerned with an error term governed by a $(1, 1)$ -tensor, η say, of Euclidean size $O(r^{-8})$, amplified by some $O(R)$ factor in \mathbf{f} -scale; we thus roughly speaking end up by Lemma 1.8 (with $a = 4$) with an error of size $O(R^{-2})$, which suffices to go on our construction in § 1.4. On the other hand, starting with $O(r^{-4})$ or $O(r^{-6})$ Euclidean precision, that is, applying Lemma 1.8 with $a = 2$ or 3 , would only provide an error of size $O(1)$ or $O(R^{-1})$ after the gluing of § 1.3.2, which would not be accurate enough to conclude.

We hence come to the statements on Ψ_{euc} and Ψ_{mxd} . We consider before starting a large constant K such that the image of Φ_Y is contained in both $\{r \geq K\} \subset \mathbb{R}^4/\mathcal{D}_k$ and $\{R \geq K\} \subset \mathbb{R}^4/\mathcal{D}_k$, and define a cut-off function $\chi : \mathbb{R} \rightarrow [0, 1]$ such that

$$\chi(t) = \begin{cases} 0 & \text{if } t \leq K - 1, \\ 1 & \text{if } t \geq K, \end{cases}$$

which will be useful when defining functions to be pulled-back to Y via Φ_Y .

The Euclidean component Ψ_{euc} . In an asymptotically Euclidean setting, a natural first candidate for the potential of a Kähler form is $1/4r^2$. Now remember we are working with I_1^Y , or more exactly with $\Phi_{Y^*}I_1^Y$, but we forget about the push-forward here for simplicity of notation; following Lemma 1.6, a straightforward computation gives, near infinity in \mathbb{R}^4 :

$$\begin{aligned} dd_{I_1^Y}^c(\tfrac{1}{4}r^2) &= \tfrac{1}{2}d[(I_1 + \iota_1^Y)rdr] = \tfrac{1}{2}d[\alpha_1 + c(|\xi_2|^2 + |\xi_3|^2)r^{-4}\alpha_1 + O(r^{-7})] \\ &= \omega_1^{\mathbf{e}} - c(|\xi_2|^2 + |\xi_3|^2)\theta_1 + O(r^{-8}), \end{aligned}$$

where the O are understood in the Euclidean way. On the other hand observe that $I_1 d(r^{-2}) = -2r^{-4}\alpha_1$, and thus

$$dd_{I_1^Y}^c(r^{-2}) = d[(I_1 + \iota_1^Y)d(r^{-2})] = d[-2r^{-4}\alpha_1 + O(r^{-7})] = 4\theta_1 + O(r^{-8}).$$

Now define

$$\Psi_{\text{euc}} = \frac{1}{4}\chi(r)(r^2 + c(|\xi_2|^2 + |\xi_3|^2 - |\xi_1|^2)r^{-2});$$

on $\mathbb{R}^4/\mathcal{D}_k$ (it is \mathcal{D}_k -invariant); it has support in the image of Φ_Y , has the growth stated in the lemma as well as its differential, and by the previous two estimates we get that $\omega_1^e - c|\xi_1|^2\theta_1 - dd_{\Phi_Y^*I_1^Y}^c\Psi_{\text{euc}} = O(r^{-8})$ for \mathbf{e} with according decay on the derivatives, as wanted.

The mixed component Ψ_{mxd} . The main reason why we could construct Ψ_{euc} such as to reach estimates (14) is essentially that θ_1 can be realised as an I_1 -complex hessian, at least away from 0. Now realising θ_2 and θ_3 as I_1 -complex Hessians as well does not seem possible: see [Joy00, p. 202] on that matter. Nonetheless, θ_2 and θ_3 may not be so problematic when looked at via \mathbf{f} . We can indeed approximate them precisely enough with respect to this metric by the I_1 or I_1^Y -complex Hessians of some well-chosen \mathcal{D}_k -invariant functions, provided that we partially leave the Euclidean world and use also functions coming from Taub-NUT geometry, e.g. y_1 and R (hence the previous dichotomy ‘Euclidean/mixed’).

LEMMA 1.10. *Consider the complex valued function*

$$\psi_c := -2\frac{(y_2 + iy_3)\sinh(4my_1)}{r^2R}$$

on $\mathbb{R}^4 \setminus \{0\}$. Then near infinity:

- (i) $|(\nabla^{\mathbf{f}})^\ell \psi_c|_{\mathbf{f}} = O(R^{-1})$ for $\ell = 0, \dots, 4$;
- (ii) $|(\nabla^{\mathbf{f}})^\ell (dd_{I_1^Y}^c \psi_c - (\theta_2 + i\theta_3))|_{\mathbf{f}} = O(R^{-2})$ for $\ell = 0, 1, 2$, and these estimates hold with I_1 replaced by I_1^Y as well.

The proof of this crucial lemma is essentially computational, which is why we postpone it to § 1.5; let us just say at this stage that it will appear clearly along this proof that the main point is to identify some function ψ such that $\partial\psi/\partial y_1$ is proportional to $1/r^4$, up to higher order terms, and this turns out to happen precisely for $\psi = \sinh(4my_1)/r^2R$, see (30) below.

For now set $\psi_2 = \Re(\psi_c)$ and $\psi_3 = \Im(\psi_c)$, and define

$$\Psi_{\text{mxd}} := -2c\chi(R)(\langle \xi_1, \xi_2 \rangle \psi_2 + \langle \xi_1, \xi_3 \rangle \psi_3).$$

In view of Lemma 1.10, such a function, defined on the image of Φ_Y , verifies the growth assertions of Proposition 1.7, as well as the estimates (15): Proposition 1.7 is proved. \square

We are now in position to perform the gluing advertised in point (iv) of the program of § 1.1. This is done in the next section to which the reader may jump directly, since we conclude the current section by the proof of Lemma 1.1, assuming Lemma 1.6 (and more precisely the assertion on I_1^Y in that statement).

Proof of Lemma 1.1 following Lemma 1.6. We fix Φ_Y as in Lemma 1.6; we work on \mathbb{R}^4 , and to simplify notation we forget about the push-forwards by Φ_Y .

We are thus looking for a diffeomorphism \beth of \mathbb{R}^4 such that $|I_1^Y - \beth^*I_1|_{\mathbf{e}} = O(r^{-8})$, with according decay on Euclidean derivatives, until the end of this proof we forget about ALF

geometry and stick to the Euclidean setting; we will thus content ourselves with using O in this Euclidean meaning. An explicit formula is given for \beth in the statement of Lemma 1.1, which is $\beth : (z_1, z_2) \mapsto (1 + (a/r^4))(z_1, z_2)$ with (z_1, z_2) the standard complex coordinates on (\mathbb{C}^2, I_1) ; up to determining the value of the constant a , we could thus simply check that such a \beth meets our requirement, in light of the asymptotics for I_1^Y stated in Lemma 1.6.

We prefer nonetheless the following more constructive approach. In terms of Kodaira–Spencer theory, if we set $\beth = \text{id}_{\mathbb{C}^2} + \varepsilon$, seeing thus $\varepsilon = (\varepsilon_1(\partial/\partial z_1), \varepsilon_2(\partial/\partial z_2))$ as the direction of a deformation of $\text{id}_{\mathbb{C}^2}$, the condition $I_1^Y - \beth^* I_1 = O(r^{-8})$ becomes, neglecting the $O(r^{-8})$ error term, $\bar{\partial}_{I_1} \varepsilon = \iota_1^Y$. Now ι_1^Y is an \mathbf{e} -harmonic I_1 -(0, 1) form with values in $T^{(1,0)}\mathbb{C}^2$ and with r^{-4} -decay; a multiple of the (1, 0)-gradient of the Green function $1/r^2$ is thus the best-placed candidate for the role of ε . This actually works, with the choice

$$\varepsilon = \frac{c}{8} (|\xi_2|^2 + |\xi_3|^2) \left[\text{grad}^{\mathbf{e}} \left(\frac{1}{r^2} \right) \right]^{1,0},$$

that is

$$\varepsilon_1 = -\frac{c(|\xi_2|^2 + |\xi_3|^2)z_1}{4r^4} \quad \text{and} \quad \varepsilon_2 = -\frac{c(|\xi_2|^2 + |\xi_3|^2)z_2}{4r^4},$$

with c the constant given by Lemma 1.6.

The last point to be dealt with is $\beth = \text{id}_{\mathbb{C}^2} + (\varepsilon_1, \varepsilon_2)$ being a diffeomorphism between infinities of \mathbb{C}^2 ; we leave it to the reader as an easy exercise.

The estimate $\beth^* \Omega_{\mathbf{e}} - \Omega_{\mathbf{e}} = O(r^{-8})$ amounts to seeing that $\Re \varepsilon (\partial \varepsilon_1 / \partial z_1 + \partial \varepsilon_2 / \partial z_2) = O(r^{-8})$: extend $\text{id}(z_1 + \varepsilon_1) \wedge d(\bar{z}_1 + \bar{\varepsilon}_1) \wedge \text{id}(z_2 + \varepsilon_2) \wedge d(\bar{z}_2 + \bar{\varepsilon}_2)$, and look at the linear terms in $\varepsilon_1, \varepsilon_2$. Since after multiplication by $a := -c(|\xi_2|^2 + |\xi_3|^2)/4$ the error would again be $O(r^{-8})$, we can do this computation with z_1/r^4 and z_2/r^4 playing the respective roles of ε_1 and ε_2 . Now $(\partial/\partial z_j)(z_j/r^4) = 1/r^4 - 2|z_j|^2/r^6, j = 1, 2$. Since these are real, we only need to compute the sum $(\partial/\partial z_1)(z_1/r^4) + (\partial/\partial z_2)(z_2/r^4)$, which is $2/r^4 - 2|z_1|^2/r^6 - 2|z_2|^2/r^6 = 0$.

The \mathcal{D}_k -invariance of \beth thus constituted is clear. □

Remark 1.11. According to the preceding proof, \beth as we construct it depends only on $c(|\xi_2|^2 + |\xi_3|^2)$. If now ξ is chosen as an $A\zeta, A \in \text{SO}(3), \zeta \in \mathfrak{h} - D$, so as to satisfy condition (11) as is evoked in point (iii) in the program of § 1.1, by Remark 1.4, $|\xi_2|^2 = |\xi_3|^2$ does not depend on A , and has to be the middle eigenvalue of the matrix $(\langle \zeta_j, \zeta_\ell \rangle)$. Consequently, $\beth = \beth_{A\zeta}$ does not depend on $A \in \text{SO}(3)$.

1.3.2 *The gluing.* We keep the notation of the previous section: $(Y, g_Y, (I_j^Y)_{j=1,2,3})$ is a \mathcal{D}_k -ALE instanton with parameter ξ verifying (11), Φ_Y an asymptotic isometry between infinities of Y and $\mathbb{R}^4/\mathcal{D}_k$ fixed by Lemma 1.6, and \beth is given by Lemma 1.1 which we may also see as as diffeomorphism of $(\mathbb{R}^4 \setminus \{0\})/\mathcal{D}_k$.

As alluded to above, the form we want to glue $\omega_1^Y = g_Y(I_1^Y \cdot, \cdot)$ with at infinity is $dI_1^Y d\varphi^b$, where $\varphi^b = \beth^* \varphi$, with $\varphi = \varphi_m$ LeBrun’s I_1 -potential for \mathbf{f} given by (6). We set likewise $\mathbf{f}^b = \beth^* \mathbf{f}$, both on \mathbb{R}^4 and its quotient. Recall that $\Psi = \Psi_{\text{euc}} + \Psi_{\text{mxd}}$ is defined in Proposition 1.7 as an approximate I_1^Y -complex potential of ω_1^Y . The next proposition explains how to glue $dI_1^Y d\varphi^b$ to ω_1^Y , so as to obtain an ALF metric on Y at the end; as we need it below, let us mention here that we see Ψ_{mxd} on Y via Φ_Y , and we more precisely consider a smooth extension of $\Phi_Y^* \Psi_{\text{mxd}}$, which is assumed to be bounded in absolute value by $\frac{1}{2}$.

PROPOSITION 1.12. Take $K \geq 0$ so that the identification Φ_Y between infinities of $\mathbb{R}^4/\mathcal{D}_k$ and Y , as well as the diffeomorphism \sqsupset , are defined on $\varphi \geq K$. Consider $r_0, R_0 \gg 1$, $\beta \in (0, 1]$ and set

$$\Phi_m^b = \Phi_{\text{alf}}^+ - \Phi_{\text{ale}} - \Phi_{\text{alf}}^- \tag{16}$$

where

$$\begin{cases} \Phi_{\text{alf}}^+ = \Phi_{\text{alf}}^+(\cdot; \beta, K) = \kappa \circ (\varphi^b + \beta^{-1}(\varphi^b)^{1-\beta} - \Psi_{\text{mxd}} - (K + \beta^{-1}K^{1-\beta})), \\ \Phi_{\text{ale}} = \Phi_{\text{ale}}(\cdot; r_0) = \chi(r - r_0)\Psi_{\text{euc}}, \\ \Phi_{\text{alf}}^- = \Phi_{\text{alf}}^-(\cdot; \beta, R_0) = \chi\left(\frac{R - R_0}{R_0}\right)\beta^{-1}(\varphi^b)^{1-\beta}, \end{cases} \tag{17}$$

with $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ a convex function which is constant on $(-\infty, \frac{1}{2}]$ and equal to $\text{id}_{\mathbb{R}}$ on $[1, \infty)$ and χ the cut-off function $d\kappa/dt$. Then if the parameters K, r_0 and R_0 (respectively β) are chosen large enough (respectively small enough), the symmetric 2-tensor g_m associated via I_1^Y to the I_1^Y -(1, 1)-form

$$\omega_m := \omega_1^Y + dd_{I_1^Y}^c \Phi_m^b$$

is well defined on the whole Y , is a Kähler metric for I_1^Y , is ALF in the sense that $|(\nabla^{\mathbf{f}^b})^\ell(g_m - \mathbf{f}^b)|_{\mathbf{f}^b} = O(R^{-2})$ for $\ell = 0, 1, 2$, and its volume form Ω_m verifies

$$|(\nabla^{\mathbf{f}^b})^\ell(\Omega_m - \Omega_Y)|_{\mathbf{f}^b} = O(R^{-2}) \tag{18}$$

for $\ell = 0, 1, 2$, where Ω_Y is the volume form of the ALE metric g_Y .

Remark 1.13. As will be clear from the proof below, the role of the component Φ_{alf}^+ is to bring (slightly more than) the ALF part in ω_m ; the role of Φ_{ale} is to remove the ALE part, coming from ω_1^Y ; finally, Φ_{alf}^- is used to correct the term $\beta^{-1}(\varphi^b)^{1-\beta}$ of Φ_{alf}^+ near infinity (where its contribution to positivity is no longer needed) so as to end up with the announced asymptotics.

Proof. To begin with, we mention the following comparison between \mathbf{f} and its correction $\mathbf{f}^b = \sqsupset^*\mathbf{f}$, that we will keep in mind.

LEMMA 1.14. For $\ell = 0, 1, 2$, we have $|(\nabla^{\mathbf{f}})^\ell(\mathbf{f}^b - \mathbf{f})|_{\mathbf{f}} = O(R^{-1})$ on \mathbb{R}^4 . Moreover $\sqsupset^*R = R + O(R^{-1})$.

The proof of this lemma is postponed to § 1.5.2, as we focus on that of Proposition 1.12.

Step 1.1. For now, we work first with the parameter β in (17) equal to 1, and consider the closed I_1^Y -hermitian form $dd_{I_1^Y}^c \Phi_{\text{alf}}^+ = dd_{I_1^Y}^c \kappa \circ (\varphi^b - \Psi_{\text{mxd}} - K)$ on Y . Even though K is not fixed yet, this form is equal to $dd_{I_1^Y}^c(\varphi^b - \Psi_{\text{mxd}})$ on $\{\varphi^b - \Psi_{\text{mxd}} \geq K+1\}$ seen on Y via Φ_Y ; this is possible for K large enough since $\varphi^b - \Psi_{\text{mxd}}$ is proper on \mathbb{R}^4 as $\varphi^b \geq \sqsupset^*R \sim R$ (by Lemma 1.14) and $\Psi_{\text{mxd}} = O(R^{-1})$. Moreover κ is convex, and thus $dd_{I_1^Y}^c[\kappa \circ (\varphi^b - \Psi_{\text{mxd}} - K)]$ is non-negative wherever $dd_{I_1^Y}^c(\varphi^b - \Psi_{\text{mxd}})$ is, which we claim is the case near infinity. Since indeed $|dd_{I_1^Y}^c \Psi_{\text{mxd}}|_{\mathbf{f}} = O(R^{-1})$, our claim will be checked if we prove the estimate

$$|dd_{I_1^Y}^c \varphi^b - \frac{1}{2}[\mathbf{f}^b(I_1^Y \cdot, \cdot) - \mathbf{f}^b(\cdot, I_1^Y \cdot)]|_{\mathbf{f}^b} = O(R^{-2}), \tag{19}$$

as $\varpi_{\mathbf{f}^b} := \frac{1}{2}[\mathbf{f}^b(I_1^Y \cdot, \cdot) - \mathbf{f}^b(\cdot, I_1^Y \cdot)]$ is nothing but the I_1^Y -hermitian form associated to the I_1^Y -hermitian metric $\frac{1}{2}[\mathbf{f}^b + \mathbf{f}^b(I_1^Y \cdot, I_1^Y \cdot)]$; notice $\varpi_{\mathbf{f}^b}$ is not closed in general. Pushing-forward by \mathfrak{Q} , proving estimate (19) amounts to seeing that

$$|dd_{\mathfrak{Q}_* I_1^Y}^c \varphi - \frac{1}{2}[\mathbf{f}(\mathfrak{Q}_* I_1^Y \cdot, \cdot) - \mathbf{f}(\cdot, \mathfrak{Q}_* I_1^Y \cdot)]|_{\mathbf{f}^b} = O(R^{-2}).$$

Now $dd_{\mathfrak{Q}_* I_1^Y}^c \varphi = d\mathfrak{Q}_* I_1^Y d\varphi = \omega_{\mathbf{f}} + dj d\varphi$, where $\omega_{\mathbf{f}} = \mathbf{f}(I_1 \cdot, \cdot)$ and $j = \mathfrak{Q}_* I_1^Y - I_1$. Let us estimate $dj d\varphi$; by Lemma 1.1 and by Lemma 1.8, for all $\ell = 0, 1$, $|(\nabla^{\mathbf{f}})^\ell j|_{\mathbf{f}} = O(R^{-3})$, whereas $|(\nabla^{\mathbf{f}})^\ell \varphi|_{\mathbf{f}} = O(R^{2-\ell})$, $\ell = 1, 2$; therefore $|dj d\varphi|_{\mathbf{f}} = O(R^{-2})$. On the other hand, still from $\mathfrak{Q}_* I_1^Y = I_1 + j$, $\mathbf{f}(\mathfrak{Q}_* I_1^Y \cdot, \cdot) - \mathbf{f}(\cdot, \mathfrak{Q}_* I_1^Y \cdot) = 2\omega_{\mathbf{f}} + \mathbf{f}(j \cdot, \cdot) - \mathbf{f}(\cdot, j)$. The error term $\mathbf{f}(j \cdot, \cdot) - \mathbf{f}(\cdot, j)$ is controlled by $|j|_{\mathbf{f}}$, which is $O(R^{-3})$. We have thus proved estimate (19). Thanks to the general formal formula

$$\nabla^{g+h} T = \nabla^g T + (g+h)^{-1} * \nabla^g h * T, \tag{20}$$

(see e.g. [GV16], formula (3.39)) for any metrics g and $g+h$ (h is thus seen here as a perturbation) and any tensor T , with Lemma 1.14 we can take $g = \mathbf{f}$, $g+h = \mathbf{f}^b$ and T the tensor in play, and prove with the same techniques an estimate similar to (19) up to order 2, that is

$$|(\nabla^{\mathbf{f}^b})^\ell (dd_{\mathfrak{Q}_* I_1^Y}^c \varphi - \varpi_{\mathbf{f}^b})|_{\mathbf{f}^b} = O(R^{-2}),$$

for $\ell = 1, 2$; this also uses $|(\nabla^{\mathbf{f}})^\ell \varphi|_{\mathbf{f}} = O(R^{-2-\ell})$ for $\ell = 0, \dots, 4$, and $|(\nabla^{\mathbf{f}})^\ell j|_{\mathbf{f}} = O(R^{-3})$ for $\ell = 0, \dots, 4$, which follows from Lemma 1.8. If therefore K is chosen large enough, and taking moreover the contribution of Ψ_{mxd} into account, $\omega_1^Y + dd_{I_1^Y}^c \kappa \circ (\varphi^b - \Psi_{\text{mxd}} - K)$ is well defined and is an I_1^Y -Kähler form, and is equal to $(\omega_1^Y - dd_{I_1^Y}^c \Psi_{\text{mxd}}) + \varpi_{\mathbf{f}^b}$ up to a $O(R^{-2})$ error at orders 0, 1 and 2 for \mathbf{f}^b .

Step 1.2. We can now deal with the $\beta < 1$ case (our intention is to make β small); as

$$dd_{I_1^Y}^c (\beta^{-1}(\varphi^b)^{1-\beta}) = (1-\beta)(\varphi^b)^{-\beta} (\beta^{-1} dd_{I_1^Y}^c \varphi^b - (\varphi^b)^{-1} d\varphi^b \wedge d_{I_1^Y}^c \varphi^b),$$

as φ^b grows at least like $m(R^b)^2$, and as $d\varphi^b$ and $d_{I_1^Y}^c \varphi^b$ are $O(R^b)$ for \mathbf{f}^b according to what precedes, we have $|(\varphi^b)^{-1} d\varphi^b \wedge d_{I_1^Y}^c \varphi^b|_{\mathbf{f}^b} \leq A$ near infinity for some $A > 0$ (independent of K, β, R_0, r_0), whereas $dd_{I_1^Y}^c \varphi^b \geq \frac{1}{2} \varpi_{\mathbf{f}^b}$ near infinity. This way, assuming that $\beta \in (0, A^{-1})$,

$$dd_{I_1^Y}^c (\beta^{-1}(\varphi^b)^{1-\beta}) \geq (1-\beta)(\varphi^b)^{-\beta} (\beta^{-1} - A) dd_{I_1^Y}^c \varphi^b \geq \frac{1}{2} (1-\beta)(\varphi^b)^{-\beta} (\beta^{-1} - A) \varpi_{\mathbf{f}^b}$$

near infinity, and, more precisely, as soon as $|(\varphi^b)^{-1} d\varphi^b \wedge d_{I_1^Y}^c \varphi^b|_{\mathbf{f}^b} \leq A$ and $dd_{I_1^Y}^c \varphi^b \geq \frac{1}{2} \varpi_{\mathbf{f}^b}$, hence on $\{\varphi^b \geq K\}$, say; we fix such a K once and for all. Therefore, as $|\Psi_{\text{mxd}}| \leq \frac{1}{2}$ on Y :

- $dd_{I_1^Y}^c \Phi_{\text{alf}}^+$ is trivial where $(\varphi^b + \beta^{-1}(\varphi^b)^{1-\beta} - \Psi_{\text{mxd}} - K - \beta^{-1} K^{1-\beta}) < \frac{1}{2}$;
- on $\{(\varphi^b + \beta^{-1}(\varphi^b)^{1-\beta} - \Psi_{\text{mxd}} - K - \beta^{-1} K^{1-\beta}) \geq \frac{1}{2}\} \subset \{\varphi^b \geq K\}$ (this inclusion being independent of β), $dd_{I_1^Y}^c [\varphi^b + \beta^{-1}(\varphi^b)^{1-\beta} - \Psi_{\text{mxd}}] \geq \frac{1}{2} \varpi_{\mathbf{f}^b} + (\beta^{-1} - A)(\varphi^b)^{-1} \varpi_{\mathbf{f}^b} > 0$, and thus $dd_{I_1^Y}^c \Phi_{\text{alf}}^+ \geq 0$. More precisely, on $\{\varphi^b \geq K+2\}$, where $(\varphi^b + \beta^{-1}(\varphi^b)^{1-\beta} - \Psi_{\text{mxd}} - K - \beta^{-1} K^{1-\beta}) \geq 1$, $dd_{I_1^Y}^c \Phi_{\text{alf}}^+ = dd_{I_1^Y}^c [\varphi^b + \beta^{-1}(\varphi^b)^{1-\beta} - \Psi_{\text{mxd}}]$. Thus $\omega_1^Y + dd_{I_1^Y}^c \Phi_{\text{alf}}^+ \geq \omega_1^Y$ on Y , refined as $\omega_1^Y + dd_{I_1^Y}^c \Phi_{\text{alf}}^+ \geq \omega_1^Y + \frac{1}{2}(1 + (1-\beta)(\beta^{-1} - A)(\varphi^b)^{-\beta}) \varpi_{\mathbf{f}^b}$ on $\{\varphi^b \geq K+2\}$;

- $\omega_1^Y + dd_{I_1^Y}^c \Phi_{\text{alf}}^+$ is furthermore asymptotic to

$$(\omega_1^Y - dd_{I_1^Y}^c \Psi_{\text{mxd}}) + \varpi_{\mathbf{f}^b} + \beta^{-1} dd_{I_1^Y}^c (\varphi^b)^{1-\beta}, \tag{21}$$

with an error of size $O(R^{-2})$, as well as its first and second $\nabla^{\mathbf{f}^b}$ -derivatives. Notice that at this stage, we have not *used* Ψ_{mxd} , but instead only checked it did not impede us to reach a desired positivity assertion on $\omega_1^Y + dd^c \Phi_{\text{alf}}^+$.

Step 2. We now deal with the summand $-\Phi_{\text{ale}}$ of Φ_m^b , which is meant to kill the ALE part of the Kähler form $\omega_1^Y + dd^c \Phi_{\text{alf}}^+$, or equivalently of the I_1^Y -hermitian form $(\omega_1^Y - dd_{I_1^Y}^c \Psi_{\text{mxd}}) + \varpi_{\mathbf{f}^b}$ (we see the remaining terms in (21) as ALF perturbations, corrected below with the help of Φ_{alf}^-). There are again two issues here: the positivity of the resulting I_1^Y - $(-1, 1)$ form $\omega_1^Y + dd_{I_1^Y}^c (\Phi_{\text{alf}}^+ - \Phi_{\text{ale}})$ on Y , and its asymptotics. The asymptotics are independent of r_0, R_0 and, to a certain extent, β , since we are only looking at what happens near infinity. Indeed, for any value of these parameters, and provided that r_0 (respectively R_0) is chosen much larger than K (respectively r_0), we have $\{R \geq R_0\} \subset \{r \geq r_0 + 1\} \subset \{\varphi^b \geq K + 2\}$ and on that region, by definition of Φ_{alf}^+ and Φ_{ale} ,

$$\omega_1^Y + dd_{I_1^Y}^c (\Phi_{\text{alf}}^+ - \Phi_{\text{ale}}) = (\omega_1^Y - dd_{I_1^Y}^c \Psi) + dd_{I_1^Y}^c (\varphi^b + \beta^{-1} (\varphi^b)^{1-\beta}),$$

with that $\Psi = \Psi_{\text{mxd}} + \Psi_{\text{euc}}$ of Proposition 1.7. The term $(\omega_1^Y - dd_{I_1^Y}^c \Psi)$ in the right-hand side is thus $O(R^{-2})$ for \mathbf{f} by this proposition, which is exactly where Ψ_{mxd} , as well as Ψ_{euc} as we defined it, are *used*; again, these asymptotics hold for \mathbf{f}^b by Lemma 1.14. We have already dealt with the asymptotics of $dd_{I_1^Y}^c (\varphi^b + \beta^{-1} (\varphi^b)^{1-\beta})$ in the previous step. As in (21), we hence have that $\omega_1^Y + dd_{I_1^Y}^c (\Phi_{\text{alf}}^+ - \Phi_{\text{ale}})$ is asymptotic to $\varpi_{\mathbf{f}^b} + \beta^{-1} dd_{I_1^Y}^c (\varphi^b)^{1-\beta}$, with an error of size $O(R^{-2})$ for \mathbf{f}^b up to its second $\nabla^{\mathbf{f}^b}$ -derivatives.

We are therefore left with the positivity assertion, which has to be proved carefully since we essentially have to subtract a metric to another one; this is where we fix r_0 and β . This boils down to the following:

- take r_0 so that on $r \geq r_0$, $dd_{I_1^Y}^c (\varphi^b + \beta^{-1} (\varphi^b)^{1-\beta} - \Psi_{\text{mxd}}) \geq \frac{1}{2} [1 + (1 - \beta)(\beta^{-1} - A)(\varphi^b)^{-\beta}] \varpi_{\mathbf{f}^b}$ (as underlined above, this can be done *independently of* β);
- consider $\omega_1^Y - dd_{I_1^Y}^c \Phi_{\text{ale}}$, rewritten as $[1 - \chi(r - r_0)]\omega_1^Y + \chi(r - r_0)(\omega_1^Y - dd_{I_1^Y}^c \Psi_{\text{euc}}) + \mathcal{R}_{r_0}$, where $\mathcal{R}_{r_0} = \chi'(r - r_0)(\Psi_{\text{euc}} dd_{I_1^Y}^c r + dr \wedge d_{I_1^Y}^c \Psi_{\text{euc}} + d\Psi_{\text{euc}} \wedge d_{I_1^Y}^c r) + \chi''(r - r_0)\Psi_{\text{euc}} dr \wedge d_{I_1^Y}^c r$ has support in $\{r_0 \leq r \leq r_0 + 1\}$;
- as $|\omega_1^Y - dd_{I_1^Y}^c \Psi_{\text{euc}}|_{\mathbf{e}} = O(r^{-4})$, we get $|\omega_1^Y - dd_{I_1^Y}^c \Psi_{\text{euc}}|_{\mathbf{f}^b} = O(R^{-1})$ by Lemma 1.8; we can thus *fix* r_0 *once and for all* so that $|\chi(r - r_0)(\omega_1^Y - dd_{I_1^Y}^c \Psi_{\text{euc}})|_{\mathbf{f}^b} \leq \frac{1}{6} \varpi_{\mathbf{f}^b}$ on Y ;
- *we now fix* $\beta > 0$ small enough so that $(1 - \beta)(\beta^{-1} - A)(\varphi^b)^{-\beta} \geq 6 \sup_{r_0 \leq r \leq r_0 + 1} |\mathcal{R}_{r_0}|_{\mathbf{f}^b}$; this way, $\omega_1^Y + dd_{I_1^Y}^c (\Phi_{\text{alf}}^+ - \Phi_{\text{ale}})$, which equals $\omega_1^Y + dd_{I_1^Y}^c \Phi_{\text{alf}}^+ > 0$ on $Y \setminus \{r \geq r_0\}$, is bounded below by $(\frac{1}{2} - \frac{1}{6} - \frac{1}{6}) \varpi_{\mathbf{f}^b} > 0$ on $\{r_0 \leq r \leq r_0 + 1\}$, and by $(\frac{1}{2} - \frac{1}{6}) \varpi_{\mathbf{f}^b} > 0$ on $\{r \geq r_0 + 1\}$.

Step 3. We conclude by analysing the term Φ_{alf}^- of Φ_m^b , and by fixing R_0 . First, by the known asymptotics on $\omega_1^Y + dd_{I_1^Y}^c (\Phi_{\text{alf}}^+ - \Phi_{\text{ale}})$ before the above list, and the very shape of Φ_{alf}^- , we get that $\omega_m = \omega_1^Y + dd_{I_1^Y}^c (\Phi_{\text{alf}}^+ - \Phi_{\text{ale}}) - dd_{I_1^Y}^c \Phi_{\text{alf}}^-$ is asymptotic to $\varpi_{\mathbf{f}^b}$ (the error being of size $O(R^{-2})$ for \mathbf{f}^b up to two $\nabla^{\mathbf{f}^b}$ -derivatives). Second, recall that R_0 is supposed large enough so that $\{R \geq R_0\} \subset \{r \geq r_0 + 1\}$; running the first three points in the above list, and taking supports into account, one

can also say that $\omega_1^Y + dd_{I_1^Y}^c(\Phi_{\text{ALF}}^+ - \Phi_{\text{ALE}})$, which equals $dd^c\varphi^b + dd_{I_1^Y}^c(\beta^{-1}(\varphi^b)^{1-\beta}) + (\omega_1^Y - dd_{I_1^Y}^c\Psi)$ on $\{R \geq R_0\}$, is thus $\geq \frac{1}{2}\varpi_{\mathbf{f}^b} + dd_{I_1^Y}^c(\beta^{-1}(\varphi^b)^{1-\beta}) - \frac{1}{6}\varpi_{\mathbf{f}^b}$, i.e. $\geq \frac{1}{3}\varpi_{\mathbf{f}^b} + dd_{I_1^Y}^c(\beta^{-1}(\varphi^b)^{1-\beta})$ on this region, where we recall that $dd_{I_1^Y}^c(\beta^{-1}(\varphi^b)^{1-\beta}) \geq 0$. Now, $dd_{I_1^Y}^c\Phi_{\text{alf}}^-$ (which has support in $\{R \geq R_0 + 1\}$) can be rewritten as $\chi((R - R_0)/R_0)dd_{I_1^Y}^c(\beta^{-1}(\varphi^b)^{1-\beta}) + \mathcal{R}_{R_0}^-$, where $\mathcal{R}_{R_0}^- = (1/\beta R_0)\chi'((R - R_0)/R_0)\mathcal{R}_1^- + (1/\beta R_0^2)\chi''((R - R_0)/R_0)(\varphi^b)^{1-\beta}\mathcal{R}_2^-$ has support in $\{R_0 \leq R \leq 2R_0\}$; here \mathcal{R}_1^- stands for $(\varphi^b)^{1-\beta}dd_{I_1^Y}^cR + dR \wedge d_{I_1^Y}^c(\varphi^b)^{1-\beta} + d(\varphi^b)^{1-\beta} \wedge d_{I_1^Y}^cR$ and \mathcal{R}_2^- for $dR \wedge d_{I_1^Y}^cR$. As $\mathcal{R}_1^- = O(R^{1-2\beta})$, and $\mathcal{R}_2^- = O(R^{2-2\beta})$, independently of R_0 , we can fix R_0 large enough so that $\sup_Y |\mathcal{R}_{R_0}^-|_{\mathbf{f}^b} = \sup_{R_0 \leq R \leq 2R_0} |\mathcal{R}_{R_0}^-|_{\mathbf{f}^b} \leq \frac{1}{6}$.

We sum all this up as: on $\{R \geq R_0\}$, $\omega_m = \omega_1^Y + dd_{I_1^Y}^c(\Phi_{\text{alf}}^+ - \Phi_{\text{ale}}) - \chi((R - R_0)/R_0)dd_{I_1^Y}^c(\beta^{-1}(\varphi^b)^{1-\beta}) - \mathcal{R}_{R_0}^- \geq \frac{1}{3}\varpi_{\mathbf{f}^b} + [1 - \chi((R - R_0)/R_0)]dd_{I_1^Y}^c(\beta^{-1}(\varphi^b)^{1-\beta}) - \frac{1}{6}\varpi_{\mathbf{f}^b} \geq \frac{1}{6}\varpi_{\mathbf{f}^b} > 0$; moreover, on $Y \setminus \{R \geq R_0\}$, ω_m equals $\omega_1^Y + dd_{I_1^Y}^c(\Phi_{\text{alf}}^+ - \Phi_{\text{ale}})$, positive on the whole Y . In conclusion, $\omega_m > 0$ on Y .

The last part of the statements concerns volume forms, and is a direct consequence of the estimates on the metrics, after observing that (on \mathbb{R}^4 , say; recall that $\Phi_{Y,*}\Omega_Y = \Omega_{\mathbf{e}}$): $\text{vol}^{\mathbf{f}^b} - \Omega_Y = \sqsupset^* \text{vol}^{\mathbf{f}} - \Omega_{\mathbf{e}} = \sqsupset^* \Omega_{\mathbf{e}} - \Omega_{\mathbf{e}}$, which can be written as $\varepsilon \Omega_{\mathbf{e}}$ with $|(\nabla^{\mathbf{e}})^\ell \varepsilon|_{\mathbf{e}} = O(r^{-8})$, $\ell \geq 0$, by Lemma 1.1. This converts into $|(\nabla^{\mathbf{f}^b})^\ell \varepsilon|_{\mathbf{f}^b} = O(R^{-4})$, $\ell \geq 0$, which is better than wanted. \square

Remark 1.15. We can now make our motivational Remark 1.9 more precise; indeed, in the above proof, one can see that the dominant term in $g_m - \mathbf{f}^b$ comes from $\sqsupset^* d(jd\varphi)$, and more precisely, from its component with shape $\nabla^{\mathbf{f}^b} j * d\varphi^b$ and this, for $\nabla^{\mathbf{f}^b}$ -derivatives up to order 2. Now, we need in the next section an error between g_m and \mathbf{f}^b of size $O(R^{-\delta})$ ($\delta > 0$) at order 0, but of size $O(R^{-\delta-1})$ ($\delta > 0$) at order 1 (and of size $O(R^{-\delta-\alpha-1})$, $\alpha > 0$, at order $1 + \alpha$) for \mathbf{f}^b ; an $O(R^{-1})$ -error in the statement of 1.12 (as would be the case with, say, an $O(r^{-6})$ in Lemma 1.6 or in (14), by Lemma 1.8 and using that $d\varphi = O(R)$) would make our procedure fail.

1.4 Corrections on the glued metric

1.4.1 *A Calabi–Yau type theorem.* We want to correct our I_1^Y -Kähler metric g_m from Proposition 1.12 into a Ricci-flat Kähler metric. For this it is sufficient to correct it into an I_1^Y -Kähler metric with volume form Ω_Y , since this is the volume of the I_1^Y -Kähler metric g_Y , and, as is well known, once the complex structure is fixed, the Ricci tensor of a Kähler metric depends only on its volume form. As suggested by the program ending to Theorem 1.3, at the level of I_1^Y -Kähler forms, we want to stay in the same class; in other words, we are looking for the I_1^Y -complex hessian of some function to be the desired correction.

The tool we are willing to use to determine this function is the ALF Calabi–Yau type theorem of the Introduction, which we state precisely now (we call the manifold in play \mathcal{Y} for more genericity).

THEOREM 1.16. *Let $\beta \in (0, 1)$ and let $(\mathcal{Y}, g_{\mathcal{Y}}, J_{\mathcal{Y}}, \omega_{\mathcal{Y}})$ an ALF Kähler 4-manifold of dihedral type of order β . Let $f \in C_{\beta+2}^\infty(\mathcal{Y}, g_{\mathcal{Y}})$. Then there exists $\varphi \in C_\beta^\infty(\mathcal{Y}, g_{\mathcal{Y}})$ such that $\omega_{\mathcal{Y}} + dd_{J_{\mathcal{Y}}}^c \varphi$ is Kähler, and*

$$(\omega_{\mathcal{Y}} + dd_{J_{\mathcal{Y}}}^c \varphi)^2 = e^f \omega_{\mathcal{Y}}^2. \tag{22}$$

The weighted spaces of this statement follow a classical definition, and our statement simply means that for all $\ell \geq 0$, $|(\nabla^{g_{\mathcal{Y}}})^\ell f|_{\mathcal{Y}} = O(R^{-2-\beta-\ell})$ and $|(\nabla^{g_{\mathcal{Y}}})^\ell \varphi|_{\mathcal{Y}} = O(R^{-\beta-\ell})$; the proof of Theorem 1.16, rather classical yet a bit involved, is postponed to Part 3 below.

Let us now make the following remark: since we want to construct a metric with volume form Ω_Y , this is tempting to take $f = \log(\Omega_Y/\text{vol}_{g_m})$ to apply Theorem 1.16. But so far we only control such an f up to two derivatives (see Proposition 1.12, estimates (18)); also, even in the C^0 sense, we only have $f = O(R^{-2})$ instead of $O(R^{-2-\beta})$.

The other issue is that \mathcal{Y} being a ‘ALF Kähler manifold of dihedral type of order β ’ means that outside a compact subset, \mathcal{Y} is diffeomorphic to the complement of a ball in $\mathbb{R}^4/\mathcal{D}_k$, and that one can choose the diffeomorphism $\Phi_{\mathcal{Y}}$ between infinities of \mathcal{Y} and $\mathbb{R}^4/\mathcal{D}_k$ such that for all $\ell \geq 0$, $|(\nabla^{g_{\mathcal{Y}}})^\ell(\Phi_{\mathcal{Y}*}g_{\mathcal{Y}} - \mathbf{f})|_{g_{\mathcal{Y}}} = O(\rho^{-\beta-\ell})$, and $|(\nabla^{g_{\mathcal{Y}}})^\ell(\Phi_{\mathcal{Y}*}J^{\mathcal{Y}} - I_1)|_{g_{\mathcal{Y}}} = O(\rho^{-\beta-\ell})$. Here again, a reading of Proposition 1.12 indicates that the asymptotics at our disposal do not allow us to take immediately $\Phi_{\mathcal{Y}} = \Phi_Y$ or $\Phi_Y \circ \mathfrak{J}$.

We remedy to these technical problems as follows. First we correct g_m into an I_1^Y -Kähler metric with volume form Ω_Y , which is nothing but a Ricci-flat I_1^Y -Kähler metric, outside a compact subset of Y , which gives us an f with compact support; then we put this corrected metric into so-called *Bianchi gauge* with respect to $\Phi_Y^*\mathbf{f}^b$, which corresponds to correct Φ_Y itself so as to fit into the definition of an ALF Kähler manifold of dihedral type up to the desired order.

1.4.2 *Ricci-flatness outside a compact subset.* To correct g_m into an I_1^Y -Kähler metric with volume form Ω_Y outside a compact subset of Y , we use the inverse function theorem on Monge–Ampère operators, between relevant Hölder spaces. Namely, we extend \mathbf{f}^b on Y as a smooth metric and define on Y the following *weighted Hölder spaces*:

$$C_\delta^{\ell,\alpha}(Y, \mathbf{f}^b) := \{f \in C_{\text{loc}}^{\ell,\alpha} \mid \|f\|_{C_\delta^{\ell,\alpha}(\mathbf{f}^b)} < \infty\}, \tag{23}$$

for $\ell \in \mathbb{N}$, $\alpha \in (0, 1]$, $\delta \in \mathbb{R}$, and where

$$\|f\|_{C_\delta^{\ell,\alpha}(\mathbf{f}^b)} := \|R^\delta f\|_{C^0} + \dots + \|R^{\delta+\ell}(\nabla^{\mathbf{f}^b})^\ell f\|_{C^0} + \sup_{x \in Y} [R^\ell(\nabla^{\mathbf{f}^b})^\ell f]_\delta^\alpha,$$

with

$$[u]_\delta^\alpha = \sup_{\substack{(x,y) \in Y, \\ d_{\mathbf{f}^b}(x,y) < \text{inj}_{\mathbf{f}^b}}} \left| \max(R(x)^{\alpha+\delta}, R(y)^{\alpha+\delta}) \frac{u(x) - u(y)}{d_{\mathbf{f}^b}(x,y)^\alpha} \right| \tag{24}$$

for u a $C_{\text{loc}}^{\ell,\alpha}$ tensor ($u(x) - u(y)$ interpreted via parallel transport), with R a smooth positive extension of Φ_Y^*R on Y , and C^0 -norms of the tensors computed with \mathbf{f}^b .

We then state the following, indicating the type of functions which can help correcting ω_m in the sense raised above.

PROPOSITION 1.17. *Fix $(\alpha, \delta) \in (0, 1)^2$ such that $\alpha + \delta < 1$. There exists a smooth function $\psi \in C_{\delta-1}^{2,\alpha}(\mathbf{f}^b) \cap C_{\delta-2}^{3,\alpha}(\mathbf{f}^b)$ such that $\omega_\psi := \omega_m + dd_{I_1^Y}^c \psi$ is Kähler for I_1^Y , and such that $\frac{1}{2}\omega_\psi^2 = \Omega_Y$ outside a compact set.*

In the statement, the intention is to take the orders of regularity and decay as small as possible, or, more precisely, to keep α in the range $(0, 1 - \delta)$, to make the rest of the argument (a forthcoming use of the implicit function theorem, followed by a gauge process) work, so as to minimise the efforts involved in the construction of the reference metric g_m of Proposition 1.12.

Proof. Taking χ a cut-off function as in Proposition 1.12 and setting $\chi_{R_1} = \chi(R - R_1)$, we are done if we solve the problem $(\omega_m + dd_{I_1^Y}^c \psi)^2 = (1 - \chi_{R_1})\omega_m^2 + 2\chi_{R_1}\Omega_Y$ for R_1 large enough. This is manageable, with the help of the inverse function theorem, since:

- $\omega_m^2 - ((1 - \chi_{R_1})\omega_m^2 + 2\chi_{R_1}\Omega_Y) = \chi_{R_1}(\omega_m^2 - 2\Omega_Y)$, and $\|\chi_{R_1}((\omega_m^2 - 2\Omega_Y)/\Omega_Y)\|_{C_\delta^{k,\alpha}}$ tends to 0 as R_1 goes to ∞ thanks to estimates (18) for $k = 0, 1$;
- the linearisation of the Monge–Ampère operators $C_{\delta-1-\varepsilon}^{2+\varepsilon,\alpha} \rightarrow C_{\delta+1-\varepsilon}^{\varepsilon,\alpha}$, $\varepsilon = 0, 1$, $\psi \mapsto (\omega_m + dd_{I_1^Y}^c \psi)^2/\omega_m^2$, at $\psi = 0$, are the scalar Laplacians $\Delta_{g_m} : C_{\delta-1-\varepsilon}^{2+\varepsilon,\alpha} \rightarrow C_{\delta+1-\varepsilon}^{\varepsilon,\alpha}$. These are surjective, with kernel reduced to constant functions, according to the appendix of [BM11]. More precisely, such statements hold for $\Delta_{\mathbf{f}^b}$ by [BM11]. Indeed, one has the formal representation (using the Kähler property near infinity)

$$\begin{aligned} \Delta_{g_m} &= \Delta_{\mathbf{f}^b} + (g_m^{-1} - (\mathbf{f}^b)^{-1}) * [(\nabla^{\mathbf{f}^b})^2 \cdot] + g_m^{-1} * \nabla^{\mathbf{f}^b} j * (d \cdot) * I_1^Y \\ &\quad + g_m^{-1} * j * (\nabla^{\mathbf{f}^b} d \cdot) * I_1^Y + g_m^{-1} * \nabla^{\mathbf{f}^b} j * (d \cdot) * (I_1^Y - j), \end{aligned}$$

where $j = I_1^Y - \mathfrak{I}^* I_1$, as in the proof of Proposition 1.12. From this formula, together with the estimates $(\nabla^{\mathbf{f}^b})^\ell(\mathbf{f}^b - g_m) = O(R^{-2})$, $\ell = 0, 1, 2$, and $(\nabla^{\mathbf{f}^b})^\ell j = O(R^{-3})$, $\ell = 0, \dots, 3$, we get that Δ_{g_m} is well defined as an operator $C_{\delta-1}^{2,\alpha} \rightarrow C_{\delta+1}^{0,\alpha}$ or $C_{\delta-2}^{3,\alpha} \rightarrow C_\delta^{1,\alpha}$, and differs from $\Delta_{\mathbf{f}^b}$ by an operator of respective sizes $O(R^{-2+\alpha})$ and $O(R^{-1+\alpha})$ in the appropriate operator norms. This gives us the desired mapping properties for Δ_{g_m} , first for Dirichlet problems on exterior domains by a perturbation argument, next on the whole Y by a classical argument (based e.g. on a parametrix and a Poincaré inequality).

Once R_1 is chosen large enough to apply the inverse function theorem simultaneously, and once ψ is fixed in $C_{\delta-1}^{2,\alpha}(\mathbf{f}^b) \cap C_{\delta-2}^{3,\alpha}(\mathbf{f}^b)$ so that $(\omega_m + dd_{I_1^Y}^c \psi)^2 = (1 - \chi_{R_1})\omega_m^2 + 2\chi_{R_1}\Omega_Y$, the last point to be checked is the positivity of $\omega_\psi := \omega_m + dd_{I_1^Y}^c \psi$. As $dd_{I_1^Y}^c \psi = O(R^{-\delta})$, ω_ψ is asymptotic to ω_m , hence positive near infinity. Since its determinant $((1 - \chi_{R_1})\omega_m^2 + 2\chi_{R_1}\Omega_Y)/\omega_m^2$ relatively to ω_m never vanishes, it is positive on the whole Y . The smoothness of ψ is local. \square

1.4.3 Bianchi gauge for ω_ψ .

Motivation. We are now willing to deduce regularity statements on g_ψ , using its Ricci-flatness near infinity. However this cannot be done immediately. The reason is that the Ricci-flatness condition is invariant under diffeomorphisms, and consequently *the linearisation of the Ricci tensor seen as an operator on metrics is not (strongly) elliptic*, which is problematic when looking for regularity.

One can however bypass this difficulty by fixing a gauge, which infinitesimally corresponds to looking at metrics with good diffeomorphisms. We introduce the diffeomorphisms we shall work with in next paragraph; then the gauge is fixed, and regularity is deduced from this process (Propositions 1.21 and 1.23). Notice that the Ricci-flatness of g_ψ is an indispensable prerequisite in this procedure, since the gauge alone is not enough in general to obtain the regularity statement we are seeking here.

ALF diffeomorphisms of \mathbb{C}^2 . The class of diffeomorphisms we work with to perform our gauge enters into the following definition; we define the dual frames (e_0^b, \dots, e_3^b) and $((e_0^*)^b, \dots, (e_3^*)^b)$ as the pull-backs by \mathfrak{I} of the frames (e_i) and (e_i^*) defined in § 1.2.2 by (8), (9). We fix $R_0 \geq 1$ so that \mathfrak{I} induces a diffeomorphism between $\{R^b \geq R_0\}$ and $\{R \geq R_0\}$.

DEFINITION 1.18. Let $(\ell, \alpha) \in \mathbb{N}^* \times (0, 1)$, and let $\nu > -1$. We denote by $\mathbf{Diff}_{\nu, R_0}^{\ell, \alpha}$ the class of diffeomorphisms ϕ of $\{R^b \geq R_0\}$ such that:

- ϕ has regularity $C_{\text{loc}}^{\ell, \alpha}$, and induces the identity on $\{R^b = R_0\}$;
- there exists a constant C such that for any $x \in \{R^b \geq R_0\}$, $d_{\mathbf{f}^b}(x, \phi(x)) \leq C(1 + R^b(x))^{-\nu}$;

- consider the in $C_{\text{loc}}^{\ell-1,\alpha}$ maps $\phi_{ij} : \{R^b \geq R_0\} \rightarrow \mathbb{R}$ given by

$$\phi_{ij}(x) = (e_i^b)^*((\phi^* e_j^b)_x - (e_j^b)_x); \quad i, j = 0, \dots, 3.$$

We then ask: $\phi_{ij} \in C_{\nu+1}^{\ell-1,\alpha}(\{R^b \geq R_0\}, \mathbf{f}^b)$.

We endow $\mathbf{Diff}_{\nu,R_0}^{\ell,\alpha}$ with the natural topology.

The Hölder spaces are those defined for (some smooth extension of) \mathbf{f}^b on \mathbb{C}^2 , in the same way as those of defining equation (23). Notice that we authorise the distance between a point and its image by a $\phi \in \mathbf{Diff}_{\nu,R_0}^{\ell,\alpha}$ to go to ∞ when $\nu < 0$; observe nonetheless that the allowed rate of blow-up is sub-linear, and thus the diffeomorphisms are proper.

Diffeomorphisms as Riemannian exponential maps. We now parametrise our diffeomorphisms via vector fields.

LEMMA 1.19. *There exists a neighbourhood $\mathcal{V}_\nu^{\ell,\alpha}$ of 0 in $C_{\nu,0}^{\ell,\alpha}(\{R^b \geq R_0\}, \mathbf{f}^b)$ such that for any Z in that neighbourhood, the map $\phi_Z : x \mapsto \exp_x^{\mathbf{f}^b}(Z(x))$ is in $\mathbf{Diff}_{\nu,R_0}^{\ell,\alpha}$.*

If moreover $\nu \geq 0$, one can choose $\mathcal{V}_\nu^{\ell,\alpha}$ so that $Z \mapsto \phi_Z$ realises a diffeomorphism of $\mathcal{V}_\nu^{\ell,\alpha}$ onto a neighbourhood of the identity map in $\mathbf{Diff}_{\nu,R_0}^{\ell,\alpha}$.

Apart from the 0 indices indicating vanishing of the vector fields along $\{R^b = R_0\}$, the weighted spaces of vector fields are defined analogously to that of the previous paragraph, or equivalently $Z \in C_{\nu,0}^{\ell,\alpha}(\{R \geq R_0\}, \mathbf{f}^b)$ if and only if $Z \in C_{\text{loc}}^{\ell,\alpha}(\{R^b \geq R_0\})$, $Z|_{R^b=R_0} \equiv 0$ and $\chi(R^b - R_0)(e_i^b)^*(Z) \in C_\nu^{\ell,\alpha}(\mathbb{C}^2, \mathbf{f}^b)$, $i = 0, \dots, 3$ (with χ a cut-off function as in Proposition 1.12).

Proof. The regularity assertions are rather standard; we moreover use the fact that the injectivity radius of \mathbf{f}^b is bounded from below (that of \mathbf{f} is), to get that any diffeomorphism C^0 -close to identity can be written as a ϕ_Z for some (small) continuous Z , which boils down to joining any two points x and y with $d_{\mathbf{f}^b}(x, y) \leq \frac{1}{2} \text{inj}_{\mathbf{f}^b}$ by a *unique* minimising geodesic $t \mapsto \exp_x(tZ_x)$. We shall nonetheless pay particular attention to the fact that in the general case, we authorise vector fields blowing up at infinity, when verifying the injectivity of ϕ_Z for a given Z close to 0 in $C_\nu^{\ell,\alpha}$; in the same vein, one should keep in mind that *even on compact manifolds*, a ϕ_W might not be injective, unless $|\nabla W|$ is small. In this respect, the key here is the decay of the derivatives of Z at infinity, combined with the decay of $\text{Rm}^{\mathbf{f}^b}$. Suppose $(\ell, \alpha) = (1, 0)$ to fix ideas; for simplicity, we work on the whole \mathbb{C}^2 , where we extend \mathbf{f}^b smoothly. For the injectivity of ϕ_Z with fixed $Z \in C_\nu^{1,0}$ (defined on \mathbb{C}^2) and $\|Z\|_{C_\nu^{1,0}} \leq 1$ say, we claim that there exists a constant C independent of Z such that for any triple (x, y, z) such that $\phi_Z(x) = \phi_Z(y) =: z$,

$$d_{\mathbf{f}^b}(x, y) \leq C(1 + R(z))^{-4-3\nu} \|Z\|_{C_\nu^{1,0}} d_{\mathbf{f}^b}(x, y),$$

from which the injectivity of ϕ_Z follows at once provided $\|Z\|_{C_\nu^{1,0}}$ is small enough. We reach this claim thanks to the estimate $|\text{Rm}^{\mathbf{f}^b}| = O(R^{-3})$, as follows. For x, y as in the claim, call respectively γ_x and γ_y the geodesics $t \mapsto \exp_x^{\mathbf{f}^b}(tZ(x))$ and $t \mapsto \exp_y^{\mathbf{f}^b}(tZ(y))$, and denote by $p_{\gamma_x}, p_{\gamma_y}$ the attached parallel transports. Using [BK81, Proposition 6.6], control first $d_{\mathbf{f}^b}(x, y)$ by $|p_{\gamma_x}(1)(Z(x)) - p_{\gamma_y}(1)(Z(y))|_{\mathbf{f}^b} (1 + R(z))^{-3-2\nu}$. Then control $|p_{\gamma_x}(1)(Z(x)) - p_{\gamma_y}(1)(Z(y))|_{\mathbf{f}^b}$ by $d_{\mathbf{f}^b}(x, y)(1 + R(z))^{-1-\nu} \|Z\|_{C_\nu^{1,0}}$; for this interpolate between γ_x and γ_y by $\gamma_s(t) := \exp_{\alpha(s)}^{\mathbf{f}^b}[tZ(\alpha(s))]$, where α is a minimising geodesic for \mathbf{f}^b joining x and y . This is where one uses the estimates on the derivatives of Z . □

With similar techniques, one establishes the following lemma.

LEMMA 1.20. *With the notation of Lemma 1.19, for $Z \in \mathcal{V}_\nu^{\ell,\alpha}$, one has $(\phi_Z)^*\mathbf{f}^b - \mathbf{f}^b$ is $C_{\nu+1}^{\ell-1,\alpha}$; more precisely, one has the estimate $\|(\phi_Z)^*\mathbf{f}^b - \mathbf{f}^b - \delta^{\mathbf{f}^b}[\mathbf{f}^b(Z, \cdot)]\|_{C_{2\nu+2}^{\ell-1,\alpha}} \leq C\|Z\|_{C_\nu^{\ell,\alpha}}$.*

The gauge. Denote by $B^h = \delta^h + \frac{1}{2}d \operatorname{tr}^h$ the Bianchi operator associated to any smooth metric h on (an open subset of) \mathbb{R}^4 . The gauge process now states the following.

PROPOSITION 1.21. *Let (α, δ) fixed in Proposition 1.17 and δ assumed $> \frac{1}{2}$. If R_0 is large enough, there exists a smooth diffeomorphism $\phi \in \mathbf{Diff}_{\delta-1, R_0}^{1,\alpha}$, commuting with the action of \mathcal{D}_k hence descending to $\mathbb{R}^4/\mathcal{D}_k$ near infinity, such that*

$$B^{\phi^*\mathbf{f}^b}((\Phi_Y)_*g_\psi) = 0$$

near infinity on \mathbb{C}^2 , where g_ψ stands for the I_1^Y -Kähler metric associated to the Kähler form ω_ψ of Proposition 1.17. As a consequence, $\mathbf{f}^b - (\phi \circ \Phi_Y)_*g_\psi \in C_\delta^{1,\alpha}(X, \mathbf{f}^b)$.

Remark 1.22. The assumption $\delta > \frac{1}{2}$ is actually superfluous; we chose to keep it nonetheless as it authorises a shorter proof (we only need *one* iteration in *Step 1* below).

Proof. Fix α_2 as in the statement, and consider the map

$$\begin{aligned} \Xi : \mathcal{V}_{\delta-1}^{2,\alpha_2} \times \mathbf{Met}_\delta^{1,\alpha_2}(\mathbf{f}^b) &\longrightarrow C_{\delta+1}^{0,\alpha_2}(T^*\mathbb{C}^2, \mathbf{f}^b) \\ (Z, g) &\longmapsto B^{\phi_Z^*\mathbf{f}^b}(g). \end{aligned}$$

We would like to solve the equation

$$B^{\phi_Z^*\mathbf{f}^b}(g_\psi) = 0, \quad \text{i.e. } \Xi(Z, g_\psi) = 0, \tag{25}$$

near infinity, and for this use the implicit function theorem near $(0, \mathbf{f}^b)$, since the differential of Ξ with respect to Z is $(\nabla^{\mathbf{f}^b})^*\nabla^{\mathbf{f}^b}$, which as we shall see enjoys surjectivity properties. We first solve a linearised version (25), to get in the $\nu \geq 0$ position of Lemma 1.19, better adapted to solve (a duly modified version of) this nonlinear equation.

Step 1. Let $Z \in C_{\delta-1,0}^{2,\alpha}(\{R^b \geq R_0\}, \mathbf{f}^b)$ be a smooth vector field such that $(\nabla^{\mathbf{f}^b})^*\nabla^{\mathbf{f}^b}[\mathbf{f}^b(Z, \cdot)] = B^{\mathbf{f}^b}(g_\psi)$ near infinity. As we shall see below, such an equation can be solved and, picking some $\delta' \in (0, \delta)$, and up to using some cut-off function (which does not affect the equation verified by Z near infinity), we can assume that $\|Z\|_{C_{\delta'-1}^{2,\alpha}(\mathbf{f}^b)}$ is small enough so that $Z \in \mathcal{V}_{\delta'-1}^{2,\alpha}$, i.e., $\phi_Z \in \mathbf{Diff}_{\delta'-1, R_0}^{1,\alpha}$ (in particular, it is a diffeomorphism); as Z has regularity $C_{\delta-1}^{2,\alpha}(\mathbf{f}^b)$, we moreover get $\phi_Z \in \mathbf{Diff}_{\delta-1, R_0}^{1,\alpha}$. Taking a mean along the action of \mathcal{D}_k , we can also assume that Z is \mathcal{D}_k -invariant, and hence that ϕ_Z commutes to the \mathcal{D}_k -action. The smoothness of Z , purely local, comes at once from that of \mathbf{f}^b and $B^{\mathbf{f}^b}(g)$, and ellipticity of $(\nabla^{\mathbf{f}^b})^*\nabla^{\mathbf{f}^b}$.

Now, given metrics g, g' and $g+h$ (with g' and $g+h$ seen as perturbations of g), repeated use of equation (20) provides

$$B^{g+h}(g') = B^g(g') - B^g(h) + Q_g(h, g'),$$

where $Q_g(h, g')$ admits a formal expansion

$$Q_g(h, g') = \nabla^g h * \{(g + h)^{-1} - g^{-1}\} + (g + h)^{-2} * \nabla^g h * h + \nabla^g g' * \{(g + h)^{-1} - g^{-1}\}(g + h)^{-2} * \nabla^g h * (g' - g).$$

Taking $g = \mathbf{f}^b$, $h = h_Z := (\phi_Z)^* \mathbf{f}^b - \mathbf{f}^b$ and $g' = g_\psi$, we get that $Q_{\mathbf{f}^b}(h_Z, g_\psi)$ factors through $(\nabla^{\mathbf{f}^b} h_Z) * h_Z$, $(\nabla^{\mathbf{f}^b} h_Z) * (\mathbf{f}^b - g_\psi)$ and $(\nabla^{\mathbf{f}^b} g_\psi) * h_Z$. This way, by Lemma 1.20 and as Z is smooth, $Q_{\mathbf{f}^b}(h_Z, g_\psi) \in C_{\text{loc}}^\infty \cap C_{1+2\delta}^{0,\alpha}$. Moreover, as $h_Z = -\delta^{\mathbf{f}^b}[\mathbf{f}^b(Z, \cdot)] + h'_Z$ with $h'_Z \in C_{\text{loc}}^\infty \cap C_{2\delta}^{1,\alpha}$,

$$B^{\mathbf{f}^b}(h_Z) = B^g(\delta^{\mathbf{f}^b}[\mathbf{f}^b(Z, \cdot)]) + B^{\mathbf{f}^b}(h'_Z) = (\nabla^{\mathbf{f}^b})^* \nabla^{\mathbf{f}^b}[\mathbf{f}^b(Z, \cdot)] + B^{\mathbf{f}^b}(h'_Z),$$

with $B^{\mathbf{f}^b}(h'_Z) \in C_{\text{loc}}^\infty \cap C_{1+2\delta}^{0,\alpha}$. In conclusion, $B^{(\phi_Z)^* \mathbf{f}^b}(g_\psi) = B^{\mathbf{f}^b}(g_\psi) - B^{\mathbf{f}^b}(h_Z) + Q_{\mathbf{f}^b}(h_Z, g_\psi) = \underbrace{B^{\mathbf{f}^b}(g_\psi) - (\nabla^{\mathbf{f}^b})^* \nabla^{\mathbf{f}^b}[\mathbf{f}^b(Z, \cdot)]}_{= 0 \text{ near infinity}} + B^{\mathbf{f}^b}(h'_Z) + Q_{\mathbf{f}^b}(h_Z, g_\psi)$ is now in $C_{\text{loc}}^\infty \cap C_{1+2\delta}^{0,\alpha}$. Pushing forward

by ϕ_Z , we have $B^{\mathbf{f}^b}[(\phi_Z)^* g_\psi] \in C_{\text{loc}}^\infty \cap C_{1+2\delta}^{0,\alpha}(\mathbf{f}^b)$, where we had $B^{\mathbf{f}^b}(g_\psi) \in C_{\text{loc}}^\infty \cap C_{1+\delta}^{0,\alpha}(\mathbf{f}^b)$ (and $1 + 2\delta > 2 > 1 + \delta$; this is where $\delta > \frac{1}{2}$ is used).

Step 2. Set $\bar{\delta}_1 = 2\delta - 1 \in (0, 1)$, and take $\delta_1 \in (0, \bar{\delta}_1)$. As $B^{\mathbf{f}^b}[(\phi_Z)^* g_\psi] \in C_{\text{loc}}^\infty \cap C_{2+\bar{\delta}_1}^{0,\alpha}(\mathbf{f}^b)$, $\|\chi_{R_2}(R^b) B^{\mathbf{f}^b}[(\phi_Z)^* g_\psi]\|_{C_{2+\delta_1}^{0,\alpha}(\mathbf{f}^b)}$ goes to 0 as R_2 goes to infinity; here $\chi_{R_2} = \chi(-R_2)$, with χ a cut-off function as in Proposition 1.12. In other words, given any neighbourhood of $B^{\mathbf{f}^b}[(\phi_Z)^* g_\psi]$ in $C_{2+\delta_1}^{0,\alpha}(\mathbf{f}^b)$, the 1-form $(1 - \chi_{R_2}(R^b)) B^{\mathbf{f}^b}[(\phi_Z)^* g_\psi]$ lies in this neighbourhood for R_2 large enough; by construction it moreover *vanishes on* $\{R^b \geq R_2 + 1\}$. This being said, we now work for simplicity with the operator $z \mapsto \Xi[z, (\phi_Z)^* g_\psi] = B^{\phi_Z^* \mathbf{f}^b}[(\phi_Z)^* g_\psi]$ seen as a map $\mathcal{V}_{\delta_1}^{2,\alpha} \rightarrow C_{\delta_1+2}^{0,\alpha}(T^*\mathbb{C}^2, \mathbf{f}^b)$ thanks to the formulas on $B^{g+h}(g')$ and $Q_g(h, g')$ of Step 1. Finding *one* solution z to the equation $\Xi[z, (\phi_Z)^* g_\psi] = 0$ near infinity, that is, finding a solution to the equation $\Xi[z, (\phi_Z)^* g_\psi] = (1 - \chi(R^b - R_2)) B^{\mathbf{f}^b}[(\phi_Z)^* g_\psi]$ for some arbitrarily large R_2 , with $z \in C_{\delta_1,0}^{2,\alpha}(\{R^b \geq R_0\}, \mathbf{f}^b)$, thus amounts by the implicit function theorem to establishing the surjectivity of the operator

$$\begin{aligned} & (\nabla^{\mathbf{f}^b})^* \nabla^{\mathbf{f}^b} + [(\phi_Z)^* g_\psi - \mathbf{f}^b] * (\nabla^{\mathbf{f}^b})^2 \cdot + \nabla^{\mathbf{f}^b} [(\phi_Z)^* g_\psi] * \nabla^{\mathbf{f}^b} \\ &= \frac{\partial \Xi}{\partial z} \Big|_{(0, (\phi_Z)^* g_\psi)} : C_{\delta_1,0}^{2,\alpha}(\{R^b \geq R_0\}, \mathbf{f}^b) \longrightarrow C_{\delta_1+2}^{0,\alpha}(\{R^b \geq R_0\}, T^*\mathbb{C}^2, \mathbf{f}^b) \cong C_{\delta_1+2}^{0,\alpha}(\{R^b \geq R_0\}, \mathbf{f}^b). \end{aligned} \tag{26}$$

Now, as $[(\phi_Z)^* g_\psi - \mathbf{f}^b]$ is $C_\delta^{1,\alpha}(\mathbf{f}^b)$ and $\nabla^{\mathbf{f}^b}[(\phi_Z)^* g_\psi]$ is $C_{\delta+1}^{0,\alpha}(\mathbf{f}^b)$, the operator (26) differs from $(\nabla^{\mathbf{f}^b})^* \nabla^{\mathbf{f}^b}$ by some asymptotically vanishing term in the $C_{\delta_1}^{2,\alpha}(\mathbf{f}^b)$ -to- $C_{\delta_1+2}^{0,\alpha}(\mathbf{f}^b)$ norm. Hence (up to working in Step 2 with an R_0 possibly larger than in Step 1), it is enough to see that $(\nabla^{\mathbf{f}^b})^* \nabla^{\mathbf{f}^b} : C_{\vartheta,0}^{2,\alpha}(\{R^b \geq R_0\}, \mathbf{f}^b) \rightarrow C_{\vartheta+2}^{0,\alpha}(\{R^b \geq R_0\}, \mathbf{f}^b)$ admits a bounded right inverse for $\vartheta \in (0, 1)$ (playing the role of δ_1), with norm independent of the domain as we let R_0 grow; we also do the case $\vartheta \in (-1, 0)$ (playing the role of δ) without the independence of the bounds, needed to get Z in the beginning of the proof. We work with \mathbf{f} , diffeomorphic to \mathbf{f}^b (and replace R^b by R , and so on). As for $i = 0, \dots, 3$, $(\nabla^{\mathbf{f}})^* \nabla^{\mathbf{f}}(ve_i^*)$ equals $(\Delta_{\mathbf{f}} v)e_i^*$ plus a linear combination of the $(e_j \cdot v) \nabla_{e_k}^{\mathbf{f}} e_\ell$ and the $v(\nabla_{e_j, e_k}^{\mathbf{f}})^2 e_\ell$, up to increasing R_0 , by a standard perturbation argument, the existence of a bounded right inverse for

$$(\nabla^{\mathbf{f}})^* \nabla^{\mathbf{f}} : C_{\vartheta,0}^{2,\alpha}(\{R \geq R_0\}, T^*\mathbb{C}^2, \mathbf{f}) \longrightarrow C_{\vartheta+2}^{0,\alpha}(\{R \geq R_0\}, T^*\mathbb{C}^2, \mathbf{f})$$

amounts to the analogue for the scalar Laplacian with Dirichlet condition

$$\Delta_{\mathbf{f}} : C_{\vartheta,0}^{2,\alpha}(\{R \geq R_0\}, \mathbb{R}, \mathbf{f}) \longrightarrow C_{\vartheta+2}^{0,\alpha}(\{R \geq R_0\}, \mathbb{R}, \mathbf{f}).$$

Using a similar procedure as in [Min09, § 2.1] (where the ‘ δ ’ there should be thought of as ‘ $\frac{3}{2} - \vartheta$ ’ with our ϑ), we first get, for $\vartheta \in (-1, 1) \setminus \{0\}$, the existence of a bounded linear $\mathcal{G}_\vartheta : C_{\vartheta+2}^{0,\alpha}(\{R \geq R_0\}, \mathbf{f}) \rightarrow C_{\vartheta}^{2,\alpha}(\{R \geq R_0\}, \mathbf{f})$ satisfying $\Delta_{\mathbf{f}} \circ \mathcal{G}_\vartheta = \text{id}_{C_{\vartheta+2}^{0,\alpha}(\{R \geq R_0\})}$; beware we momentarily drop the 0 index to the target space $C_{\vartheta}^{2,\alpha}$ of \mathcal{G}_ϑ . Moreover, a careful reading of Minerbe’s construction shows that for $\vartheta < 1$ (which prevents us to be in the case ‘ $\delta < \delta_j < 2 - \delta$ ’ of [Min09, bottom of p. 937]; remember δ plays the role of $\frac{3}{2} - \vartheta$, and that δ_j has shape $\frac{3}{2} + j$, $j \in \mathbb{N}$ [Min09, p. 931]) gives that the above \mathcal{G}_ϑ has image in $C_{\vartheta,0}^{2,\alpha}(\{R \geq R_0\}, \mathbf{f})$. Finally, when $\vartheta \in (0, 1)$, a bound on \mathcal{G}_ϑ (of shape $C/\vartheta(1 - \vartheta)$ and) uniform in R_0 can also be extracted from this construction; a similar bound, uniform in R_0 , is also valid on $\mathcal{G}_{\vartheta-1}$. Consequently, for all $\vartheta \in (-1, 1) \setminus \{0\}$,

$$(\nabla^{\mathbf{f}})^* \nabla^{\mathbf{f}} : C_{\vartheta,0}^{2,\alpha}(\{R \geq R_0\}, T^* \mathbb{C}^2, \mathbf{f}) \longrightarrow C_{\vartheta+2}^{0,\alpha}(\{R \geq R_0\}, T^* \mathbb{C}^2, \mathbf{f})$$

is a surjective map, with a bounded right inverse \mathcal{G}_ϑ enjoying a bound independent of R_0 , as claimed.

We conclude Step 2 by taking a smooth solution $z \in C_{\delta_1,0}^{2,\alpha}(\{R^b \geq R'_0\}, \mathbf{f}^b)$ to $\Xi(z, (\phi_Z)_* g_\psi) = 0$ near infinity (chosen \mathcal{D}_k -invariant), producing a smooth $\phi_z \in \mathbf{Diff}_{\delta_1, R'_0}^{1,\alpha}$, and by putting $\phi = (\phi_z \circ \phi_{Z'})^{-1} \in \mathbf{Diff}_{\delta-1, R'_0}^{1,\alpha}$ (which is smooth), with $Z' = \chi(R^b - R'_0)Z$. □

Regularity of g_ψ . We conclude this paragraph by the following statement, which finally allows us to apply Theorem 1.16.

PROPOSITION 1.23. *With the same notation as in Proposition 1.21, $\mathbf{f}^b - (\phi \circ \Phi_Y)_* g_\psi \in C_\delta^\infty(X, \mathbf{f}^b)$ near infinity, and in particular, $|\text{Rm}^{g_\psi}|_{g_\psi} = O(R^{-2-\delta})$.*

Proof. The assertion on the curvature of g_ψ directly follows from the estimate stated on $\varepsilon := \phi^* \mathbf{f}^b - g_\psi$ (or $\phi_* \varepsilon$), and the fact that $|\text{Rm}^{\mathbf{f}^b}|_{\mathbf{f}^b} = O(R^{-3})$. For the regularity statement on ε , proceed as follows: set $\mathbf{F} = \phi^* \mathbf{f}^b$, and define the operator $\Phi^{\mathbf{F}}$ by

$$\Phi^{\mathbf{F}}(h) = \text{Ric}(h) + (\delta^h)^{*h} [h_{\mathbf{F}} \cdot B^{\mathbf{F}} h]$$

on C_{loc}^2 metrics, where $h_{\mathbf{F}}$ is the endomorphism of T^*X such that $(h_{\mathbf{F}} \cdot \alpha, \cdot)_h = (\alpha, \cdot)_{\mathbf{F}}$ on 1-forms. This way, $\Phi^{\mathbf{F}}(\mathbf{F}) = \Phi^{\mathbf{F}}(g_\psi) = 0$ near infinity. Now $\Phi^{\mathbf{F}}$ is of order 2, hence schematically,

$$0 = \Phi^{\mathbf{F}}(\mathbf{F}) - \Phi^{\mathbf{F}}(g_\psi) = (d_{\mathbf{F}} \Phi^{\mathbf{F}})(\varepsilon) + P_{\mathbf{F}}(\varepsilon),$$

with $P_{\mathbf{F}}(\varepsilon)$ an at-least-quadratic combination of ε , its first and its second derivatives, with coefficients depending on \mathbf{F} . Quoting [Bam11, p. 16], one has more precisely:

$$\begin{aligned} 2P_{\mathbf{F}}(\varepsilon) &= (g_\psi)^{uv} (g_\psi)^{pq} (\nabla_u^\psi \varepsilon_{pa} \nabla_v^\psi \varepsilon_{qb} - \nabla_p^\psi \varepsilon_{ua} \nabla_v^\psi \varepsilon_{qb} + \frac{1}{2} \nabla_a^\psi \varepsilon_{up} \nabla_b^\psi \varepsilon_{vq}) \\ &+ (g_\psi)^{uv} (-\nabla_u^\psi \varepsilon_{vp} + \frac{1}{2} \nabla_p^\psi \varepsilon_{uv}) (g_\psi)^{pq} (\nabla_a^\psi \varepsilon_{qb} + \nabla_b^\psi \varepsilon_{qa} - \nabla_q^\psi \varepsilon_{ab}) \\ &+ \mathbf{F}^{uv} (-\nabla_u^\psi \varepsilon_{vp} + \frac{1}{2} \nabla_p^\psi \varepsilon_{uv}) \mathbf{F}^{pq} \nabla_q^\psi \varepsilon_{ab} \\ &+ \mathbf{F}^{uv} \mathbf{F}^{pq} (-\varepsilon_{ap} (\nabla_{bu}^\psi)^2 \varepsilon_{vq} - \varepsilon_{bp} (\nabla_{au}^\psi)^2 \varepsilon_{vq} + \frac{1}{2} \varepsilon_{ap} (\nabla_{bq}^\psi)^2 \varepsilon_{uv} + \frac{1}{2} \varepsilon_{bp} (\nabla_{aq}^\psi)^2 \varepsilon_{uv}) \\ &+ ((g_\psi)^{uv} - \mathbf{F}^{uv}) ((\nabla_{ua}^\psi)^2 \varepsilon_{bv} + (\nabla_{ab}^\psi)^2 \varepsilon_{av} - (\nabla_{uv}^\psi)^2 \varepsilon_{ab} + (\nabla_{ab}^\psi)^2 \varepsilon_{uv}). \end{aligned} \tag{27}$$

The interest of this formula lies in the following: in $P_{\mathbf{F}}(\varepsilon)$,

- (i) the only occurrence of the second derivatives of $\varepsilon = g_{\psi} - \mathbf{F}$ with respect to g_{ψ} , denoted by $(\nabla^{\psi})^2\varepsilon$, in (27), is via tensors factoring through $\varepsilon * (\nabla^{\psi})^2\varepsilon$; using (20), we can moreover rewrite $(\nabla^{\psi})^2\varepsilon$ as $(1 + \mathbf{F}^{-1} * \varepsilon) * (\nabla^{\mathbf{F}})^2\varepsilon$ plus some terms factorising through $(\nabla^{\mathbf{F}}\varepsilon)^{*2}$;
- (ii) all other terms factor through $(\nabla^{\psi})^{*2}\varepsilon$; according to (20), one can say these terms factor through $(\nabla^{\mathbf{F}})^{*2}\varepsilon$ as well;
- (iii) the algebraic coefficients are controlled (for \mathbf{F} say) in $C^{1,\alpha}$.

We sum these three points up by writing

$$\frac{1}{2}\mathcal{L}_{\phi*\mathbf{f}^b}\varepsilon + \varepsilon * (\nabla^{\mathbf{F}})^2\varepsilon = (\nabla^{\mathbf{F}}\varepsilon)^{*2} * Q(\varepsilon, \nabla^{\mathbf{F}}\varepsilon), \tag{28}$$

where $\mathcal{L}_{\phi*\mathbf{f}^b} = d_{\mathbf{F}}\Phi^{\mathbf{F}}$ is the Lichnerowicz Laplacian of $\mathbf{F} = \phi*\mathbf{f}^b$, the symbols $*$ denote algebraic operations. Since $\varepsilon \in C_{\delta}^{1,\alpha}(\mathbb{R}^4, \mathbf{F})$ the right-hand side of (28) is in $C_{2\delta+2}^{0,\alpha}(\mathbb{R}^4, \mathbf{F})$. Again since $\varepsilon \in C_{\delta}^{1,\alpha}$, the linear operator $\eta \mapsto \frac{1}{2}\mathcal{L}_{\phi*\mathbf{f}^b}\eta + \varepsilon * (\nabla^{\mathbf{F}})^2\eta$ is elliptic and one can draw for this operator weighted estimates similar to those for $\mathcal{L}_{\phi*\mathbf{f}^b}$. From this we deduce that $\varepsilon \in C_{\delta}^{2,\alpha}$. Repeating this argument, we get $\varepsilon \in C_{\delta}^{\infty}(\mathbf{F})$, hence $\mathbf{f}^b - \phi_*g_{\psi} = \phi_*\varepsilon \in C_{\delta}^{\infty}(\mathbf{f}^b)$. Notice that one could even get from this scheme $\varepsilon = \varepsilon_0 + \varepsilon'$, with ε_0 an $\mathcal{L}_{\phi*\mathbf{f}^b}$ -harmonic tensor in $C_{\delta}^{\infty}(\mathbf{F})$, hence in $C_1^{\infty}(\mathbf{F})$, and $\varepsilon' \in C_{2\delta}^{\infty}(\mathbf{F})$; this yields in the end $\phi_*\varepsilon \in C_1^{\infty}(\mathbf{f}^b)$, which is better than needed. \square

1.4.4 *Conclusion: proof of Theorem 1.3.* We have proved that \mathbf{f}^b and $(\phi \circ \Phi_Y)_*g_{\psi}$ are C_{β}^{∞} -close, provided that $\beta = \delta$; to fulfil completely the requirements of Theorem 1.16, we are only left with checking that $(\phi \circ \Phi_Y)_*I_1^Y$ is also C_{β}^{∞} close to the complex structure $I_1^b := \mathfrak{I}^*I_1$.

The estimate $(\phi \circ \Phi_Y)_*I_1^Y - I_1^b \in C_{\beta}^0$ follows easily from the decomposition $(\Phi_Y)_*I_1^Y - \phi^*I_1^b = ((\Phi_Y)_*I_1^Y - I_1) + (I_1 - I_1^b) + (I_1^b - \phi^*I_1^b)$, from the estimates $|(\Phi_Y)_*I_1^Y - I_1|_{\mathbf{e}} = O(r^{-4})$ and $|I_1 - I_1^b|_{\mathbf{e}} = O(r^{-4})$ converted into $|(\Phi_Y)_*I_1^Y - I_1|_{\mathbf{f}}, |I_1 - I_1^b|_{\mathbf{f}^b} = O(R^{-1})$, and $|I_1^b - \phi^*I_1^b|_{\mathbf{f}^b} = O(R^{-\delta}) = O(R^{-\beta})$ following from $\phi \in \mathbf{Diff}_{\delta^{-1}, R_0}^{1,\alpha}$ in Proposition 1.21.

For higher order estimates, remember that g_{ψ} is Kähler for I_1^Y , and \mathbf{f}^b for I_1^b . It is thus enough for instance to evaluate the successive $(\nabla^{\mathbf{f}^b})^{\ell}((\phi \circ \Phi_Y)_*I_1^Y)$. In view of formula (20) and dropping Φ_Y , we thus write formally for $\ell = 1$,

$$\begin{aligned} \nabla^{\mathbf{f}^b}(\phi_*I_1^Y) &= \underbrace{\nabla^{\phi_*g_{\psi}}(\phi_*I_1^Y)}_{= 0 \text{ since } g_{\psi} \text{ is } I_1^Y\text{-Kähler}} + (\mathbf{f}^b)^{-1} * \nabla^{\phi_*g_{\psi}}(\mathbf{f}^b - \phi_*g_{\psi}) * (\phi_*I_1^Y), \end{aligned}$$

which easily gives $\nabla^{\mathbf{f}^b}((\phi \circ \Phi_Y)_*I_1^Y) \in C_{\beta+1}^0$ in view of $(\mathbf{f}^b - \phi_*g_{\psi}) \in C_{\beta}^1$. For $\ell \geq 2$, simply use inductively formula (20), and the estimate $(\mathbf{f}^b - \phi_*g_{\psi}) \in C_{\beta}^{\ell}$.

As sketched in the introduction of this section, we now apply Theorem 1.16, with $(\mathcal{Y}, g_{\mathcal{Y}}, J_{\mathcal{Y}}, \omega_{\mathcal{Y}}) = (Y, g_{\psi}, I_1^Y, \omega_{\psi})$ and $f = \log(\Omega_Y / \text{vol}^{g_{\psi}})$, which is smooth and has compact support. This gives us an I_1^Y -metric $g_{RF,m}$ on Y , with volume form Ω_Y and which is thus Ricci-flat, and with Kähler form $\omega_{\psi} + dd_{I_1^Y}^c\varphi$ for some $\varphi \in C_{\beta}^{\infty}(Y, g_{\psi})$ with β close to 1. At this stage, $(g_{RF,m} - g_{\psi}) \in C_{\beta}^{\infty}(\phi*\mathbf{f}^b)$; from this, $(\nabla^{\mathbf{f}^b})^{\ell}(\mathbf{f}^b - g_m) = O(R^{-2})$, $\ell = 0, 1, 2$, and $(\nabla^{\mathbf{f}^b})^{\ell}(\mathbf{f}^b - g_m) = O(R^{-1-\beta})$, $\ell = 0, 1$ (Proposition 1.17), we also have $(\nabla^{\mathbf{f}^b})^{\ell}(\mathbf{f}^b - g_{RF,m}) = O(R^{-1-\beta})$, $\ell = 0, 1$.

We need two more complex structures for Theorem 1.3. Recall we have two more symplectic forms coming with the ALE hyperkähler structure $(Y, g_Y, I_1^Y, I_2^Y, I_3^Y)$, namely $\omega_2^Y := g_Y(I_2^Y \cdot, \cdot)$ and $\omega_3^Y := g_Y(I_3^Y \cdot, \cdot)$. We simply define J_2^Y and J_3^Y as the endomorphisms verifying

$g_{RF,m}(J_2^Y \cdot, \cdot) = \omega_2^Y$ and $g_{RF,m}(J_3^Y \cdot, \cdot) = \omega_3^Y$; one then checks these are almost complex structures, satisfying the quaternionic relations with I_1^Y , using $(\omega_2^Y)^2 = (\omega_3^Y)^2 = 2 \text{vol}^{g_{RF,m}}$ and that the $I_1^Y - (1, 1)$ part of ω_2^Y and ω_3^Y is 0. To check J_2^Y and J_3^Y are integrable, use moreover that the holomorphic symplectic 2-form $\omega_2^Y + i\omega_3^Y$, whose $g_{RF,m}$ -norm is constant, is $g_{RF,m}$ -parallel.

The cubic decay of $\text{Rm}^{g_{RF,m}}$ comes as follows: first, an over-quadratic decay is easily deduced from $(g_\psi - g_{RF,m}) \in C_{\beta+2}^2(Y, g_\psi)$ and $\text{Rm}^{g_\psi} = O(R^{-2-\beta})$ (Proposition 1.23). Then a result of Minerbe [Min07, Theorem 2.5.9] (see also [CC15a]) asserts that we automatically end up with a cubic rate decay of the curvature. \square

Remark 1.24. We conclude this section by completing Remark 1.15, about the order of approximation on the prototype metric g_m needed prior to its corrections. We can indeed observe that making g_m Ricci-flat near infinity, or constructing a Bianchi gauge for g_ψ would only require to start with an error term $g_m - \mathbf{f}^b$ in $C_\delta^{0,\alpha}(\mathbf{f}^b)$; now, improving the regularity after the gauge (in view of making our metric globally Ricci-flat by Theorem 1.16) does require a $C_\delta^{1,\alpha}(\mathbf{f}^b)$ error term, hence the $O(r^{-8})$ of Lemma 1.6 and (14), as sketched above. Also, the improvement of regularity up to infinite differentiation order is fairly automatic once the Bianchi gauge is reached, hence our choice of stating Theorem 1.16 under this shape, which does not require much more effort than a possible lower-order version.

1.5 Verification of the technical Lemmas 1.10 and 1.14

We conclude this part by the left-over proofs of Lemmas 1.10 and 1.14, both useful in the gluing performed in § 1.3. Recall that on the one hand, Lemma 1.10 is about verifying the asymptotics at different orders of a function ψ_c , the hessian of which is meant to approximate the 2-form $\theta_2 + i\theta_3$ in the Taub-NUT framework, although such an approximation is likely to be vain in the Euclidean setting; and that on the other hand, Lemma 1.14 consists of saying that even though $\mathbf{f}^b = \beth^* \mathbf{f}$, with \beth a diffeomorphism of \mathbb{R}^4 better adapted to the Euclidean scope, the transition between \mathbf{f} and \mathbf{f}^b is relatively harmless.

1.5.1 Proof of Lemma 1.10.

Asymptotics of ψ_c and its successive derivatives. We first look at the first point of the statement of Lemma 1.10. Since ψ_c is \mathbb{S}^1 -invariant when looked at on \mathbb{C}^2 (recall that the \mathbb{S}^1 -action on \mathbb{C}^2 is given by $\alpha \cdot (z_1, z_2) = (e^{i\alpha} z_1, e^{-i\alpha} z_2)$), or in other words is a function of y_1, y_2, y_3 (recall in particular that $2r^2 = R \cosh(4my_1) + y_1 \sinh(4my_1)$, following (5) and the definitions of y_1 and R given in § 1.2.2), we have $d\psi_c = (\partial\psi_c/\partial y_1)dy_1 + (\partial\psi_c/\partial y_2)dy_2 + (\partial\psi_c/\partial y_3)dy_3$, and one can see as well the partial derivatives $\partial\psi_c/\partial y_\ell$ as functions of the y_j only. If we thus prove here that for any $p, q, s \geq 0$ such that $p + q + s \leq 4$,

$$\frac{\partial^{p+q+s}\psi_c}{\partial y_1^p \partial y_2^q \partial y_3^s} = O(R^{-1-q-s}), \tag{29}$$

we will get the desired estimates, since we moreover know that $|(\nabla^{\mathbf{f}})^\ell dy_j|_{\mathbf{f}} = O(R^{-1-\ell})$ for all $\ell \geq 1$ and $j = 1, 2, 3$.

The estimate (29) at order 0 is immediate, since $\sinh(4my_1) = O(R^{-1}r^2)$; this follows from the identity $2r^2 = R \cosh(4my_1) + y_1 \sinh(4my_1)$. What is thus clearly to be seen is that each time we differentiate with respect to y_2 or y_3 , we win an R^{-1} , and each time we differentiate with respect to y_1 , we lose nothing. Let us see how it goes at order 1, that is when $p + q + s = 1$. If $p = 1$ and $q = s = 0$, then (near infinity, where $\chi(R) \equiv 1$)

$$\frac{\partial\psi_c}{\partial y_1} = -4(y_2 + iy_3) \left(\frac{4m \cosh(4my_1)}{2Rr^2} - \frac{y_1 \sinh(4my_1)}{2r^2R^3} - \frac{\sinh(4my_1)}{4r^4R} \frac{\partial(2r^2)}{\partial y_1} \right) \tag{30}$$

and $\partial(2r^2)/\partial y_1 = 2V(y_1 \cosh(4my_1) + R \sinh(4my_1))$ (recall that $V = (1 + 4mR)/2R$), so that, after simplifying

$$\frac{\partial\psi_c}{\partial y_1} = -4(y_2 + iy_3) \left(\frac{m}{r^4} - \frac{1}{R^3} + \frac{1}{4r^4R} \right),$$

and this is $O(R^{-1})$, since $r^{-2} = O(R^{-1})$ (as $R = O(r^2)$).

If $q = 1$ and $p = s = 0$, then

$$\frac{\partial\psi_c}{\partial y_2} = -2 \frac{\sinh(4my_1)}{r^2R} - 2(y_2 + iy_3) \sinh(4my_1) \left(\frac{y_2}{r^2R^3} + \frac{y_2 \cosh(4my_1)}{2r^4R^2} \right),$$

since $\partial(2r^2)/\partial y_2 = (y_2/R) \cosh(4my_1)$. As $\sinh(4my_1)$ and $\cosh(4my_1)$ are $O(r^2R^{-1})$, we end up with $\partial\psi_c/\partial y_2 = O(r^2/(R^2r^2)) + O(R \cdot r^2/R \cdot (r^{-2}R^{-2} + r^2/R \cdot r^{-4}R^{-1})) = O(R^{-2})$. The case $s = 1$ and $p = s = 0$, i.e. the estimate on $\partial\psi_c/\partial y_3$, is done by substituting y_3 to y_2 .

In a nutshell, we win one order each time we differentiate y_2, y_3, R and r^2 with respect to y_2 or y_3 , which moreover kills functions of y_1 such as $\sinh(4my_1)$; we win one order as well when differentiating y_2, y_3 and R with respect to y_1 , but this does not hold any more for r^2 or functions like $\sinh(4my_1)$. More formally, using explicit formulas for the $\partial(2r^2)/\partial y_j, j = 1, 2, 3$, we can easily prove by induction that for any p, q, s there exists a polynomial $Q_{p,q,s}$ of total degree $\leq (1 + p + q + s)$ in its first two variables, and $2 + 3p + 2(q + s)$ in total, such that

$$\frac{\partial^{p+q+s}\psi_c}{\partial y_1^p \partial y_2^q \partial y_3^s} = \frac{Q_{p,q,s}(Re^{\pm 4my_1}, y_1 e^{\pm 4my_1}, R, y_1, y_2, y_3)}{(2r^2)^{1+p+q+s} R^{2(1+p+q+s)}},$$

for instance, $Q_{1,0,0}(Re^{\pm 4my_1}, y_1 e^{\pm 4my_1}, R, y_1, y_2, y_3) = 4(y_2 + iy_3)[(R \cosh(4my_1) + y_1 \sinh(4my_1))^2 - R^2 - 4R^3]$. If now $P(\xi_1, \xi_2, \eta_1, \dots, \eta_4) = \xi_1^{a_1} \xi_2^{a_2} \eta_1^{b_1} \dots \eta_4^{b_4}$ is one of the monomials appearing in $Q_{p,q,s}$ and $a := a_1 + a_2, b := b_1 + \dots + b_4$ so that $a \leq 2(1 + p + q + s)$ and $a + b \leq 2 + 3p + 2(q + s)$, since $Re^{\pm 4my_1}, y_1 e^{\pm 4my_1} = O(r^2)$, we get that

$$\frac{P(Re^{\pm 4my_1}, y_1 e^{\pm 4my_1}, R, y_1, y_2, y_3)}{(r^2)^{1+p+q+s} R^{2+2(p+q+s)}} = O\left(\frac{(r^2)^a R^b}{(r^2)^{1+p+q+s} R^{2(1+p+q+s)}} \right),$$

and this is $O(r^{2a-2(1+p+q+s)} R^{b-2(1+p+q+s)})$; since $a \leq 1 + p + q + s$ and $r^{-2} = O(R^{-1})$, this is finally $O(R^{a+b-3(1+p+q+s)})$, which in turn is $O(R^{-(1+q+s)})$ since $a + b \leq 2 + 3p + 2(q + s)$. Therefore $\partial^{p+q+s}\psi_c/\partial y_1^p \partial y_2^q \partial y_3^s = O(R^{-(1+q+s)})$, and this settles the proof of point (i) of the statement.

Asymptotics of $\theta_2 + i\theta_3$, and comparison with $dd_{I_1}^c \psi_c$ and $dd_{I_1^Y}^c \psi_c$. We thus come now to point (ii) of this statement. We do it for $\ell = 0$; it will become clear from this that the subsequent estimates could be dealt with in an analogous way. Our strategy for proving the desired estimate is the following: first we restrict ourselves to $dd_{I_1}^c \psi_c$; next we decompose $dd_{I_1}^c \psi_c - (\theta_2 + i\theta_3)$ into its $dy_1 \wedge \eta$ -component and its $dy_1 \wedge \eta$ -free component; we then observe that the $dy_1 \wedge \eta$ -free components of both $dd_{I_1}^c \psi_c$ and $(\theta_2 + i\theta_3)$ have already the size we want, whereas we need to look at the $dy_1 \wedge \eta$ -component of the very difference $[dd_{I_1}^c \psi_c - (\theta_2 + i\theta_3)]$ to reach the desired estimate. We conclude by collecting together these estimates, and settling the case of the error term $d(I_1^Y - I_1)d\psi_c$.

Since ψ_c is S^1 -invariant,

$$\begin{aligned} dd_{I_1}^c \psi_c &= V^{-1} \left(\frac{\partial^2 \psi_c}{\partial y_1^2} - V^{-1} \frac{\partial V}{\partial y_1} \frac{\partial \psi_c}{\partial y_1} \right) dy_1 \wedge \eta + \left(\frac{\partial^2 \psi_c}{\partial y_2^2} + \frac{\partial^2 \psi_c}{\partial y_3^2} + V^{-1} \frac{\partial V}{\partial y_1} \frac{\partial \psi_c}{\partial y_1} \right) dy_2 \wedge dy_3 \\ &+ V^{-1} \left(\frac{\partial^2 \psi_c}{\partial y_1 \partial y_2} - V^{-1} \frac{\partial V}{\partial y_2} \frac{\partial \psi_c}{\partial y_1} \right) (dy_2 \wedge \eta - V dy_3 \wedge dy_1) \\ &+ V^{-1} \left(\frac{\partial^2 \psi_c}{\partial y_1 \partial y_3} - V^{-1} \frac{\partial V}{\partial y_3} \frac{\partial \psi_c}{\partial y_1} \right) (dy_3 \wedge \eta - V dy_1 \wedge dy_2), \end{aligned}$$

and since $(\xi, -I_1 V \xi, \zeta, I_1 \zeta)$ is the dual frame of (η, dy_1, dy_2, dy_3) and $(\theta_2 + i\theta_3)$ is $(1, 1)$ for I_1 ,

$$\begin{aligned} \theta_2 + i\theta_3 &= V(\theta_2 + i\theta_3)(\xi, I_1 \xi) dy_1 \wedge \eta + (\theta_2 + i\theta_3)(\zeta, I_1 \zeta) dy_2 \wedge dy_3 \\ &+ (\theta_2 + i\theta_3)(\xi, I_1 \zeta) (V dy_1 \wedge dy_2 - dy_3 \wedge \eta) \\ &+ (\theta_2 + i\theta_3)(\xi, \zeta) (V dy_3 \wedge dy_1 - dy_2 \wedge \eta). \end{aligned} \tag{31}$$

We already know that (on $R \geq K$), $\partial \psi_c / \partial y_1 = -4(y_2 + iy_3)(m/r^4 - 1/R^3 + 1/4r^4 R)$, thus (recall that $\partial(2r^2)/\partial y_1 = V(|z_1|^2 - |z_2|^2)$)

$$\frac{\partial^2 \psi_c}{\partial y_1^2} = -4(y_2 + iy_3) \left(-\frac{2mV(|z_1|^2 - |z_2|^2)}{r^6} + \frac{3y_1}{R^5} - \frac{y_1}{4r^4 R^3} - \frac{V(|z_1|^2 - |z_2|^2)}{4r^6 R} \right),$$

the main term of which is $8mV(y_2 + iy_3)(|z_1|^2 - |z_2|^2)/r^6$, in the sense that it is $O(R^{-1})$, whereas the other summands are $O(R^{-2})$. Moreover, from the estimates of point (i) and the fact that $\partial V / \partial y_j = O(R^{-2})$, $j = 1, 2, 3$, we get that

$$dd_{I_1}^c \psi_c = \frac{8mV(y_2 + iy_3)(|z_1|^2 - |z_2|^2)}{r^6} dy_1 \wedge \eta + O(R^{-2}),$$

when estimated with respect to \mathbf{f} .

Now recall that $\alpha_j = I_j r dr$, $j = 1, 2, 3$, and observe that

$$\begin{aligned} \theta_2 + i\theta_3 &= \frac{rdr \wedge \alpha_2 - \alpha_3 \wedge \alpha_1 + irdr \wedge \alpha_3 - i\alpha_1 \wedge \alpha_2}{r^6} = \frac{(rdr - i\alpha_1) \wedge (\alpha_2 + i\alpha_3)}{r^6} \\ &= \frac{(z_1 d\bar{z}_1 + z_2 d\bar{z}_2) \wedge (-z_2 dz_1 + z_1 dz_2)}{r^6} = \frac{\vartheta \wedge \phi}{r^6}, \end{aligned}$$

if we set $\vartheta = z_1 d\bar{z}_1 + z_2 d\bar{z}_2$ and $\phi = -z_2 dz_1 + z_1 dz_2$. Direct computations, use e.g. (10), give

$$\begin{aligned} \vartheta(\xi) &= -(|z_1|^2 - |z_2|^2), & \vartheta(\zeta) &= \frac{2z_1 z_2}{iR} \cosh(4my_1), \\ \phi(\xi) &= -2iz_1 z_2, & \phi(\zeta) &= -\frac{y_1}{2iR}. \end{aligned}$$

In particular, $\vartheta(\xi) = O(r^2)$, $\vartheta(\zeta) = O(r^2 R^{-1})$, $\phi(\xi) = O(R)$ and $\phi(\zeta) = O(1)$. Moreover, since ϑ (respectively ϕ) is $(0, 1)$ (respectively $(1, 0)$) for I_1 , $(\theta_2 + i\theta_3)(\xi, I_1 \xi) = -(2i/r^6)\vartheta(\xi)\phi(\xi) = 8mz_1 z_2(|z_1|^2 - |z_2|^2)/r^6$. Therefore, from (31) and since ϑ (respectively ϕ) has type $(0, 1)$ (respectively $(1, 0)$) for I_1 , using $r^{-2} = O(R^{-1})$ when necessary, we get

$$\theta_2 + i\theta_3 = \frac{8mV z_1 z_2 (|z_1|^2 - |z_2|^2)}{r^6} dy_1 \wedge \eta + O(R^{-2}).$$

with respect to \mathbf{f} . Since $y_2 + iy_3 = -iz_1 z_2$, we thus have $|dd_{I_1}^c \psi_c - (\theta_2 + i\theta_3)|_{\mathbf{f}} = O(R^{-2})$.

We set $\iota_1^Y := I_1^Y - I_1$, and conclude with an estimate on $|d(I_1^Y - I_1)d\psi_c|_{\mathbf{f}} = |d\iota_1^Y d\psi_c|_{\mathbf{f}}$, which is controlled by $|\iota_1^Y|_{\mathbf{f}}|\nabla^{\mathbf{f}}d\psi_c|_{\mathbf{f}} + |\nabla^{\mathbf{f}}\iota_1^Y|_{\mathbf{f}}|d\psi_c|_{\mathbf{f}}$. But $|\iota_1^Y|_{\mathbf{f}}$ and $|\nabla^{\mathbf{f}}\iota_1^Y|_{\mathbf{f}}$ are $O(r^{-2})$ hence $O(R^{-1})$ (see e.g. the proof of Proposition 1.12), and $|d\psi_c|_{\mathbf{f}}$ and $|\nabla^{\mathbf{f}}d\psi_c|_{\mathbf{f}}$ are $O(R^{-1})$ as well from point (i), and as a result $|d(I_1^Y - I_1)d\psi_c|_{\mathbf{f}} = O(R^{-2})$.

This settles the case $\ell = 0$ of the statement. Cases $\ell = 1$ and 2 are done in the same way, noticing in particular that when letting $\nabla^{\mathbf{f}}$ act on the $(\nabla^{\mathbf{f}})^j\psi_c$ or the $(\nabla^{\mathbf{f}})^j\iota_1^Y$, we keep the same order of precision. □

Remark 1.25. The function ψ_c is not so small with respect to \mathbf{e} , at least at positive orders; for instance, the best one seems able to do on its differential is $|d\psi_c|_{\mathbf{e}} = O(rR^{-1})$.

1.5.2 *Comparison between \mathbf{f} and \mathbf{f}^b : proof of Lemma 1.14.* Before comparing the metrics, and for this the 1-forms $dy_j^b := \beth^*dy_j$, $j = 1, 2, 3$, and $\eta^b := \beth^*\eta$ to their natural ('unflat') analogues, we shall compare the $y_j^b := \beth^*y_j$ to the y_j , $j = 1, 2, 3$.

LEMMA 1.26. *We have $y_j^b - y_j = O(R^{-1})$, $j = 1, 2, 3$. Consequently if $R^b := \beth^*R$, then $R^b - R = O(R^{-1})$.*

Proof of Lemma 1.26, estimates on $(y_2^b - y_2)$ and $(y_3^b - y_3)$. Since $y_2 = (1/2i)(z_1z_2 - \overline{z_1z_2})$, and $\beth(z_1, z_2) = (\alpha z_1, \alpha z_2)$ with $\alpha = 1 + a/r^4$ by Lemma 1.1, it is clear that $y_2^b = \alpha^2 y_2 = y_2 + O(y_2 r^{-4})$, that is $y_2^b - y_2 = O(Rr^{-4})$, and this is $O(R^{-1})$; recall that $R = O(r^2)$.

Similarly, $y_3 = -\frac{1}{2}(z_1z_2 + \overline{z_1z_2})$, thus $y_3^b - y_3 = y_3(\alpha^2 - 1)$, which is $O(R^{-1})$.

Estimate on $(y_1^b - y_1)$. The case of y_1^b is slightly more subtle, and for this we shall use the very definition of y_1 . We fix $(z_1, z_2) \in \mathbb{C}^2$. Since $\beth^*z_1 = \alpha z_1$, $\beth^*z_2 = \alpha z_2$, if one sets $u^b = \beth^*u$ and $v^b = \beth^*v$, LeBrun's formulas (5) become

$$\begin{aligned} \alpha^2|z_1|^2 &= e^{2m[(u^b)^2 - (v^b)^2]}(u^b)^2, \\ \alpha^2|z_2|^2 &= e^{2m[(v^b)^2 - (u^b)^2]}(v^b)^2, \end{aligned} \tag{32}$$

which we rewrite as

$$\begin{aligned} |z_1|^2 &= e^{2m\alpha^2[(u^b/\alpha)^2 - (v^b/\alpha)^2]}(u^b/\alpha)^2, \\ |z_2|^2 &= e^{2m\alpha^2[(v^b/\alpha)^2 - (u^b/\alpha)^2]}(v^b/\alpha)^2. \end{aligned}$$

These are precisely the equations verified by $u_{m\alpha^2}$ and $v_{m\alpha^2}$ instead of u^b/α and v^b/α ; by uniqueness of the solutions when $|z_1|$ and $|z_2|$ are fixed, $u^b/\alpha = u_{m\alpha^2}$ and $v^b/\alpha = v_{m\alpha^2}$, that is $u^b = \alpha u_{m\alpha^2}$ and $v^b = \alpha v_{m\alpha^2}$, and consequently $y_1^b = \frac{1}{2}[(u^b)^2 - (v^b)^2] = (\alpha^2/2)(u_{m\alpha^2}^2 - v_{m\alpha^2}^2) = \alpha^2 y_{1,m\alpha^2}$.

Now still with (z_1, z_2) fixed, differentiating LeBrun's equations *with respect to the mass parameter*, μ say, since we also see m as fixed, and rearranging them gives

$$\frac{\partial y_{1,\mu}}{\partial \mu} = -\frac{4R_\mu y_{1,\mu}}{1 + 4\mu R_\mu};$$

in particular $y_{1,\mu}$ is a non-increasing (respectively non-decreasing) function of μ on $\{|z_1| \geq |z_2|\}$ (respectively on $\{|z_2| \geq |z_1|\}$).

Since $\alpha \leq 1$, we have for instance on $\{|z_1| \geq |z_2|\}$ the estimate

$$0 \leq y_{1,m} - y_{1,m\alpha^2} = \int_m^{m\alpha^2} \frac{4R_\mu y_{1,\mu}}{1 + 4\mu R_\mu} d\mu \leq y_{1,m} \int_m^{m\alpha^2} \frac{d\mu}{\mu} = 2y_{1,m} \log \alpha,$$

and similarly $0 \leq y_{1,m\alpha^2} - y_{1,m} \leq -2y_{1,m} \log \alpha$ on $\{|z_2| \geq |z_1|\}$. Since in both cases $\log \alpha = O(r^{-4}) = O(R^{-2})$, we have

$$y_{1,m\alpha^2} - y_{1,m} = O(y_{1,m}R^{-2}) = O(R^{-1}).$$

Therefore $y_1^b - y_1 = \alpha^2(y_{1,m\alpha^2} - y_1) + (\alpha^2 - 1)y_1 = O(R^{-1})$ as claimed, since $\alpha - 1 = O(r^{-4}) = O(R^{-2})$ and in particular $\alpha \sim 1$ near infinity.

The estimate $R^b - R = O(R^{-1})$ comes as follows: $(R^b - R)(R^b + R) = (R^b)^2 - R^2 = (y_1^b)^2 - y_1^2 + (y_2^b)^2 - y_2^2 + (y_3^b)^2 - y_3^2 = O(1)$ from the previous estimates, and thus $R^b - R = O(1/(R^b + R))$, which in particular is $O(R^{-1})$. \square

Estimates on the $dy_j^b - dy_j$, $j = 1, 2, 3$, and $\eta^b - \eta$. We come back to the proof of Lemma 1.14 itself, and start with analysing the transition involved by \sqsupset at the level of 1-forms. We adopt by places the following elementary strategy to evaluate the gap between our fundamental 1-forms and their pull-backs by \sqsupset : for γ one of the dy_j or η , we write

$$\gamma^b = \gamma^b(\xi)\eta + V^{-1}\gamma^b(-I_1\xi)dy_1 + \gamma^b(\zeta)dy_2 + \gamma^b(I_1\zeta)dy_3,$$

and then evaluate the difference $\gamma^b(\xi) - \gamma(\xi)$, and the subsequent ones. We start with the easy cases of dy_2 and dy_3 ; for more concision, we use the complex expression $\gamma = dy_2 + idy_3$.

Keep the notation $\sqsupset(z_1, z_2) = (\alpha z_1, \alpha z_2)$; then $\sqsupset^*(dy_2 + idy_3) = d(\alpha^2(y_2 + iy_3)) = \alpha^2(dy_2 + idy_3) + (y_2 + iy_3)d(\alpha^2)$. Since $\alpha = 1 + O(r^{-4})$, we focus on $d(\alpha^2)$, or rather on $d\alpha$. As α is invariant under the usual action of \mathbb{S}^1 , we already know that $d\alpha(\xi) = 0$. Moreover,

$$d\alpha = -2a \frac{r^2 d(r^2)}{(r^4)^2}, \tag{33}$$

which we keep under this shape since $d(r^2) = \bar{z}_1 dz_1 + z_1 d\bar{z}_1 + \bar{z}_2 dz_2 + z_2 d\bar{z}_2$ is easy to evaluate against $I_1\xi$, ζ and $I_1\zeta$. As a matter of fact, all computations done:

$$\begin{aligned} d\alpha(-I_1\xi) &= -4a \frac{|z_1|^4 - |z_2|^4}{(\kappa + r^4)^2}, & d\alpha(\zeta) &= \frac{-8ar^2}{(r^4)^2} \frac{y_2 \cosh(4my_1)}{R}, \\ d\alpha(I_1\zeta) &= \frac{-8ar^2}{(r^4)^2} \frac{y_3 \cosh(4my_1)}{R}. \end{aligned} \tag{34}$$

In particular, $d\alpha(-I_1\xi) = O(r^{-4}) = O(R^{-2})$, and $d\alpha(\zeta) = O(R^{-1}r^{-4})$ and $d\alpha(I_1\zeta) = O(R^{-1}r^{-4})$, which are $O(R^{-3})$. Since $\alpha \sim 1$ and $y_2 + iy_3 = O(R)$, we end up with $(dy_2^b + idy_3^b)(-I_1\zeta) = O(R^{-2})$, $(dy_2^b + idy_3^b)(\zeta) = 1 + O(R^{-1})$ and $(dy_2^b + idy_3^b)(I_1\zeta) = i + O(R^{-1})$. In other words,

$$|(dy_2^b + idy_3^b) - (dy_2 + idy_3)|_{\mathfrak{f}} = O(R^{-1}).$$

In a way similar to what is done above on $y_1^b - y_1$, the estimate on $dy_1^b - dy_1$ requires little extra care. First, likewise y_1, y_1^b is invariant under the action of \mathbb{S}^1 , since \sqsupset commutes to this action; therefore $dy_1^b(\xi) = 0$. Next, pulling-back (A.5) for dy_1 (proof of Proposition A.9 below) by \sqsupset gives

$$dy_1^b = \frac{1}{4mR^b} (e^{-4my_1^b} d(\alpha^2|z_1|^2) - e^{4my_1^b} d(\alpha^2|z_2|^2)).$$

When evaluating dy_1^b , we decompose the term $e^{-4my_1^b} d(\alpha^2|z_1|^2) - e^{4my_1^b} d(\alpha^2|z_2|^2)$ into $\sigma := \alpha^2(e^{-4my_1^b} d(|z_1|^2) - e^{4my_1^b} d(|z_2|^2))$ and $\rho := (e^{-4my_1^b}|z_1|^2 - e^{4my_1^b}|z_2|^2)d(\alpha^2) = \alpha^{-2}((u^b)^2 - (v^b)^2)d(\alpha^2) = 4\alpha^{-1}y_1^b d\alpha$.

Now $\sigma(-I_1\xi) = 2\alpha^2(|z_1|^2e^{-4my_1^b} + |z_2|^2e^{4my_1^b}) = 4R^b$, and by (34), $\rho(-I_1\xi) = 4\alpha^{-1}y_1^b d\alpha(-I_1\xi) = -16a\alpha^{-1}y_1^b(|z_1|^4 - |z_2|^4)/(r^4)^2$; this way

$$dy_1^b(-I_1\xi) = (V^b)^{-1} - 8a\alpha^{-1} \frac{y_1^b}{1 + 4mR^b} \frac{|z_1|^4 - |z_2|^4}{(r^4)^2}, \tag{35}$$

where $V^b = \square^*V = (1 + 4mR^b)/2R^b$. Since the last summand is $O(r^{-4})$ and thus $O(R^{-2})$, and $(V^b)^{-1} - V^{-1} = 2R^b/(1 + 4mR^b) - 2R/(1 + 4mR) = 2(R^b - R)/(1 + 4mR^b)(1 + 4mR) = O(R^{-3})$, we have $dy_1^b(-I_1\xi) = V + O(R^{-2})$.

Moreover $\sigma(\zeta) = (\alpha^2/2iR)(e^{4m(y_1 - y_1^b)}(z_1z_2 - \bar{z}_1\bar{z}_2) - e^{-4m(y_1 - y_1^b)}(z_1z_2 - \bar{z}_1\bar{z}_2)) = \alpha^2(y_2/R) \sinh[4m(y_1 - y_1^b)]$, and $\rho(\zeta) = 4\alpha^{-1}y_1^b d\alpha(\zeta) = -32a\alpha^{-1}(r^2/(r^4)^2)y_1^by_2 \cosh(4my_1)/R$ by (34). Thus

$$dy_1^b(\zeta) = \alpha^2 \frac{y_2}{2R(1 + 4mR^b)} \sinh[4m(y_1 - y_1^b)] - 16a\alpha^{-1} \frac{r^2}{(r^4)^2} \frac{y_1^by_2 \cosh(4my_1)}{R(1 + 4mR^b)}; \tag{36}$$

since $y_1 - y_1^b = O(R^{-1})$, the first summand is $O(R^{-2})$, whereas since $\cosh(4my_1) = O(r^2R^{-1})$, the second summand is $O(R^{-1}r^{-4})$, that is $O(R^{-3})$, and as a result $dy_1^b(\zeta) = O(R^{-2})$. Similarly $dy_1^b(I_1\zeta) = O(R^{-2})$ (just replace y_2 by y_3 in the last equality above).

Estimate on η^b . We conclude our estimate of $|\mathbf{f}^b - \mathbf{f}|_{\mathbf{f}}$ by the estimate on η^b . We start with a formula for η^b ; since on $\{z_1 \neq 0\}$, $d(\square^*\bar{z}_1)/\square^*\bar{z}_1 - d(\square^*z_1)/\square^*z_1 = d(\alpha^2\bar{z}_1)/\alpha^2\bar{z}_1 - d(\alpha^2z_1)/\alpha^2z_1 = d\bar{z}_1/\bar{z}_1 + d(\alpha^2)/\alpha^2 - dz_1/z_1 - d(\alpha^2)/\alpha^2 = d\bar{z}_1/\bar{z}_1 - dz_1/z_1$, and similarly $\square^*(d\bar{z}_2/\bar{z}_2 - dz_2/z_2) = d\bar{z}_2/\bar{z}_2 - dz_2/z_2$ on $\{z_2 \neq 0\}$, we have on $\{z_1z_2 \neq 0\}$, according to the identity $\eta^b = (i/4R)[u^2(d\bar{z}_1/\bar{z}_1 - dz_1/z_1) - v^2(d\bar{z}_2/\bar{z}_2 - dz_2/z_2)]$ (Lemma A.6 in the appendix):

$$\eta^b = \frac{i}{4R^b} \left[(u^b)^2 \left(\frac{d\bar{z}_1}{\bar{z}_1} - \frac{dz_1}{z_1} \right) - (v^b)^2 \left(\frac{d\bar{z}_2}{\bar{z}_2} - \frac{dz_2}{z_2} \right) \right].$$

From this we compute $\eta^b(\xi) = 1$ and $\eta^b(-I_1\xi) = 0$. We also compute $\eta^b(\zeta)$ as follows:

$$\begin{aligned} \eta^b(\zeta) &= \frac{i}{4R^b} \frac{1}{2iR} \left[(u^b)^2 e^{4my_1} \left(\frac{z_2}{\bar{z}_1} - \frac{\bar{z}_2}{z_1} \right) - (v^b)^2 e^{-4my_1} \left(\frac{z_1}{\bar{z}_2} - \frac{\bar{z}_1}{z_2} \right) \right] \\ &= \frac{i}{4R^b} \frac{\alpha^2}{2iR} \left[(u^b)^2 e^{4my_1} \frac{z_1z_2 - \bar{z}_1\bar{z}_2}{\alpha^2|z_1|^2} - (v^b)^2 e^{-4my_1} \frac{z_1z_2 - \bar{z}_1\bar{z}_2}{\alpha^2|z_2|^2} \right] \\ &= \frac{i\alpha^2y_2}{2R^bR} \sinh[4m(y_1 - y_1^b)], \end{aligned}$$

since from the pulled-back LeBrun's equations (32), $(u^b)^2/\alpha^2|z_1|^2 = e^{-4my_1^b}$ and $(v^b)^2/\alpha^2|z_2|^2 = e^{4my_1^b}$. Similarly $\eta^b(I_1\zeta) = (i\alpha^2y_3/2R^bR) \sinh[4m(y_1 - y_1^b)]$, and since $(y_1 - y_1^b) = O(R^{-1})$, both $\eta^b(\zeta)$ and $\eta^b(I_1\zeta)$ are $O(R^{-2})$. Gathering those estimates, we get that

$$|\eta^b - \eta|_{\mathbf{f}} = O(R^{-2}),$$

which is better than needed.

Recall that $\mathbf{f} = V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta$; since $V^{-1} - (V^b)^{-1}$, and similarly $V - V^b$, are $O(R^{-3})$, in view of the estimates we have just proved on the $dy_j - dy_j^b$ and $\eta^b - \eta$, we have

$$|\mathbf{f}^b - \mathbf{f}|_{\mathbf{f}} = O(R^{-1}).$$

Estimate on $\nabla^{\mathbf{f}}(\mathbf{f} - \mathbf{f}^b)$. We now prove that $|\nabla^{\mathbf{f}}(\mathbf{f} - \mathbf{f}^b)|_{\mathbf{f}} = O(R^{-1})$, which is the same as proving that $|\nabla^{\mathbf{f}}\mathbf{f}^b|_{\mathbf{f}} = O(R^{-1})$. In view of the previous estimates on $V - V^b$, $V^{-1} - (V^b)^{-1}$, on the $dy_j - dy_j^b$ and on $\eta - \eta^b$, and since the $\nabla^{\mathbf{f}}dy_j$ and $\nabla^{\mathbf{f}}\eta$ are $O(R^{-2})$ for \mathbf{f} , it will be sufficient for our purpose to see that the $\nabla^{\mathbf{f}}(dy_j - dy_j^b)$ and $\nabla^{\mathbf{f}}(\eta - \eta^b)$ are $O(R^{-1})$ for \mathbf{f} .

We start with $\nabla^{\mathbf{f}}(dy_2 - dy_2^b)$ and $\nabla^{\mathbf{f}}(dy_3 - dy_3^b)$. We have $d(y_2 + iy_3) - d(y_2^b + iy_3^b) = (\alpha^2 - 1)d(y_2 + iy_3) + 2(y_2 + iy_3)\alpha d\alpha$, we know that $\alpha - 1 = O(r^{-4}) = O(R^{-2})$, and we actually proved that $|d\alpha|_{\mathbf{f}} = O(r^{-4}) = O(R^{-2})$. Similarly, we will be done if we prove that $|\nabla^{\mathbf{f}}d\alpha|_{\mathbf{f}}$ is still $O(r^{-4})$.

Since α is \mathbb{S}^1 -invariant, $d\alpha = (\partial\alpha/\partial y_1)dy_1 + (\partial\alpha/\partial y_2)dy_2 + (\partial\alpha/\partial y_3)dy_3$; the $\partial\alpha/\partial y_j$ are \mathbb{S}^1 -invariant as well, and thus $\nabla^{\mathbf{f}}d\alpha = \sum_{j,\ell=1}^3 (\partial^2\alpha/\partial y_j\partial y_\ell)dy_j \otimes dy_\ell + \sum_{j=1}^3 (\partial\alpha/\partial y_j)\nabla^{\mathbf{f}}dy_j$. The last summand is $O(R^{-2}r^{-4})$, since the $\partial\alpha/\partial y_j$ are $O(r^{-4})$ and the $|\nabla^{\mathbf{f}}dy_j|_{\mathbf{f}}$ are $O(R^{-2})$; we thus focus on the hessian $\sum_{j,\ell=1}^3 (\partial^2\alpha/\partial y_j\partial y_\ell)dy_j \otimes dy_\ell$, and all we need to prove is $\partial^2\alpha/\partial y_j\partial y_\ell = O(r^{-4})$ (actually, $O(R^{-2})$) for all j, ℓ . Now in terms of the y_j variables,

$$\alpha = 1 + \frac{a}{((y_1^2 + y_2^2 + y_3^2)^{1/2} \cosh(4my_1) + y_1 \sinh(4my_1))^2},$$

and using that $e^{4m|y_1|} = O(Rr^{-2})$, proving that $\partial^2\alpha/\partial y_j\partial y_\ell = O((R \cosh(4my_1) + y_1 \sinh(4my_1))^{-2}) = O(r^{-4})$ for all j, ℓ amounts to an easy exercise. This settles the cases of $\nabla^{\mathbf{f}}(dy_2 - dy_2^b)$ and $\nabla^{\mathbf{f}}(dy_3 - dy_3^b)$.

Since our treatment of $dy_1 - dy_1^b$ is a little less conventional, we shall see now how goes that of $\nabla^{\mathbf{f}}(dy_1 - dy_1^b)$. According to (35) and (36) and the previous estimates on the derivatives of r^2 , it is enough to see that $dy_1^b = O(1)$ and $dR^b = O(1)$, which are known for the previous step, giving in particular $d \sinh[4m(y_1 - y_1^b)] = \cosh[4m(y_1 - y_1^b)]d(y_1 - y_1^b)$, which is $O(R^{-1})$ (actually $O(R^{-2})$) for \mathbf{f} since $\cosh[4m(y_1 - y_1^b)] \sim 1$ and $|d(y_1 - y_1^b)|_{\mathbf{f}} = O(R^{-2})$.

The treatment of η^b is similar.

We prove finally that $|(\nabla^{\mathbf{f}})^2(\mathbf{f} - \mathbf{f}^b)|_{\mathbf{f}} = O(R^{-1})$ with the same techniques. □

2. Asymptotics of ALE hyperkähler metrics

We prove in this part an explicit version of Theorem 0.3; we indeed compute explicitly the first non-vanishing perturbative terms of the hyperkähler data of the ALE gravitational instantons seen as deformations of Kleinian singularities. This gives in particular the asymptotics stated in the previous part, Lemma 1.6, which are crucial in our construction of ALF metrics, as mentioned already.

2.1 Kronheimer’s ALE instantons

2.1.1 Basic facts and notation. We introduce a few notions about the ALE gravitational instantons constructed by Kronheimer in [Kro89a], and which is exhaustive in the sense that any ALE gravitational instanton is isomorphic to one of Kronheimer’s list, so as to state properly the main result of this part, i.e. Theorem 2.1 of the next paragraph, dealing with precise asymptotics of those asymptotically Euclidean spaces.

Finite subgroups of $SU(2)$, and McKay correspondence. The classification of the finite subgroup of $SU(2)$ is well known: up to conjugation, in addition to the binary dihedral groups \mathcal{D}_k used in Part 1, one has the cyclic groups of order $k \geq 2$, generated by $(\begin{smallmatrix} e^{2i\pi/k} & 0 \\ 0 & e^{-2i\pi/k} \end{smallmatrix})$, on the one hand, and the binary tetrahedral, octahedral and icosahedral groups of respective orders 24, 48 and 120,

which admit more complicated generators; all we need to notice for further purpose is that they respectively contain \mathcal{D}_2 , \mathcal{D}_3 and \mathcal{D}_5 (among others) as subgroups. When no specification is needed, we shall adopt the notation Γ for any fixed group among these finite subgroups of $SU(2)$.

ALE instantons modelled on \mathbb{R}^4/Γ . Kronheimer’s construction now consists in producing asymptotically Euclidean hyperkähler metrics on *smooth deformations* of the Kleinian singularity \mathbb{C}^2/Γ , which are diffeomorphic to the *minimal resolution of \mathbb{C}^2/Γ* . More precisely, the hyperkähler manifolds Kronheimer produces are parametrised as follows: since Γ is a finite subgroup of $SU(2)$, McKay’s correspondence [McK79] associates a simple Lie algebra, \mathfrak{g}_Γ say, to this group; for instance, the Lie algebra associated to \mathcal{D}_k is $\mathfrak{so}(2k + 4)$ (this Lie algebra is also referred to as D_{k+2} ; we prefer the \mathfrak{so} notation which is less confusing when working with binary dihedral groups!). Pick a (real) Cartan subalgebra \mathfrak{h} of \mathfrak{g}_Γ . Then:

For any $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$ outside a codimension 3 set D , there exists an ALE gravitational instanton $(X_\zeta, g_\zeta, I_1^\zeta, I_2^\zeta, I_3^\zeta)$ modelled on \mathbb{R}^4/Γ at infinity in the sense that there exists a diffeomorphism Φ_ζ between infinities of X_ζ and \mathbb{R}^4/Γ such that $\Phi_{\zeta*}g_\zeta - \mathbf{e} = O(r^{-4})$, $\Phi_{\zeta*}I_j^\zeta - I_j = O(r^{-4})$, $j = 1, 2, 3$.

The O are here understood in the asymptotically Euclidean setting, i.e. $\varepsilon = O(r^{-a})$ means that for all $\ell \geq 0$, $|(\nabla^e)^\ell \varepsilon|_{\mathbf{e}} = O(r^{-a-\ell})$; since we remain in this setting until the end of this part, we shall keep this convention throughout the following §§ 2.3 and 2.4.

2.1.2 *Asymptotics of ALE instantons: statement of the theorem.* Up to a judicious choice of the ALE diffeomorphism Φ_ζ , which actually is obtained from Kronheimer’s construction, one can be more accurate about the $O(r^{-4})$ -error term evoked above. This is the purpose of the main result of this part.

THEOREM 2.1. *Given $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, one can choose the diffeomorphism Φ_ζ between infinities of X_ζ and \mathbb{R}^4/Γ such that $\Phi_{\zeta*}g_\zeta - \mathbf{e} = h_\zeta + O(r^{-6})$, $\Phi_{\zeta*}I_1^\zeta - I_1 = \iota_1^\zeta + O(r^{-6})$ and if $\omega_1^\zeta := g_\zeta(I_1^\zeta \cdot, \cdot)$, then $\Phi_{\zeta*}\omega_1^\zeta - \omega_1^e = \varpi_1^\zeta + O(r^{-6})$, where h_ζ , ι_1^ζ and ϖ_1^ζ are given by*

$$\begin{aligned}
 h_\zeta = & -\|\Gamma\| \sum_{(j,k,\ell) \in \mathfrak{I}_3} |\zeta_j|^2 \frac{(rdr)^2 + \alpha_j^2 - \alpha_k^2 - \alpha_\ell^2}{r^6} - 2\|\Gamma\| \langle \zeta_1, \zeta_2 \rangle \frac{\alpha_1 \cdot \alpha_2 - rdr \cdot \alpha_3}{r^6} \\
 & - 2\|\Gamma\| \langle \zeta_1, \zeta_3 \rangle \frac{\alpha_1 \cdot \alpha_3 + rdr \cdot \alpha_2}{r^6} - 2\|\Gamma\| \langle \zeta_2, \zeta_3 \rangle \frac{\alpha_2 \cdot \alpha_3 - rdr \cdot \alpha_1}{r^6}, \tag{37}
 \end{aligned}$$

with $\mathfrak{I}_3 = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$; ι_1^ζ is \mathbf{e} -symmetric and satisfies the coupling

$$\begin{aligned}
 \mathbf{e}(\iota_1^\zeta \cdot, \cdot) = & \|\Gamma\| (|\zeta_3|^2 - |\zeta_2|^2) \frac{\alpha_2 \cdot \alpha_3}{r^6} - \|\Gamma\| (|\zeta_3|^2 + |\zeta_2|^2) \frac{rdr \cdot \alpha_1}{r^6} \\
 & - 2\|\Gamma\| \langle \zeta_2, \zeta_3 \rangle \frac{(rdr)^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2}{r^6}; \tag{38}
 \end{aligned}$$

and

$$\varpi_1^\zeta = -\|\Gamma\| |\zeta_1|^2 \theta_1 - 2\|\Gamma\| \langle \zeta_1, \zeta_2 \rangle \theta_2 - 2\|\Gamma\| \langle \zeta_1, \zeta_3 \rangle \theta_3; \tag{39}$$

here $\|\Gamma\| = \mathbf{c}|\Gamma|$ for a universal constant $\mathbf{c} > 0$.

Moreover, $\Phi_{\zeta*} \text{vol}^{g_\zeta} = \Omega_{\mathbf{e}}$, and if Γ is binary dihedral, tetrahedral, octahedral or icosahedral, the error term can be taken of size $O(r^{-8})$.

Recall the notation $\alpha_j = I_j r dr$, $j = 1, 2, 3$, and $\theta_a = (rdr \wedge \alpha_a - \alpha_b \wedge \alpha_c)/r^6$, $(a, b, c) \in \mathfrak{I}_3$. The scalar product on \mathfrak{h} used in this statement is the one induced by the Killing form.

The rest of this part is devoted to the proof of this result. In the next section we specify the meaning of the space of parameters $\mathfrak{h} - D$; in particular we see how \mathfrak{h} is identified to the degree 2 homology of our Kronheimer’s instantons, which is helpful in computing the constant \mathbf{c} of the statement, as well as the coefficients appearing in (37)–(39). We also fix the choice of the diffeomorphisms Φ_ζ , and check their properties on volume forms (Lemma 2.5). To make the rest of our strategy a bit more explicit, let us mention here:

- (i) the diffeomorphisms we fix are a natural by-product of Kronheimer’s construction;
- (ii) besides their volume properties, their main feature is to put automatically the instanton metrics, seen as deformations of the Euclidean metric, in a special gauge at leading order; with the notation of Theorem 2.1, this can be stated as:

$$\text{tr}^e(h_\zeta) = 0 \quad \text{and} \quad \delta^e h_\zeta = 0$$

(Proposition 2.7), and exploits the volume conservation property;

- (iii) thanks to the gauge (and its proof), we prove that the ϖ_1^ζ (and their analogues ϖ_2^ζ and ϖ_3^ζ) are linear combinations of the $dd_{I_j}^\zeta(1/r^2) = 4\theta_j$, $j = 1, 2, 3$ (Proposition 2.10); from the way Kronheimer’s diffeomorphisms are constructed (compositions of asymptotically isometric biholomorphisms for successive complex structures in a precise order), we explicitly compute the coefficients of the θ_j (Proposition 2.12);
- (iv) we finally convert these results in the explicit writing of h_ζ and ι_1^ζ ; these different steps are the object of § 2.3, and we conclude the proof of Theorem 2.1 in § 2.4 by ruling out the $O(r^{-6})$ -error terms, which pertains to the same circle of ideas.

As a conclusion to this program, let us notice that, in general (that is, using asymptotically isometric diffeomorphism *a priori* different from Kronheimer’s ones), the gauge of point (ii) is likely to be recovered from an analytic Bianchi gauge process, which would allow us to deal with point (ii) and point (iii), first half; however, Kronheimer’s charts volume condition reveals quite useful in point (iii), second half, and these charts readily allow us to pass to point (iv), hence our choice to use them here.

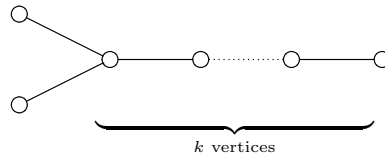
2.2 Precisions on Kronheimer’s construction

2.2.1 The degree 2 homology/cohomology.

The ‘forbidden set’ D. We keep the notation Γ for one of the subgroups of $SU(2)$ mentioned in the previous section. We saw that Kronheimer’s ALE instantons asymptotic to \mathbb{R}^4/Γ are parametrised by a triple $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, with \mathfrak{h} a real Cartan subalgebra of the Lie algebra associated to Γ by McKay correspondence; for instance, if $\Gamma = \mathcal{D}_k$, $k \geq 2$, then one can take \mathfrak{h} as the Cartan subalgebra of $\mathfrak{so}(2k + 4)$ constituted by matrices of shape $\text{diag}(\lambda_1, \dots, \lambda_{k+2}, -\lambda_1, \dots, -\lambda_{k+2})$. We shall first be more specific about the ‘forbidden set’ D ; according to [Kro89a, Corollary 2.10], it is the union of codimension 3 subspaces $D_\theta \otimes \mathbb{R}^3$ over a positive root system of \mathfrak{h} , with D_θ the kernels of the concerned roots; as such, it thus has codimension 3 in \mathfrak{h} .

Topology of X_ζ . Recall the notation $(X_\zeta, g_\zeta, I_1^\zeta, I_2^\zeta, I_3^\zeta)$ for the hyperkähler manifold of admissible parameter ζ (this is actually also defined as a hyperkähler orbifold if $\zeta \in D$). Those spaces are diffeomorphic to the minimal resolution of \mathbb{C}^2/Γ (for I_1 , say) [Kro89a, Corollary 3.12]; as such

they are simply connected and, again when $\Gamma = \mathcal{D}_k$, their rank 2 topology is given by the diagram



(which is nothing but the Dynkin diagram associated to $\mathfrak{so}(2k+4)$), where each vertex represents the class of a sphere of -2 self-intersection, and where two vertices are linked by an edge if and only if the corresponding spheres intersect, in which case they intersect normally at one point.

Furthermore, there is an identification between \mathfrak{h} and $H^2(X_\zeta, \mathbb{R}) \simeq H^2_{\text{cpct}}(X_\zeta, \mathbb{R})$ (see [Joy00, p. 183] for this ‘ \simeq ’) such that:

- the cohomology class of the Kähler form $\omega_j^\zeta := g_\zeta(I_j^\zeta, \cdot)$ is ζ_j , $j = 1, 2, 3$;
- $H_2(X_\zeta, \mathbb{Z})$ is identified with the root lattice of \mathfrak{h} ; more precisely, given simple roots of \mathfrak{h} and the corresponding basis of $H_2(X_\zeta, \mathbb{Z})$, the intersection matrix of this basis is exactly the opposite of the Cartan matrix of the simple roots, see [Kro89a, p. 678]; in the case $\Gamma = \mathcal{D}_k$, $k \geq 2$, this matrix is thus the $(k+2) \times (k+2)$ matrix

$$\begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 2 & -1 & 0 & & \vdots \\ -1 & -1 & 2 & -1 & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

From the latter, we deduce the following lemma, identifying the cup-product \cup on $H^2_{\text{cpct}}(X_\zeta, \mathbb{R})$, or between $H^2_{\text{cpct}}(X_\zeta, \mathbb{R})$ and $H^2(X_\zeta, \mathbb{R})$, and the scalar product on \mathfrak{h} induced by the Killing form, up to signs.

LEMMA 2.2. Consider $\alpha, \beta \in H^2(X_\zeta, \mathbb{R})$, such that α or β has compact support. Then $\alpha \cup \beta = \int_{X_\zeta} \alpha \wedge \beta = -\langle \alpha, \beta \rangle$, where the latter is computed with seeing α and β in \mathfrak{h} via the above identification between $H^2(X_\zeta, \mathbb{R})$ and \mathfrak{h} .

Proof. We do it for $\Gamma = \mathcal{D}_k$, $k \geq 2$. By Poincaré duality, the computation of $\alpha \cup \beta$ amounts to that of intersection numbers for a basis of $H_2(X_\zeta, \mathbb{Z})$. But through the identification between $H_2(X_\zeta, \mathbb{Z})$ and the root lattice of \mathfrak{h} above, the matrix of intersection numbers on the one hand and that of scalar products of the corresponding basis (or dually, of the simple roots) are the same up to signs. \square

Period matrix. For $\zeta \in \mathfrak{h} - D$, consider as above a basis Σ_ℓ , $\ell = 1, \dots, r$ say, of $H_2(X_\zeta, \mathbb{Z})$; from the previous paragraph, the period matrix

$$P(\zeta) = (P_{j\ell}(\zeta))_{\substack{1 \leq j \leq 3 \\ 1 \leq \ell \leq r}} := \left(\int_{\Sigma_\ell} \omega_j^\zeta \right)_{\substack{1 \leq j \leq 3 \\ 1 \leq \ell \leq r}}$$

can be computed thanks to the identities $[\omega_j^\zeta] = \zeta_j$. One easily sees that these $P(\zeta) = P(\xi)$ if and only if $\zeta = \xi$. With this formalism Kronheimer’s classification [Kro89b, Theorem 1.3]

can be stated as: *two ALE gravitational instantons are isomorphic as hyperkähler manifolds if and only if they have the same period matrix.* From this we deduce the following (see also [BR12, p. 8, (4)]).

LEMMA 2.3. *Let $\zeta \in \mathfrak{h} - D$, and let $A \in \text{SO}(3)$ act on ζ and the complex structures I_j^ζ as in § 1.1. Then there exists a tri-holomorphic isometry between $(X_\zeta, g_\zeta, (AI^\zeta)_1, (AI^\zeta)_2, (AI^\zeta)_3)$ and $(X_{A\zeta}, g_{A\zeta}, I_1^{A\zeta}, I_2^{A\zeta}, I_3^{A\zeta})$.*

Proof. Simply check that in both cases, the period matrix is $AP(\zeta)$, and apply Kronheimer’s classification theorem. □

2.2.2 Analytic expansions.

Choice of the chart at infinity. Consider a parameter $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \otimes \mathbb{R}^3$ and set

$$\zeta' = (0, \zeta_2, \zeta_3), \quad \zeta'' = (0, 0, \zeta_3);$$

we will keep this notation below. As described in [Kro89a, p. 677], there exist proper continuous maps

$$\begin{aligned} \lambda_1^\zeta &: (X_\zeta, g_\zeta, I_1^\zeta, I_2^\zeta, I_3^\zeta) \longrightarrow (X_{\zeta'}, g_{\zeta'}, I_1^{\zeta'}, I_2^{\zeta'}, I_3^{\zeta'}), \\ \lambda_2^{\zeta'} &: (X_{\zeta'}, g_{\zeta'}, I_1^{\zeta'}, I_2^{\zeta'}, I_3^{\zeta'}) \longrightarrow (X_{\zeta''}, g_{\zeta''}, I_1^{\zeta''}, I_2^{\zeta''}, I_3^{\zeta''}), \\ \lambda_3^{\zeta''} &: (X_{\zeta''}, g_{\zeta''}, I_1^{\zeta''}, I_2^{\zeta''}, I_3^{\zeta''}) \longrightarrow (\mathbb{R}^4/\Gamma, \mathbf{e}, I_1, I_2, I_3), \end{aligned}$$

which are diffeomorphisms (at least) on $(\lambda_3^{\zeta''} \circ \lambda_2^{\zeta'} \circ \lambda_1^\zeta)^{-1}(\{0\})$, $(\lambda_3^{\zeta''} \circ \lambda_2^{\zeta'})^{-1}(\{0\})$, and $(\lambda_3^{\zeta''})^{-1}(\{0\})$ respectively. As soon as $\zeta'' \notin D$ (respectively $\zeta', \zeta \notin D$), $\lambda_3^{\zeta''}$ (respectively $\lambda_2^{\zeta'}, \lambda_1^\zeta$) is a resolution of singularities for the third (respectively the second, the first) pair of complex structures; in particular, if $\zeta' \notin D$ (respectively if $\zeta \notin D$), then $\lambda_2^{\zeta'}$ (respectively λ_1^ζ) is smooth, and holomorphic for the appropriate pair of complex structures.

To get a ‘coordinate chart’ on X_ζ (or rather, to view objects on \mathbb{R}^4/Γ), one sets

$$F_\zeta := (\lambda_3^{\zeta''} \circ \lambda_2^{\zeta'} \circ \lambda_1^\zeta)^{-1} : (\mathbb{R}^4 \setminus \{0\})/\Gamma \longrightarrow X_\zeta$$

(beware this is not exactly the same order of composition as Kronheimer’s ‘coordinate chart’, but this is not a problem by symmetry).

‘Homogeneity’ and consequences. We shall see that the F_ζ are going to be the Φ_ζ of Theorem 2.1. For now, according to [Kro89a, Proposition 3.14] and its proof, we have for any ζ the converging expansion

$$F_\zeta^* g_\zeta = \mathbf{e} + \sum_{j=2}^{\infty} h_\zeta^{(j)},$$

with $h_\zeta^{(j)}$ a homogeneous polynomial of degree j in ζ with coefficients homogeneous symmetric 2-tensors on \mathbb{R}^4/Γ , more precisely, if κ_s is the dilation $x \mapsto sx$ of \mathbb{R}^4 for any positive s , $\kappa_s^* h_\zeta^{(j)} = s^{-2(j-1)} h_\zeta^{(j)}$. We will thus be concerned with determining explicitly the term $h_\zeta^{(2)}$, and moreover with showing that when Γ contains a binary dihedral group then $h_\zeta^{(3)} = 0$. For now, observe that Kronheimer’s arguments, consisting in analyticity and homogeneity properties of his construction, can also be used to give the existence of analogous expansions of other tensors

such as the complex structures, and therefore the Kähler forms, or the volume forms as well. We can write for instance

$$F_\zeta^* I_1^\zeta = I_1 + \sum_{j=1}^\infty \iota_{1,j}^\zeta, \tag{40}$$

where $\iota_{1,j}^\zeta$ is a homogeneous polynomial of degree j in ζ with coefficients $(1, 1)$ -tensors, satisfying $\kappa_s^* \iota_{1,j}^\zeta = s^{-2j} \iota_{1,j}^\zeta$ (and again, the lower-order term $\iota_{1,1}^\zeta$ vanishes, but we will find this fact again below).

2.2.3 Minimal resolutions, invariance of the holomorphic symplectic structure. We know that as soon as $\zeta \notin D$, $\lambda_1^\zeta : (X_\zeta, I_1^\zeta) \rightarrow (X_{\zeta'}, I_1^{\zeta'})$ is a minimal resolution, and a similar statement holds for $\lambda_2^{\zeta'} : (X_{\zeta'}, I_2^{\zeta'}) \rightarrow (X_{\zeta''}, I_2^{\zeta''})$ and $\lambda_3^{\zeta''} : (X_{\zeta''}, I_3^{\zeta''}) \rightarrow (\mathbb{R}^4/\Gamma, I_3)$ whenever $\zeta' \notin D$ or $\zeta'' \notin D$, respectively [Kro89a, p. 675].

As seen already, those maps can happen to be smooth, for instance λ_1^ζ is, when $\zeta, \zeta' \notin D$; we are then only left with their holomorphicity property. This can be used nevertheless with their asymptotic preserving of the hyperkähler structure, to see that they do preserve the appropriate holomorphic symplectic structure.

LEMMA 2.4. *Fix $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$, and assume that $\zeta'' \notin D$. Then the map $\lambda_3^{\zeta''}$ verifies $(\lambda_3^{\zeta''})^*(\omega_1^e + i\omega_2^e) = \omega_1^{\zeta''} + i\omega_2^{\zeta''}$.*

Similarly, if $\zeta', \zeta'' \notin D$, then $(\lambda_2^{\zeta'})^(\omega_3^e + i\omega_1^e) = \omega_3^{\zeta'} + i\omega_1^{\zeta'}$; if $\zeta, \zeta' \notin D$, then $(\lambda_1^\zeta)^*(\omega_2^e + i\omega_3^e) = \omega_2^\zeta + i\omega_3^\zeta$.*

Proof. The assertion on $\lambda_3^{\zeta''}$ is actually classical, and can be settled in the following elementary way. Call θ the 2-form $(\lambda_3^{\zeta''})^*(\omega_1^{\zeta''} + i\omega_2^{\zeta''})$, well defined on $(\mathbb{R}^4 \setminus \{0\})/\Gamma$, pulled-back to $\mathbb{R}^4 \setminus \{0\}$. Since $\lambda_3^{\zeta''}$ is holomorphic for the pair $(I_3^{\zeta''}, I_3)$ and $\omega_1^{\zeta''} + i\omega_2^{\zeta''}$ is a holomorphic $(2,0)$ -form for $I_3^{\zeta''}$, θ is a holomorphic $(2,0)$ -form for I_3 , and can thus be written as $f(\omega_1^e + i\omega_2^e)$, where f is thus holomorphic for I_3 on $\mathbb{R}^4 \setminus \{0\}$. By Hartogs' lemma it can be extended to the whole \mathbb{R}^4 ; however, since $(\lambda_3^{\zeta''})^* \omega_j^{\zeta''} = F_{\zeta''}^* \omega_j^{\zeta''} \sim \omega_j^e$ near infinity on \mathbb{R}^4/Γ , $j = 1, 2$, which can be seen as a consequence of the power series expansions analogous to (40) for Kähler forms, we get that f tends to 1 at infinity. It is therefore constant, equal to 1, which exactly means that $(\lambda_3^{\zeta''})^*(\omega_1^e + i\omega_2^e) = \omega_1^{\zeta''} + i\omega_2^{\zeta''}$.

We deal with the assertion on $\lambda_2^{\zeta'}$ in a somehow similar way. Since $\zeta', \zeta'' \notin D$, $\lambda_2^{\zeta'}$ is a global diffeomorphism between the smooth $X_{\zeta'}$ and $X_{\zeta''}$, holomorphic for the pair $(I_2^{\zeta'}, I_2^{\zeta''})$; since $\omega_3^{\zeta'} + i\omega_1^{\zeta'}$ trivialises $K_{(X_{\zeta'}, I_2^{\zeta'})}$ and is a $(2,0)$ -holomorphic form for $I_2^{\zeta'}$, $(\lambda_2^{\zeta'})^*(\omega_3^{\zeta''} + i\omega_1^{\zeta''})$ can be written as $f(\omega_3^{\zeta'} + i\omega_1^{\zeta'})$ with f a holomorphic function on $(X_{\zeta'}, I_2^{\zeta'})$. Again f tends to 1 near the infinity of $X_{\zeta'}$, since there $(\lambda_2^{\zeta'})^* \omega_j^{\zeta''} \sim \omega_j^{\zeta'}$, $j = 1, 3$. Moreover $\omega_3^{\zeta''} + i\omega_1^{\zeta''}$ never vanishes on $X_{\zeta''}$, and so neither does f on $X_{\zeta'}$. We collect those observations by saying that $\log(|f|^2)$ is a $g_{\zeta'}$ -harmonic function on $X_{\zeta'}$ tending to zero at infinity, and thus identically vanishing. Since f is holomorphic, it is not hard seeing that it is therefore constant, thus $f \equiv 1$, or in other words $(\lambda_2^{\zeta'})^*(\omega_3^{\zeta''} + i\omega_1^{\zeta''}) = \omega_3^{\zeta'} + i\omega_1^{\zeta'}$.

The assertion on λ_1^ζ is done in the exact same way. □

An easy but fundamental consequence of the construction of F_ζ via the λ_j^ζ and the previous lemma is the invariance of the volume form, which we state for ζ corresponding to smooth X_ζ so as to avoid useless technicalities.

LEMMA 2.5. *The volume form $F_\zeta^* \text{vol}^{g_\zeta}$ does not depend on $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, and is equal to the standard Ω_e .*

Proof. Notice first that once we know that $F_\zeta^* \text{vol}^{g_\zeta}$ does not depend on ζ , the equality $F_\zeta^* \text{vol}^{g_\zeta} = \Omega_e$ is a direct consequence of the expansion of $F_\zeta^* \text{vol}^{g_\zeta}$ as a power series of ζ , the constant term of which is Ω_e . To prove that $F_\zeta^* \text{vol}^{g_\zeta}$ is independent of ζ , we proceed within three steps, considering first ζ'' , and then ζ' and ζ . Even if $\zeta \notin D$, ζ' or ζ'' might lie in D ; we can however assume this is not the case without loss of generality, since $F_\zeta^* \text{vol}^{g_\zeta}$ can be written as a power series of ζ . Now from the hyperkähler data $(X_{\zeta''}, g_{\zeta''}, I_1^{\zeta''}, I_2^{\zeta''}, I_3^{\zeta''})$, we know that $\text{vol}^{g_{\zeta''}} = \frac{1}{2}(\omega_1^{\zeta''})^2$. Since $F_{\zeta''}^*(\omega_1^{\zeta''}) = \omega_1^e$ (the standard Kähler form on \mathbb{C}^2), we get that $F_{\zeta''}^* \text{vol}^{g_{\zeta''}} = \Omega_e$.

Now consider $X_{\zeta'}$; we know that $\omega_3^{\zeta'}$ is ‘preserved’ by $\lambda_2^{\zeta'}$, and therefore

$$F_{\zeta'}^* \text{vol}^{g_{\zeta'}} = \frac{1}{2}F_{\zeta'}^*(\omega_3^{\zeta'})^2 = \frac{1}{2}F_{\zeta''}^*(\lambda_2^{\zeta'})_*(\omega_3^{\zeta'})^2 = \frac{1}{2}F_{\zeta''}^*(\omega_3^{\zeta''})^2 = F_{\zeta''}^* \text{vol}^{g_{\zeta''}},$$

the last equality coming from the fact that $\omega_3^{\zeta''}$ is one of the Kähler forms of the hyperkähler structure $(g_{\zeta''}, I_1^{\zeta''}, I_2^{\zeta''}, I_3^{\zeta''})$.

To conclude, we notice that ω_3^ζ is preserved by λ_1^ζ i.e. $\omega_3^\zeta = (\lambda_1^\zeta)^*\omega_3^{\zeta'}$, and thus

$$F_\zeta^* \text{vol}^{g_\zeta} = \frac{1}{2}F_\zeta^*(\omega_3^\zeta)^2 = \frac{1}{2}F_{\zeta'}^*(\lambda_1^\zeta)_*(\omega_3^\zeta)^2 = \frac{1}{2}F_{\zeta'}^*(\omega_3^{\zeta'})^2 = F_{\zeta'}^* \text{vol}^{g_{\zeta'}};$$

here we could also have used the forms ω_2^ζ and $\omega_2^{\zeta'}$. To make a long story short, the reason for the volume form invariance is that at each step of the composition of the λ_i^ζ , at least one Kähler form is preserved. □

2.3 Explicit determination of h_ζ

2.3.1 *Verifying a gauge.* We shall now work more precisely on the first possibly non-vanishing term of the expansion of $F_{t\zeta}^* g_{t\zeta}$, $t \in \mathbb{R}$, ζ fixed; this allows us to *redefine* $h_\zeta^{(2)}$ as follows.

DEFINITION 2.6. Fix $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$, and set on $\mathbb{R}^4 \setminus \{0\}$ that

$$h_\zeta := \frac{1}{2} \frac{d^2}{dt^2} \Big|_{t=0} F_{t\zeta}^* g_{t\zeta}, \tag{41}$$

which is then $O(r^{-4})$, with ∇^e - ℓ th derivatives $O(r^{-4-\ell})$, near both 0 and infinity, and verifies

$$F_\zeta^* g_\zeta = e + h_\zeta + \varepsilon_\zeta,$$

with $(\nabla^e)^\ell \varepsilon_\zeta = O(r^{-6-\ell})$. More precisely, h_ζ is a homogeneous polynomial of degree 2 in ζ , with coefficients symmetric 2-tensors homogeneous of degree 2 in the sense that $\kappa_s^* h_\zeta = s^{-2} h_\zeta$, where κ_s is the dilation $x \mapsto sx$ of $\mathbb{R}^4 \setminus \{0\}$ for any $s > 0$; as for ε_ζ , it is a sum of terms of degree at least 3 in ζ .

As indicated by the title of this section, given an admissible ζ , we want to analyse h_ζ , which is the first (*a priori*, possibly) non-vanishing term in the expansion of g_ζ (from now on, for the sake of simplicity, we forget about the F_ζ , we will be more accurate about this abuse of notation whenever needed). There already exists a rather powerful theory of deformations

of Kähler–Einstein metrics; see in particular [Bes87, ch. 12] for an overview on that subject. Nonetheless, because of the diffeomorphisms action in general, much of the theory is configured so as to work once a gauge is fixed, precisely killing the ambiguity coming from the diffeomorphisms.

The following proposition asserts that the h_ζ are indeed in some gauge, making us able for further considerations, just as is done in § 1.4.3. Let us specify though that in determining explicitly h_ζ , we will be more concerned with other specific properties of that tensor, namely with its inductive decomposition into hermitian and skew-hermitian parts with respect to I_1 , I_2 and I_3 . As we shall see though, the gauge and the decomposition are rather intricate with one another; seeing the verification of the gauge as a guiding thread, we state the following.

PROPOSITION 2.7. *Fix $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$. Then the lower-order term h_ζ of the deformation g_ζ of \mathbf{e} on $\mathbb{R}^4 \setminus \{0\}$ is in Bianchi gauge with respect to \mathbf{e} , and more precisely*

$$\text{tr}^{\mathbf{e}}(h_\zeta) = 0 \quad \text{and} \quad \delta^{\mathbf{e}}h_\zeta = 0.$$

Moreover, the I_1 -skew-hermitian part of h_ζ is $h_{\zeta'}$, the I_2 -skew-hermitian part of h_ζ is $h_{\zeta''}$, and $h_{\zeta''}$ is I_3 -hermitian, while the I_1 -hermitian part of h_ζ , the I_2 -hermitian part of $h_{\zeta'}$ and $h_{\zeta''}$ give rise to closed forms, that is

$$d(h_\zeta(I_1 \cdot, \cdot) - h_\zeta(\cdot, I_1 \cdot)) = d(h_{\zeta'}(I_2 \cdot, \cdot) - h_{\zeta'}(\cdot, I_2 \cdot)) = d(h_{\zeta''}(I_3 \cdot, \cdot)) = 0 \quad \text{on } \mathbb{R}^4 \setminus \{0\}.$$

Remark 2.8. We took the liberty of possibly having ζ in D since these statements are made on $\mathbb{R}^4 \setminus \{0\}$. More precisely, even if X_ζ is not smooth, its orbifold singularities lie above $0 \in \mathbb{R}^4$ via F_ζ , and h_ζ is smooth on the regular part of X_ζ , i.e. $(F_\zeta)^*h_\zeta$ is smooth on $\mathbb{R}^4 \setminus \{0\}$.

Proof. Let us deal first with the assertion on $\text{tr}^{\mathbf{e}}(h_\zeta)$. At any point of $(\mathbb{R}^4 \setminus \{0\})/\Gamma$, for any t ,

$$\text{vol}^{g_{t\zeta}} = \det^{\mathbf{e}}(g_{t\zeta})\Omega_{\mathbf{e}} = \det^{\mathbf{e}}(\mathbf{e} + t^2h_\zeta + O(t^3))\Omega_{\mathbf{e}} = (1 + t^2 \text{tr}^{\mathbf{e}}(h_\zeta) + O(t^3))\Omega_{\mathbf{e}}.$$

But we saw in Lemma 2.5 that for all t , $\text{vol}^{g_{t\zeta}} = \Omega_{\mathbf{e}}$; consequently, $\text{tr}^{\mathbf{e}}(h_\zeta) = 0$.

We now deal with the divergence assertion. As for the previous lemma, we proceed inductively on the shape of ζ ; the hermitian/skew-hermitian decomposition as well as the closedness property will come out along the different steps of the induction. For this we assume that $\zeta' = (0, \zeta_2, \zeta_3)$ and $\zeta'' = (0, 0, \zeta_3)$ are as well out of the ‘forbidden set’ D . Again, since h_ζ can be written as a sum of quadratic polynomials of ζ times symmetric 2-forms independent of ζ , this assumption does not actually lead to a loss of generality.

Step 1: $\delta^{\mathbf{e}}h_{\zeta''} = 0$. We hence start with $\zeta'' = (0, 0, \zeta_3)$. Since I_3 is parallel for \mathbf{e} , we have that $d^{*\mathbf{e}}[h_{\zeta''}(\cdot, I_3 \cdot)] = (\delta^{\mathbf{e}}h_{\zeta''})(I_3 \cdot)$; indeed, given any local \mathbf{e} -orthonormal frame $(e_j)_{j=1, \dots, 4}$,

$$d^{*\mathbf{e}}[h_{\zeta''}(\cdot, I_3 \cdot)] = - \sum_{j=1}^4 e_j \lrcorner [\nabla_{e_j}^{\mathbf{e}}(h_{\zeta''}(\cdot, I_3 \cdot))] \quad \text{and} \quad \delta^{\mathbf{e}}h_{\zeta''} = - \sum_{j=1}^4 (\nabla_{e_j}^{\mathbf{e}}h_{\zeta''})(e_j, \cdot), \quad (42)$$

see for instance [Biq, 1.2.11] for the first equality, and [Biq, 1.2.13] for the second one. Moreover $h_{\zeta''}$ is clearly I_3 -hermitian, since the $g_{t\zeta''}$ are, which is straightforward from the holomorphicity of the $\lambda_3^{t\zeta''}$ for the pairs $(I_3^{t\zeta''}, I_3)$; $h_{\zeta''}(\cdot, I_3 \cdot)$ is therefore a $(1, 1)$ -form for I_3 . It is furthermore closed, since the $g_{t\zeta''}(\cdot, I_3 \cdot)$ are. We can now use the Kähler identity ‘ $d^* = [\Lambda, d^c]$ ’ with the structure (\mathbf{e}, I_3) and write

$$d^{*\mathbf{e}}(h_{\zeta''}(\cdot, I_3 \cdot)) = [\Lambda_{\omega_3^{\mathbf{e}}}, d_{I_3}^c](h_{\zeta''}(\cdot, I_3 \cdot)).$$

But $\Lambda_{\omega_3^e}(h_{\zeta''}(\cdot, I_3 \cdot)) = -\frac{1}{2} \text{tr}^e(h_{\zeta''}) = 0$, and since $h_{\zeta''}(\cdot, I_3 \cdot)$ is I_3 -hermitian and closed, $d_{I_3}^c(h_{\zeta''}(\cdot, I_3 \cdot)) = d(h_{\zeta''}(\cdot, I_3 \cdot)) = 0$, hence the result.

Step 2: $\delta^e h_{\zeta'} = 0$. We go on our induction and analyse $h_{\zeta'}$, where we recall the notation $\zeta' = (0, \zeta_2, \zeta_3)$. We proceed through the following lines:

- (i) we come back momentarily to $h_{\zeta''}$ and prove it is I_2 -skew-hermitian;
- (ii) we prove that the I_2 -skew-hermitian part of $h_{\zeta'}$ is $h_{\zeta''}$, which is known to be divergence-free for \mathbf{e} ;
- (iii) we conclude by proving that the I_2 -hermitian part of $h_{\zeta'}$ is \mathbf{e} -divergence-free as well.

We tackle point (i). Recall that the map $\lambda_2^{\zeta'} : X_{\zeta'} \rightarrow X_{\zeta''}$ is holomorphic for the pair $(I_2^{\zeta'}, I_2^{\zeta''})$; since we forget about $F_{\zeta'}$ and $F_{\zeta''}$, this amounts to writing $I_2^{\zeta'} = I_2^{\zeta''}$. Recall that in the same way as for the metric, the complex structures admit an analytic expansion, which can be written as a power series of ζ with coefficients homogeneous $(1, 1)$ -tensors on $(\mathbb{R}^4 \setminus \{0\})/\Gamma$. We assume momentarily that the first-order variation vanishes, and we thus write $I_2^{\zeta''} = I_2 + \iota_2^{\zeta''} + \epsilon_2^{\zeta''}$, where $\iota_2^{\zeta''} = \frac{1}{2}(d^2/dt^2)|_{t=0} I_2^{t\zeta''}$, is $O(r^{-4})$ (with according decay on derivatives), and $(\nabla^e)^\ell \epsilon_2^{\zeta''} = O(r^{-6-\ell})$ for all $\ell \geq 0$.

Now $\iota_2^{\zeta''}$ splits into an \mathbf{e} -symmetric part and an \mathbf{e} -anti-symmetric part. But according to [Bes87, 12.96], to the anti-symmetric part, $(\iota_2^{\zeta''})^a$ say, corresponds an I_2 -holomorphic $(2, 0)$ -form θ via the coupling $\mathbf{e}(\cdot, (\iota_2^{\zeta''})^a \cdot) = \theta$; this we get by considering the second-order variation of the Kähler–Einstein deformation $(g_{t\zeta''}, I_2^{t\zeta''})$, satisfying the gauge $\text{tr}^e(h_{\zeta''}) = \delta^e h_{\zeta''} = 0$, and observing that all the statements are local. We can lift θ on $\mathbb{R}^4 \setminus \{0\}$, and then write $\theta = fdw_1 \wedge dw_2$, where w_1 and w_2 are the standard I_2 -holomorphic coordinates $x_1 + ix_3$ and $x_4 + ix_2$, and f is thus I_2 -holomorphic with decay r^{-4} at infinity. By Hartogs’ lemma we can extend f through 0; we thus have an entire function on (\mathbb{R}^4, I_2) , decaying at infinity: the only possibility is $f \equiv 0$, and therefore $(\iota_2^{\zeta''})^a = 0$, or $\iota_2^{\zeta''}$ is \mathbf{e} -symmetric.

Here we would like to follow [Bes87, 12.96] again, to see for example that $\iota_2^{\zeta''}$ then corresponds to the I_2 -skew-hermitian part of $h_{\zeta''}$, via the coupling $\omega_2^e(\cdot, \iota_2^{\zeta''} \cdot)$, this latter $(2, 0)$ -tensor being clearly I_2 -skew-hermitian, because ω_2^e is I_2 -hermitian, and since for all t , $-1 = (I_2^{t\zeta''})^2 = I_2^2 + t^2(I_2 \iota_2^{\zeta''} + \iota_2^{\zeta''} I_2) + O(t^3)$, thus $I_2 \iota_2^{\zeta''} = -\iota_2^{\zeta''} I_2$. Since in our situation, ω_2^e does not vary, we could also expect from [Bes87, 12.95] that the I_2 -hermitian part of $h_{\zeta''}$ vanishes. Nonetheless some of the quoted arguments are of global nature, and one should check they can be adapted to our framework. This can be bypassed however by a rather simple computation, which we quote here: for any t ,

$$g_{t\zeta''} = \begin{cases} \omega_2^{t\zeta''}(\cdot, I_2^{t\zeta''} \cdot) = \omega_2^e(\cdot, I_2 \cdot) + t^2 \omega_2^e(\cdot, \iota_2^{\zeta''} \cdot) + O(t^3) & \text{since } \omega_2^{t\zeta''} = \omega_2^e \\ \mathbf{e} + t^2 h_{\zeta''} + O(t^3), \end{cases}$$

and thus $h_{\zeta''} = \omega_2(\cdot, \iota_2^{\zeta''} \cdot)$ which is I_2 -skew-hermitian, as announced.

We now claim that the I_2 -skew-hermitian part of $h_{\zeta'}$ is nothing but $h_{\zeta''}$, which is point (ii) of the current step. Indeed, since for all t , $I_2^{t\zeta'} = I_2^{t\zeta''}$ (consider $\lambda_2^{\zeta'}$),

$$\begin{aligned} 0 &= g_{t\zeta'}(I_2^{t\zeta'} \cdot, I_2^{t\zeta'} \cdot) - g_{t\zeta'} = g_{t\zeta'}(I_2^{t\zeta''} \cdot, I_2^{t\zeta''} \cdot) - g_{t\zeta'} \\ &= \mathbf{e}(I_2^{t\zeta''} \cdot, I_2^{t\zeta''} \cdot) + t^2 h_{\zeta'}(I_2^{t\zeta''} \cdot, I_2^{t\zeta''} \cdot) - \mathbf{e} - t^2 h_{\zeta'} + O(t^3) \\ &= \underbrace{\mathbf{e}(I_2 \cdot, I_2 \cdot)}_{=\mathbf{e}} + t^2 \mathbf{e}(I_2 \cdot, \iota_2^{\zeta''} \cdot) + t^2 \mathbf{e}(\iota_2^{\zeta''} \cdot, I_2 \cdot) + t^2 h_{\zeta'}(I_2 \cdot, I_2 \cdot) - \mathbf{e} - t^2 h_{\zeta'} + O(t^3), \end{aligned}$$

and thus $h_{\zeta'} - h_{\zeta'}(I_2 \cdot, I_2 \cdot) = \mathbf{e}(I_2 \cdot, \iota_2^{\zeta''} \cdot) + \mathbf{e}(\iota_2^{\zeta''} \cdot, I_2 \cdot)$. We know that $\mathbf{e}(I_2 \cdot, \iota_2^{\zeta'} \cdot) = \omega_2(\cdot, \iota_2^{\zeta''} \cdot) = h_{\zeta''}$. To conclude, use that \mathbf{e} and $\mathbf{e}(\cdot, \iota_2^{\zeta''} \cdot)$ are both symmetric, that $I_2 \iota_2^{\zeta''} = -\iota_2^{\zeta''} I_2$, and that \mathbf{e} is I_2 -hermitian to see that for all X, Y ,

$$\mathbf{e}(\iota_2^{\zeta''} X, I_2 Y) = \mathbf{e}(I_2 Y, \iota_2^{\zeta''} X) = -\mathbf{e}(Y, I_2 \iota_2^{\zeta''} X) = \mathbf{e}(Y, \iota_2^{\zeta''} I_2 X) = \mathbf{e}(I_2 X, \iota_2^{\zeta''} Y),$$

i.e. $\mathbf{e}(\iota_2^{\zeta''} \cdot, I_2 \cdot) = \mathbf{e}(I_2 \cdot, \iota_2^{\zeta''} \cdot) = h_{\zeta''}$. We have proved that

$$\frac{1}{2}(h_{\zeta'} - h_{\zeta'}(I_2 \cdot, I_2 \cdot)) = h_{\zeta''},$$

as claimed. Since $\delta^e h_{\zeta''} = 0$, to see that $\delta^e h_{\zeta'} = 0$, we are only left with checking this identity on the I_2 -hermitian part of $h_{\zeta'}$, which is point (iii) of the current induction step.

For this, let us call φ this tensor twisted by I_2 , namely $\varphi = \frac{1}{2}(h_{\zeta'}(I_2 \cdot, \cdot) - h_{\zeta'}(\cdot, I_2 \cdot))$. As above, we want to see that $d^{*e}\varphi = 0$. This is clearly an I_2 -hermitian 2-form, that is an I_2 -(1, 1)-form. It is moreover trace-free with respect to \mathbf{e} , since $h_{\zeta'}$ is. If we check it is closed then we are done, using the Kähler identity $d^{*e} = [\Lambda_{\omega_2}, d_{I_2}^c]$. For this, we use an expansion of ω_2^t : for all t ,

$$\begin{aligned} \omega_2^{t\zeta'} &= \frac{1}{2}(g_{t\zeta'}(I_2^{t\zeta'} \cdot, \cdot) - g_{t\zeta'}(\cdot, I_2^{t\zeta'} \cdot)) = \frac{1}{2}(g_{t\zeta'}(I_2^{t\zeta''} \cdot, \cdot) - g_{t\zeta'}(\cdot, I_2^{t\zeta''} \cdot)) \\ &= \frac{1}{2}(\mathbf{e}(I_2 \cdot, \cdot) + t^2 \mathbf{e}(\iota_2^{\zeta''} \cdot, \cdot) + t^2 h_{\zeta'}(I_2 \cdot, \cdot) \\ &\quad - \mathbf{e}(\cdot, I_2 \cdot) - t^2 \mathbf{e}(\cdot, \iota_2^{\zeta''} \cdot) - t^2 h_{\zeta'}(\cdot, I_2 \cdot)) + O(t^3) \\ &= \omega_2^e + t^2 \varphi + O(t^3), \quad \text{since } \mathbf{e}(I_2 \cdot, \cdot) = -\mathbf{e}(\cdot, I_2 \cdot) = \omega_2^e \text{ and } \mathbf{e}(\cdot, \iota_2^{\zeta''} \cdot) = \mathbf{e}(\iota_2^{\zeta''} \cdot, \cdot); \end{aligned}$$

this expansion can be differentiated term by term, so that $t^2 d\varphi + O(t^3) = 0$, hence $d\varphi = 0$, as wanted.

Step 3: $\delta^e h_{\zeta} = 0$. We now analyse h_{ζ} . All the techniques to pass from $h_{\zeta''}$ to $h_{\zeta'}$ can actually be used again, and bring us to the desired conclusion:

- (i) we first observe that $I_1^{\zeta} = I_1^{\zeta'}$, and we define $\iota_1^{\zeta'} = (d^2/dt^2)|_{t=0} I_1^{t\zeta'}$ which we assume again to be the possibly lower-order non-vanishing variation of $I_1^{t\zeta'}$; then $(\iota_1^{\zeta'})^a = 0$, since otherwise we would have a non-trivial entire function on \mathbb{C}^2 going to 0 at infinity;
- (ii) since $\omega_1^{\zeta'} = \omega_1^{\zeta''} = \omega_1^e$, we get that $h_{\zeta'}$ is I_1 -skew-hermitian, given by $\iota_1^{\zeta'}$ via the identity $h_{\zeta'} = \omega_1^e(\cdot, \iota_1^{\zeta'} \cdot)$, and that the I_1 -skew-hermitian component of h_{ζ} coincides with $h_{\zeta'}$, the δ^e of which vanishes; we are thus left with the I_1 -hermitian component of h_{ζ} ;
- (iii) this component is \mathbf{e} -trace-free (h_{ζ} is), and gives rise to an I_1 -hermitian 2-form ψ , which is closed since the $\omega_1^{t\zeta}$ are; the Kähler identity $[\Lambda_{\omega_1^e}, d_{I_1}^c] = d^{*e}$ then leads us to $d^{*e}\psi = 0$, which is equivalent to

$$\delta^e(I_1\text{-hermitian component of } h_{\zeta}) = 0.$$

To finish this proof, we justify our assumption of the vanishing of the first-order variation of the complex structures. For instance, let us not assume that $\iota := (d/dt)|_{t=0} I_2^{t\zeta''}$ is *a priori* vanishing. Then it is defined on $\mathbb{R}^4 \setminus \{0\}$, and is $O(r^{-2})$. Now as above, since $0 = (d/dt)|_{t=0} g_{t\zeta''}$ has vanishing trace and divergence for \mathbf{e} , the \mathbf{e} -anti-symmetric part of ι has to vanish since it gives rise to a holomorphic function on (\mathbb{R}^4, I_2) decaying at infinity. And we see as above that $\omega_2^e(\cdot, \iota \cdot) = (d/dt)|_{t=0} g_{t\zeta''} = 0$, and thus $\iota = 0$. Similarly, the arguments for $\iota_1^{\zeta'}$ apply to $(d/dt)|_{t=0} I_1^{t\zeta'}$ with $(d/dt)|_{t=0} g_{t\zeta'} = 0$ instead of $h_{\zeta'}$, so that $(d/dt)|_{t=0} I_1^{t\zeta'} = 0$. \square

Remark 2.9. By contrast with what is usually done, we used properties already known of $h_{\zeta'}$ and $h_{\zeta''}$, conjugated to properties of mappings between X_ζ , $X_{\zeta'}$ and $X_{\zeta''}$ to show that indeed, our first-order deformations were in gauge, which is also retroactively used in some places, e.g. in killing tensors like $(\iota_2'')^a$.

2.3.2 Lower-order variation of the Kähler forms: general shape. As seen when proving that the gauge was verified, given $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$, $h_{\zeta''}$ is I_3 -hermitian, the I_2 -skew-hermitian part of $h_{\zeta'}$ is $h_{\zeta''}$, and the I_1 -skew-hermitian part of $h_{\zeta'}$ is $h_{\zeta''}$. In order to determine h_ζ completely, we are thus left with working on the respective I_3 , I_2 and I_1 -hermitian components of $h_{\zeta''}$, $h_{\zeta'}$ and h_ζ , or equivalently on the respectively I_3 , I_2 and I_1 - $(1, 1)$ forms

$$\varpi_3'' := h_{\zeta''}(I_3 \cdot, \cdot), \quad \varpi_2^{\zeta'} := \frac{1}{2}(h_{\zeta'}(I_2 \cdot, \cdot) - h_{\zeta'}(\cdot, I_2 \cdot)), \quad \varpi_1^\zeta := \frac{1}{2}(h_\zeta(I_1 \cdot, \cdot) - h_\zeta(\cdot, I_1 \cdot)).$$

We interpret these forms as the first (possibly) non-vanishing variation term of ω_3'' , $\omega_2^{\zeta'}$ and ω_1^ζ ; as such and as seen above, these are closed forms. More precisely, they follow a general common pattern.

PROPOSITION 2.10. *There exist real numbers $a_{1j}(\zeta)$, $a_{2j}(\zeta')$, $a_{3j}(\zeta'')$, $j = 1, 2, 3$, such that*

$$\varpi_3'' = a_{31}(\zeta'')\theta_1 + a_{32}(\zeta'')\theta_2 + a_{33}(\zeta'')\theta_3, \quad \varpi_2^{\zeta'} = a_{21}(\zeta')\theta_1 + a_{22}(\zeta')\theta_2 + a_{23}(\zeta')\theta_3,$$

and

$$\varpi_1^\zeta = a_{11}(\zeta)\theta_1 + a_{12}(\zeta)\theta_2 + a_{13}(\zeta)\theta_3,$$

where we recall the notation

$$\theta_1 = \frac{rdr \wedge \alpha_1 - \alpha_2 \wedge \alpha_3}{r^6}, \quad \theta_2 = \frac{rdr \wedge \alpha_2 - \alpha_3 \wedge \alpha_1}{r^6}, \quad \theta_3 = \frac{rdr \wedge \alpha_3 - \alpha_1 \wedge \alpha_2}{r^6}.$$

Proof. We do it for ϖ_1^ζ , as it will be clear that the arguments would apply similarly to $\varpi_2^{\zeta'}$ and ϖ_3'' ; we work on $\mathbb{R}^4 \setminus \{0\}$. As ϖ_1^ζ is of type $(1, 1)$ for I_1 , it is at any point a linear combination of $rdr \wedge \alpha_1$, $\alpha_2 \wedge \alpha_3$, $rdr \wedge \alpha_2 - \alpha_3 \wedge \alpha_1$ and $rdr \wedge \alpha_3 - \alpha_1 \wedge \alpha_2$.

The symmetric tensor $\frac{1}{2}(h_\zeta + h_\zeta(I_1 \cdot, I_1 \cdot))$ corresponding to ϖ_1^ζ is moreover trace-free for \mathbf{e} , which translates into $\varpi_1^\zeta \wedge \omega_1^{\mathbf{e}} = 0$. Since $\omega_1 = (rdr \wedge \alpha_1 + \alpha_2 \wedge \alpha_3)/r^2$, we have $(rdr \wedge \alpha_2 - \alpha_3 \wedge \alpha_1) \wedge \omega_1^{\mathbf{e}} = (rdr \wedge \alpha_3 - \alpha_1 \wedge \alpha_2) \wedge \omega_1^{\mathbf{e}} = 0$, whereas $rdr \wedge \alpha_1 \wedge \omega_1^{\mathbf{e}} = \alpha_2 \wedge \alpha_3 \wedge \omega_1^{\mathbf{e}}$. As a consequence, the pointwise coefficient of $rdr \wedge \alpha_1$ is the opposite of that of $\alpha_2 \wedge \alpha_3$. To sum up, since the θ_j are $O(r^{-4})$ with corresponding decay (or growth, near 0) of their derivatives, which are precisely the orders of ϖ_1^ζ , we know that

$$\varpi_1^\zeta = f\theta_1 + g\theta_2 + h\theta_3,$$

for three bounded functions f, g, h , with Euclidean ℓ -th order derivatives of order $O(r^{-\ell})$, near 0 and infinity. We can be more precise here: from the properties of analytic expansions in play discussed in § 2.2.2, we have that $\kappa_s^* \varpi_1^\zeta = s^{-2} \varpi_1^\zeta$, where κ_s is the dilation of factor $s > 0$ on \mathbb{R}^4 . But we exactly have $\kappa_s^* \theta_j = s^{-2} \theta_j$, $j = 1, 2, 3$; therefore, f, g, h are functions on the sphere \mathbb{S}^3 . Notice that from this point, we also know that ϖ_1^ζ is anti-self-dual (for \mathbf{e}), since the forms θ_j are. Therefore ϖ_1^ζ is \mathbf{e} -harmonic on $\mathbb{R}^4 \setminus \{0\}$, which is the same as $(\nabla^{\mathbf{e}})^*(\nabla^{\mathbf{e}})\varpi_1^\zeta = 0$. On the other hand, the forms of θ_j are harmonic as well: they are anti-self-dual, and closed, since

$$\theta_j = \frac{1}{4} dd^c_{I_j} \left(\frac{1}{r^2} \right), \quad j = 1, 2, 3. \tag{43}$$

Putting those facts together and setting $e_j = I_j(1/r)x_i(\partial/\partial x_i)$, $j = 1, 2, 3$ (forget about (8)) so that $rdr(e_j) = 0$ and $\alpha_k(e_j) = r\delta_{jk}$, $j, k = 1, 2, 3$, we get that

$$\Delta_{\mathbf{e}}(f\theta_1) = \frac{1}{r^2}(\Delta_{\mathbb{S}^3}f)\theta_1 - 2\sum_{k=1}^3(e_k \cdot f)\nabla_{e_k}^{\mathbf{e}}\theta_1.$$

The $\nabla_{e_k}^{\mathbf{e}}\theta_1$ are easy to compute: since $e_k \cdot r = 0$, $\nabla_{e_k}^{\mathbf{e}}\theta_1 = (1/r^6)\nabla_{e_k}^{\mathbf{e}}(rdr \wedge \alpha_1 - \alpha_2 \wedge \alpha_3)$. Moreover since the I_j are parallel, we just have to compute $\nabla_{e_k}^{\mathbf{e}}rdr$; since $\nabla^{\mathbf{e}}(rdr) = \mathbf{e}$, $\nabla_{e_k}^{\mathbf{e}}(rdr) = \mathbf{e}(e_k, \cdot) = (1/r)\alpha_k$. Therefore $\nabla_{e_1}^{\mathbf{e}}\theta_1 = 0$, $\nabla_{e_2}^{\mathbf{e}}\theta_1 = (2/r)\theta_3$ and $\nabla_{e_3}^{\mathbf{e}}\theta_1 = -(2/r)\theta_2$. Thus $\Delta(f\theta_1) = (1/r^2)(\Delta_{\mathbb{S}^3}f)\theta_1 - (2/r)(e_2 \cdot f)\theta_3 + (2/r)(e_3 \cdot f)\theta_2$. A circular permutation on the indices gives as well $\Delta(g\theta_2) = (1/r^2)(\Delta_{\mathbb{S}^3}g)\theta_2 - (2/r)(e_3 \cdot g)\theta_1 + (2/r)(e_1 \cdot g)\theta_3$ and $\Delta(h\theta_3) = (1/r^2)(\Delta_{\mathbb{S}^3}h)\theta_3 - (2/r)(e_1 \cdot h)\theta_2 + (2/r)(e_2 \cdot h)\theta_1$. Since the θ_j are linearly independent, $\Delta\varpi_1^{\zeta} = 0$ translates into

$$\Delta_{\mathbb{S}^3}f - 4(e_3 \cdot g - e_2 \cdot h) = \Delta_{\mathbb{S}^3}g - 4(e_1 \cdot h - e_3 \cdot f) = \Delta_{\mathbb{S}^3}h - 4(e_2 \cdot f - e_1 \cdot g) = 0. \tag{44}$$

On the other hand, $d\varpi_1^{\zeta} = 0$ is equivalent to $e_1 \cdot f + e_2 \cdot g + e_3 \cdot h = e_2 \cdot f - e_1 \cdot g = e_3 \cdot g - e_2 \cdot h = e_1 \cdot h - e_3 \cdot f = 0$; the latter three equalities, plugged into equations (44), exactly give $\Delta_{\mathbb{S}^3}f = \Delta_{\mathbb{S}^3}g = \Delta_{\mathbb{S}^3}h = 0$, hence f, g and h are constant. \square

Remark 2.11. We have not used the Γ -invariance of the tensors here; nonetheless, since the θ_j are $SU(2)$ -invariant, which comes from the identities $\theta_j = dd_{I_j}^c(1/r^2)$, this does not give us any further information.

2.3.3 Lower-order variation of the Kähler forms: determination of the coefficients. We know from the formal expansion of g_{ζ} (or those of $g_{\zeta'}$ and $g_{\zeta''}$) that the a_{jk} coefficients of Proposition 2.10 are quadratic homogeneous polynomials in their arguments. Their explicit form is given as follows.

PROPOSITION 2.12. *With the same notation as in Proposition 2.10,*

$$\begin{aligned} a_{31}(\zeta'') &= 0, & a_{32}(\zeta'') &= 0, & a_{33}(\zeta'') &= -\|\Gamma\|\|\zeta_3\|^2, \\ a_{21}(\zeta') &= 0, & a_{22}(\zeta') &= -\|\Gamma\|\|\zeta_2\|^2, & a_{23}(\zeta') &= -2\|\Gamma\|\langle\zeta_2, \zeta_3\rangle, \\ a_{11}(\zeta) &= -\|\Gamma\|\|\zeta_1\|^2, & a_{12}(\zeta) &= -2\|\Gamma\|\langle\zeta_1, \zeta_2\rangle, & a_{13}(\zeta) &= -2\|\Gamma\|\langle\zeta_1, \zeta_3\rangle, \end{aligned}$$

where $\|\Gamma\| := |\Gamma|/4 \text{Vol}(\mathbb{B}^4) = |\Gamma|/2\pi^2$.

Proof. We shall first prove the assertion on the $a_{3j}(\zeta'')$, and then generalise our technique so as to write down five equations with the eight remaining coefficients. We conclude by using once again the specificity of Kronheimer’s F_{ζ} (which essentially amounts to a gauge fixing here) to overcome this under-determinacy.

The coefficient $a_{33}(\zeta'')$. To begin with, set $a = a_{31}(\zeta'')$, $b = a_{32}(\zeta'')$ and $c = a_{33}(\zeta'')$. We consider on $X_{\zeta''}$ (which is smooth by our assumption $\zeta'' \notin D$) a closed form λ with compact support representing ζ_3 by Poincaré duality; this is possible since minimal resolutions of \mathbb{C}^2/Γ have compactly supported cohomology [Joy00, Theorem 8.4.3], and $X_{\zeta''}$ is diffeomorphic to such a resolution (this is actually a minimal resolution of $(\mathbb{C}^2/\Gamma, I_3)$, but we will not use this fact). Next, consider a smooth cut-off function χ , vanishing on $(-\infty, 1]$, equal to 1 on $[2, +\infty)$. From the equality $\omega_3^{\mathbf{e}} = \frac{1}{2}dd_{I_3}^c(r^2)$, and from (43), we have that

$$\varepsilon := \omega_3^{\zeta''} - \lambda - d[\frac{1}{4}I_3d(\chi(r)r^2) + \frac{1}{4}(aI_1 + bI_2 + cI_3)d(\chi(r)r^{-2})]$$

is well defined on $X_{\zeta'}$, has cohomology class 0, and is $O(r^{-6})$ at infinity, with appropriate decay on its derivatives; here we write r instead of $(\lambda_3'')^*r$. As we need it further, we shall also see now that ε admits a primitive which decays at infinity.

From [Joy00, Theorem 8.4.1], ε can indeed be written as $h + d\beta + d^{*g_{\zeta''}}\gamma$, where h is in $C_3^\infty(X_\zeta, \Lambda^2)$ and is $g_{\zeta''}$ -harmonic, and β and γ are in $C_2^\infty(X_\zeta, \Lambda^2)$; we used here classical notation for weighted spaces: for example, $\beta = O(r^{-2})$, $\nabla^e\beta = O(r^{-3})$, and so on. The harmonic form h is actually decaying fast enough so that we can say it is closed and co-closed: write (all the operations and tensors are computed with respect to $g_{\zeta''}$) for all r

$$0 = \int_{\mathbb{B}(r)} (h, \Delta h) \text{vol} = \int_{\mathbb{B}(r)} (|dh|^2 + |d^*h|^2) \text{vol} + \int_{\mathbb{S}(r)} (h \odot dh + h \odot d^*h) \text{vol},$$

where $\mathbb{B}(r) = \mathbb{B}_{X_{\zeta''}}(r) = (\lambda_3'')^{-1}(\mathbb{B}^4(r)/\Gamma)$, and $\mathbb{S}(r)$ is its boundary. From what precedes, the boundary integral is easily seen to be $O(r^{3-3-4}) = O(r^{-4})$, and thus $dh = d^*h = 0$. Hence $0 = d\varepsilon = dd^*\gamma$; an integration by parts similar to the previous one, but with boundary term of size $O(r^{-2})$, leads us to $d^*\gamma = 0$, and thus $\varepsilon = h + d\beta$. According to [Joy00, Theorem 8.4.1] again, $\mathcal{H}^2(X_{\zeta''}) \rightarrow H^2(X_{\zeta''})$, $h \mapsto [h]$ is an isomorphism; now here $[h] = [\varepsilon - d\beta] = 0$. Therefore $h = 0$, and $\varepsilon = d\beta$, with $\beta = O(r^{-2})$.

We shall now compute the integrals $\int_{\mathbb{B}(r)} (\omega_3'')^2$ in two different ways. First, recall that $(\omega_3'')^2 = 2 \text{vol}^e$, and thus⁴ $\int_{\mathbb{B}(r)} (\omega_3'')^2 = (2r^4/|\Gamma|) \text{Vol}(\mathbb{B}^4)$. On the other hand, since $\omega_3'' = \lambda + d\varphi + \varepsilon$, with $\varphi = \frac{1}{4}I_3d(\chi(r)r^2) + \frac{1}{4}(aI_1 + bI_2 + cI_3)d(\chi(r)r^{-2})$, we have

$$\begin{aligned} \int_{\mathbb{B}(r)} (\omega_3'')^2 &= \int_{\mathbb{B}(r)} \lambda^2 + 2 \int_{\mathbb{B}^4(r)/\Gamma} \lambda \wedge d\varphi + 2 \int_{\mathbb{B}(r)} \lambda \wedge \varepsilon \\ &\quad + \int_{\mathbb{B}(r)} (d\varphi)^2 + 2 \int_{\mathbb{B}(r)} \varepsilon \wedge d\varphi + \int_{\mathbb{B}^4(r)/\Gamma} \varepsilon^2. \end{aligned} \tag{45}$$

Let us analyse those summands separately.

For r large enough, $\int_{\mathbb{B}(r)} \lambda^2 = \int_{X_{\zeta''}} \lambda^2 = \lambda \cup \lambda = -|\zeta_3|^2$, by Lemma 2.2, and the fact that $[\lambda] = [\omega_3''] = (\zeta'')_3 = \zeta_3$.

The integral $\int_{\mathbb{B}(r)} \lambda \wedge d\varphi$ equals $\int_{\mathbb{S}(r)} \lambda \wedge \varphi$ by Stokes' theorem, and this vanishes for r large enough; similarly, $\int_{\mathbb{B}(r)} \lambda \wedge \varepsilon = \int_{\mathbb{B}(r)} \lambda \wedge d\beta = \int_{\mathbb{S}(r)} \lambda \wedge \beta = 0$ for r large enough.

We now come to $\int_{\mathbb{B}(r)} (d\varphi)^2$. By Stokes, this is equal to $\int_{\mathbb{S}(r)} d\varphi \wedge \varphi$, which we view back on \mathbb{R}^4/Γ via λ_3'' . For $r \geq 2$, the integrand is

$$\begin{aligned} &(\omega_3^e + a\theta_1 + b\theta_2 + c\theta_3) \wedge \left[\frac{1}{2}\alpha_3 - \frac{1}{2} \left(a\frac{\alpha_1}{r^4} + b\frac{\alpha_2}{r^4} + c\frac{\alpha_3}{r^4} \right) \right] \\ &= \frac{1}{2r^2} \left(1 - \frac{2c}{r^4} \right) \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + O(r^{-7}), \end{aligned}$$

since $\omega_3^e \wedge \alpha_3 = (1/r^2)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = r^3 \text{vol}^{\mathbb{S}^3(r)}$, $\omega_3^e \wedge \alpha_1 = \omega_3^e \wedge \alpha_2 = 0$ (they factor through rdr), and $\theta_3 \wedge \alpha_3 = -(\alpha_1 \wedge \alpha_2 \wedge \alpha_3)/r^6$, $\theta_1 \wedge \alpha_3 = \theta_2 \wedge \alpha_3 = 0$ (again, factorisation through rdr) on

⁴ Indeed, $\int_{\mathbb{B}(r)} (\omega_3'')^2$ is the limit as s goes to 0 of $\int_{\mathbb{B}(r)-\mathbb{B}(s)} (\omega_3'')^2$, since as an s -tubular neighbourhood of $E := (\lambda_3'')^{-1}(\{0\})$ which is of real dimension 2, $\mathbb{B}(s)$ has its volume tending to 0 when s goes to 0. Now we can also see $\int_{\mathbb{B}(r)-\mathbb{B}(s)} (\omega_3'')^2$ on \mathbb{R}^4/Γ via λ_3'' which is diffeomorphic away from E , and since $(\lambda_3'')_*(\omega_3'')^2 = 2\Omega_e$, $\int_{\mathbb{B}(r)-\mathbb{B}(s)} (\omega_3'')^2$ is twice the Euclidean volume of the annulus of radii s and r in \mathbb{R}^4/Γ , hence the result when $s \rightarrow 0$.

$\mathbb{S}^3(r)/\Gamma$. Moreover $\theta_j \wedge \alpha_k = O(r^{-7})$ for $j, k = 1, 2, 3$. As $\int_{\mathbb{S}^3(r)/\Gamma} \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = r^3 \text{Vol}(\mathbb{B}^3(r))/|\Gamma| = 4r^2 \text{Vol}(\mathbb{B}^4(r))/|\Gamma|$, we thus end up with

$$\int_{\mathbb{B}(r)} (d\varphi)^2 = 2 \left(1 - \frac{2c}{r^4} \right) \frac{\text{Vol}(\mathbb{B}^4(r))}{|\Gamma|} + O(r^{-4}) = \frac{2(r^4 - 2c)}{|\Gamma|} \text{Vol}(\mathbb{B}^4) + O(r^{-4}).$$

We conclude by the last two summands of (45). On the one hand, $\int_{\mathbb{B}(r)} \varepsilon \wedge d\varphi = \int_{\mathbb{S}(r)} \varepsilon \wedge \varphi = O(r^{3-6+1}) = O(r^{-2})$, since $\varepsilon = O(r^{-6})$ and $\varphi = O(r)$. On the other hand, $\int_{\mathbb{B}(r)} \varepsilon^2 = \int_{\mathbb{B}(r)} \varepsilon \wedge d\beta = \int_{\mathbb{S}(r)} \varepsilon \wedge \beta$: this is $O(r^{3-6-2}) = O(r^{-5})$ (and this is actually the only place where we need an estimate on the decay of a primitive of ε).

Collecting the different estimates, for r going to ∞ we have

$$\frac{2r^4}{|\Gamma|} \text{Vol}(\mathbb{B}^4) = -|\zeta_3|^2 + \frac{2(r^4 - 2c)}{|\Gamma|} \text{Vol}(\mathbb{B}^4) + O(r^{-2}),$$

hence $c = -(|\Gamma|/4 \text{Vol}(\mathbb{B}^4))|\zeta_3|^2$.

Five further equations for eight other coefficients. For the exact same reasons as for $\omega_3^{\zeta''}$, one can write

$$\begin{cases} \omega_1^\zeta = \mu + d\psi + d\zeta, \\ \omega_2^\zeta = \omega_2^{\zeta'} = \nu + d\xi + d\eta, \end{cases}$$

with μ, ν compactly supported and of respective class Poincaré-dual to ζ_1 and $\zeta_2 = (\zeta')_2$, with ψ and ξ smooth and exact such that $4\psi = d_{I_1}^c r^2 + (a_{11}(\zeta)d_{I_1}^c r^{-2} + a_{12}(\zeta)d_{I_2}^c r^{-2} + a_{13}(\zeta)d_{I_3}^c r^{-2})$ and $4\xi = d_{I_2}^c r^2 + (a_{21}(\zeta')d_{I_1}^c r^{-2} + a_{22}(\zeta')d_{I_2}^c r^{-2} + a_{23}(\zeta')d_{I_3}^c r^{-2})$ outside a compact set, and ζ, η smooth forms with $O(r^{-2})$ -decay, such that $d\zeta$ and $d\eta$ are $O(r^{-6})$.

Recall that $\omega_3^{\zeta''} = \omega_3^\zeta$, $\omega_2^{\zeta'} = \omega_2^\zeta$, and that $\text{vol}^\zeta = \text{vol}^e$; thus, integrating just as above the five remaining relations $\omega_j^\zeta \wedge \omega_k^\zeta = 2\delta_{jk} \text{vol}^\zeta$ over images via F_ζ of Euclidean balls of radius r and letting r go to ∞ yields

$$\begin{cases} a_{11}(\zeta) = -\frac{|\Gamma||\zeta_1|^2}{4 \text{Vol}(\mathbb{B}^4)}, & a_{12}(\zeta) + a_{21}(\zeta') = -\frac{|\Gamma|\langle \zeta_1, \zeta_2 \rangle}{2 \text{Vol}(\mathbb{B}^4)}, & a_{13}(\zeta) + a_{31}(\zeta'') = -\frac{|\Gamma|\langle \zeta_1, \zeta_3 \rangle}{2 \text{Vol}(\mathbb{B}^4)}, \\ a_{22}(\zeta') = -\frac{|\Gamma||\zeta_2|^2}{4 \text{Vol}(\mathbb{B}^4)} & \text{and} & a_{23}(\zeta') + a_{32}(\zeta'') = -\frac{|\Gamma|\langle \zeta_2, \zeta_3 \rangle}{2 \text{Vol}(\mathbb{B}^4)}. \end{cases} \tag{46}$$

This provides in particular the announced values of $a_{11}(\zeta)$ and $a_{22}(\zeta')$.

Three extra equations. We now conclude, using the precise way the F_ζ are constituted; namely, we replace ζ by ζ' in the first line of (46). As, on the one hand $(\zeta')' = \zeta'$ and $(\zeta'')'' = \zeta''$, but $(\zeta')_1 = 0$, we get $a_{12}(\zeta') + a_{21}(\zeta') = -|\Gamma|\langle (\zeta')_1, \zeta'_2 \rangle / 2 \text{Vol}(\mathbb{B}^4) = 0$, and likewise, $a_{13}(\zeta') + a_{31}(\zeta'') = -|\Gamma|\langle (\zeta')_1, \zeta_3 \rangle / 2 \text{Vol}(\mathbb{B}^4) = 0$. On the other hand, since $\omega_1^{\zeta'} = \omega_1^e$, we have $a_{12}(\zeta') = a_{13}(\zeta') = 0$, and thus

$$a_{21}(\zeta') = a_{31}(\zeta'') = 0,$$

so in the end, $a_{12}(\zeta) = -|\Gamma|\langle \zeta_1, \zeta_2 \rangle / 2 \text{Vol}(\mathbb{B}^4)$ and $a_{13}(\zeta) = -|\Gamma|\langle \zeta_1, \zeta_3 \rangle / 2 \text{Vol}(\mathbb{B}^4)$.

Replacing ζ by ζ'' and using that $a_{23}(\zeta'') = 0$, we get

$$a_{32}(\zeta'') = 0,$$

hence $a_{23}(\zeta') = -|\Gamma|\langle \zeta_2, \zeta_3 \rangle / 2 \text{Vol}(\mathbb{B}^4)$ in the same way. □

2.3.4 *Conclusion: proof of Theorem 2.1 (general Γ).* Let us sum the situation up. If we take $\Phi_\zeta = F_\zeta^{-1} : X_\zeta \setminus F_\zeta^{-1}(\{0\}) \rightarrow (\mathbb{R}^4 \setminus \{0\})/\Gamma$ and keep the notation introduced in this section, we have $\Phi_{\zeta_*} g_\zeta = \mathbf{e} + h_\zeta + O(r^{-6})$, $\Phi_{\zeta_*} I_1^\zeta = I_1 + \iota_1^\zeta + O(r^{-6})$ and $\Phi_{\zeta_*} \omega_1^\zeta = \omega_1^{\mathbf{e}} + \varpi_1^\zeta + O(r^{-6})$. The I_1 -hermitian component of h_ζ is $\varpi_1^\zeta(\cdot, I_1 \cdot)$, which we know, and its I_1 -skew-hermitian component is $h_{\zeta'}$. Now the I_2 -hermitian component of $h_{\zeta'}$ is $\varpi_2^{\zeta'}(\cdot, I_2 \cdot)$, which we also know, and its I_2 -skew-hermitian component is $h_{\zeta''}$. Finally, $h_{\zeta''}$ is I_3 -hermitian, equal to $\varpi_3^{\zeta''}(\cdot, I_3 \cdot)$, which we know as well. In a nutshell, we are able to write down explicitly h_ζ from Propositions 2.10 and 2.12:

$$\begin{aligned} h_\zeta &= \varpi_3^{\zeta''}(\cdot, I_3 \cdot) + \varpi_2^{\zeta'}(\cdot, I_2 \cdot) + \varpi_1^\zeta(\cdot, I_1 \cdot) \\ &= -\|\Gamma\| \left(\sum_{j=1}^3 |\zeta_j|^2 \theta_j(\cdot, I_j \cdot) + 2 \sum_{1 \leq j < k \leq 3} \langle \zeta_j, \zeta_k \rangle \theta_k(\cdot, I_j \cdot) \right), \end{aligned}$$

which gives exactly (37), with $\mathbf{c} = 1/4 \text{Vol}(\mathbb{B}^4) = \frac{1}{2}\pi^{-2}$.

From this and the formula for ϖ_1^ζ proved in 2.12, which gives (39) of Theorem 2.1, we deduce the expected formula for ι_1^ζ . We know indeed that $\iota_1^\zeta = \iota_1^{\zeta'}$, and that $h_{\zeta'} = \omega_1^{\mathbf{e}}(\cdot, \iota_1^{\zeta'} \cdot)$ and $\iota_1^{\zeta'}$ is \mathbf{e} -symmetric, hence

$$\begin{aligned} \mathbf{e}(\iota_1^\zeta, \cdot) &= \mathbf{e}(\cdot, \iota_1^{\zeta'} \cdot) = -\omega_1^{\mathbf{e}}(I_1 \cdot, \iota_1^{\zeta'} \cdot) = -h_{\zeta'}(I_1 \cdot, \cdot) \\ &= \|\Gamma\| (|\zeta_3|^2 \theta_3(I_1 \cdot, I_3 \cdot) + |\zeta_2|^2 \theta_2(I_1 \cdot, I_2 \cdot) + 2 \langle \zeta_2, \zeta_3 \rangle \theta_3(I_1 \cdot, I_2 \cdot)) \\ &= \|\Gamma\| (|\zeta_3|^2 \theta_3(\cdot, I_2 \cdot) - |\zeta_2|^2 \theta_2(\cdot, I_3 \cdot) - 2 \langle \zeta_2, \zeta_3 \rangle \theta_3(\cdot, I_3 \cdot)) \\ &= \|\Gamma\| |\zeta_3|^2 \frac{\alpha_2 \cdot \alpha_3 - r dr \cdot \alpha_1}{r^6} - \|\Gamma\| |\zeta_2|^2 \frac{\alpha_2 \cdot \alpha_3 + r dr \cdot \alpha_1}{r^6} \\ &\quad - 2 \|\Gamma\| \langle \zeta_2, \zeta_3 \rangle \frac{(r dr)^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2}{r^6}, \end{aligned}$$

of which (38) is just a rewriting.

2.4 Vanishing of the third-order terms when Γ is not cyclic

We shall see in this section that in the expansion $g_\zeta = \mathbf{e} + h_\zeta + \sum_{j=3}^\infty h_\zeta^{(j)}$, if Γ is one of the \mathcal{D}_k , $k \geq 2$, or contains one of these as is the case when Γ is binary tetrahedral, octahedral or icosahedral, then the third-order term $h_\zeta^{(3)}$ vanishes, and that this holds as well for complex structures and Kähler forms. Keeping working with the diffeomorphisms F_ζ of the previous section even if we omit them to simplify notation, we claim the following.

PROPOSITION 2.13. *Suppose Γ contains \mathcal{D}_k , $k \geq 2$, as a subgroup. Then $g_\zeta = \mathbf{e} + h_\zeta + O(r^{-8})$, $I_1^\zeta = I_1 + \iota_1^\zeta + O(r^{-8})$, $\omega_1^\zeta = \omega_1 + \varpi_1^\zeta + O(r^{-8})$, where by $O(r^{-8})$ we mean tensors whose l th-order derivatives (for $\nabla^{\mathbf{e}}$) are $O(r^{-8-\ell})$.*

Proof. We shall first see that, for a general Γ , the crucial considerations made in §2.3 on the second-order term h_ζ of the expansion of g_ζ still hold for $h_\zeta^{(3)}$; first recall that $h_\zeta^{(3)}$ is a homogeneous polynomial of ζ of order 3, with coefficients $O(r^{-6})$ symmetric 2-tensors, with according decay on the derivatives, and those coefficients are independent of ζ . We start with claiming that

$$\text{tr}^{\mathbf{e}}(h_\zeta^{(3)}) = 0 \quad \text{and} \quad \delta^{\mathbf{e}} h_\zeta^{(3)} = 0.$$

Indeed, for the trace assertion, once $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$ is fixed, one has for all t that

$$\begin{aligned} \Omega_{\mathbf{e}} &= \text{vol}^{g_{t\zeta}} = \det^{\mathbf{e}}(\mathbf{e} + t^2 h_{\zeta} + t^3 h_{\zeta}^{(3)} + O(t^4)) \Omega_{\mathbf{e}} \\ &= (1 + t^2 \text{tr}^{\mathbf{e}}(h_{\zeta}) + t^3 \text{tr}^{\mathbf{e}}(h_{\zeta}^{(3)}) + O(t^4)) \Omega_{\mathbf{e}}, \end{aligned}$$

since the higher-order contributions of $t^2 h_{\zeta}$ are included in the $O(t^4)$, hence $\text{tr}^{\mathbf{e}}(h_{\zeta}^{(3)}) = 0$.

We thus notice that $h_{\zeta}^{(3)}$ shares this property with h_{ζ} because the nonlinear contributions of the $h_{t\zeta}$, which are of order at least 4 in t , do not interfere with the linear contribution of $h_{t\zeta}^{(3)}$. We thus generalise this observation to prove that $h_{\zeta}^{(3)}$ shares other properties with h_{ζ} , and to start with, that $\delta^{\mathbf{e}} h_{\zeta}^{(3)} = 0$, as promised. Again we proceed within three steps, considering first $\zeta'' = (0, 0, \zeta_3)$, and then $\zeta' = (0, \zeta_2, \zeta_3)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3)$.

The case of $h_{\zeta''}^{(3)}$ is immediate, and merely amounts to the fact that it is an I_3 -hermitian tensor (the $g_{t\zeta''}$ are) with vanishing trace for \mathbf{e} , used with the Kähler identity $[\Lambda_{\omega_3}, d_{I_3}^c] = d^{*\mathbf{e}}$ applied to $h_{\zeta''}^{(3)}(I_3 \cdot, \cdot)$.

For the case of $h_{\zeta'}^{(3)}$, remember the following: we first saw that the second-order variation of $I_2^{\zeta'} = I_2^{\zeta''}$ was \mathbf{e} -symmetric; this still holds for the third-order term, since the only I_2 -entire function on \mathbb{C}^2 decaying (like r^{-6}) at infinity is trivial. Then we identified the I_2 -skew-hermitian part of $h_{\zeta'}$ with $h_{\zeta''}$; again, this holds for $h_{\zeta'}^{(3)}$ with $h_{\zeta''}^{(3)}$ (and the latter is indeed I_2 -skew-hermitian). This amounts to looking at the term of order 3 in t of:

- the expansion of $g_{t\zeta''} = \omega_2^{t\zeta''}(\cdot, I_2^{t\zeta''} \cdot)$ to see that $h_{\zeta''}^{(3)}$ is indeed I_2 -skew-hermitian (recall $\omega_2^{t\zeta''} = \omega_2$ for all t);
- the expansion of $g_{t\zeta'}(I_2^{t\zeta'} \cdot, I_2^{t\zeta'} \cdot) - g_{t\zeta'}$ to see that $\frac{1}{2}(h_{\zeta'}^{(3)} + h_{\zeta'}^{(3)}(I_2 \cdot, I_2 \cdot)) = h_{\zeta''}^{(3)}$.

We concluded by using the usual Kähler identity (for I_2) on the \mathbf{e} -trace-free I_2 -(1, 1) form $\frac{1}{2}(h_{\zeta'}(I_2 \cdot, \cdot) - h_{\zeta'}(\cdot, I_2 \cdot))$, after seeing it was closed; we can do the same on its analogue $\frac{1}{2}(h_{\zeta'}^{(3)}(I_2 \cdot, \cdot) - h_{\zeta'}^{(3)}(\cdot, I_2 \cdot))$, which is also an \mathbf{e} -trace-free I_2 -(1, 1) form, and is closed as seen when looking at the third order in t of the expansion of $\omega_2^{t\zeta'} = \frac{1}{2}(g_{t\zeta'}(I_2^{t\zeta'} \cdot, \cdot) - g_{t\zeta'}(I_2^{t\zeta'} \cdot, I_2 \cdot))$.

One deals with h_{ζ} in analogous way. In particular, we get in passing that the third-order variation of $I_1^{\zeta} = I_1^{\zeta'}$, J_1^{ζ} say, is \mathbf{e} -symmetric and anti-commutes to I_1 , that the I_1 -skew-hermitian part of $h_{\zeta}^{(3)}$ is $h_{\zeta'}^{(3)}$, related to J_1^{ζ} by $h_{\zeta'}^{(3)} = \omega_1^{\mathbf{e}}(\cdot, J_1^{\zeta} \cdot)$, and that its I_1 -hermitian part gives rise to an \mathbf{e} -trace-free closed I_1 -(1, 1) form.

Running backward this description, we will thus be done if we show that the third-order variations of the Kähler forms vanish when Γ contains a binary dihedral group. In general though, we know these are $O(r^{-6})$ near 0 and infinity with corresponding decay on their derivatives, that they are of type (1, 1) for one of the I_j and trace-free; they are thus $*_{\mathbf{e}}$ -anti-self-dual, and therefore can be written as $f\theta_1 + g\theta_2 + h\theta_3$, where this time, $r^2 f$, $r^2 g$ and $r^2 h$ depend only on the spherical coordinate of their argument. Our form are moreover closed, hence in particular harmonic; using again that the Laplace–Beltrami operator and the rough Laplacian coincide on $(\mathbb{R}^4, \mathbf{e})$, and that the θ_j are harmonic, we have this time that

$$\Delta_{\mathbf{e}}(f\theta_1) = (\Delta_{\mathbf{e}} f)\theta_1 - 2 \sum_{k=0}^3 (e_k \cdot f) \nabla_{e_k}^{\mathbf{e}} \theta_1,$$

with $e_0 = (x_j/r)(\partial/\partial x_j)$. We set $\tilde{f} = r^2 f$; this is a function on \mathbb{S}^3 , and $e_0 \cdot f = e_0 \cdot (r^{-2} \tilde{f}) =$

$e_0 \cdot (r^{-2})\tilde{f} = -2r^{-3}\tilde{f} = -2r^{-1}f$. Since on functions, $\Delta_e = -(1/r^3)\partial_r(r^3\partial_r \cdot) + (1/r^2)\Delta_{\mathbb{S}^3}$, one has

$$\Delta_e f = \Delta_e(r^{-2}\tilde{f}) = -\frac{1}{r^3}\partial_r(r^3\partial_r(r^{-2}))\tilde{f} + \frac{1}{r^4}\Delta_{\mathbb{S}^3}\tilde{f} = \frac{1}{r^4}\Delta_{\mathbb{S}^3}\tilde{f},$$

since $\partial_r(r^3\partial_r(r^{-2})) = 0$ (r^{-2} is the Green function on \mathbb{R}^4).

Moreover, $\nabla_{e_0}^e \theta_1 = \partial_r(r^{-6})(rdr \wedge \alpha_1 - \alpha_2 \wedge \alpha_3) + r^{-6}\nabla_{e_0}^e(rdr \wedge \alpha_1 - \alpha_2 \wedge \alpha_3) = -(6/r)\theta_1 + (2/r)\theta_1 = -(4/r)\theta_1$. We recall that $\nabla_{e_1}^e \theta_1 = 0$, $\nabla_{e_2}^e \theta_1 = (2/r)\theta_3$ and $\nabla_{e_3}^e \theta_1 = -(2/r)\theta_2$, therefore

$$\Delta_e(f\theta_1) = \frac{1}{r^4}(\Delta_{\mathbb{S}^3}\tilde{f} - 16\tilde{f})\theta_1 - \frac{2}{r^3}((e_2 \cdot \tilde{f})\theta_3 - (e_3 \cdot \tilde{f})\theta_2).$$

Writing the analogous equations on $\tilde{g} = r^2g$, $\tilde{h} = r^2h$, the equation $\Delta_e(f\theta_1 + g\theta_2 + h\theta_3) = 0$ is equivalent to the system

$$\begin{aligned} \Delta_{\mathbb{S}^3}\tilde{f} - 16\tilde{f} - 4(e_3 \cdot \tilde{g}) + 4(e_2 \cdot \tilde{h}) &= 0, \\ \Delta_{\mathbb{S}^3}\tilde{g} - 16\tilde{g} - 4(e_1 \cdot \tilde{h}) + 4(e_3 \cdot \tilde{f}) &= 0, \\ \Delta_{\mathbb{S}^3}\tilde{h} - 16\tilde{h} - 4(e_2 \cdot \tilde{f}) + 4(e_3 \cdot \tilde{g}) &= 0. \end{aligned} \tag{47}$$

Now the closure assertion on $f\theta_1 + g\theta_2 + h\theta_3$ is equivalent to $(e_1 \cdot f) + (e_2 \cdot g) + (e_3 \cdot h) = (e_0 \cdot f) - (e_3 \cdot g) + (e_2 \cdot h) = (e_0 \cdot g) - (e_1 \cdot h) + (e_3 \cdot f) = (e_0 \cdot h) - (e_2 \cdot f) + (e_1 \cdot g) = 0$.

Since $e_0 \cdot u = -(2/r^3)\tilde{u}$ and $e_k \cdot u = (1/r^2)e_k \cdot \tilde{u}$ for $u = f, g, h$ and $k = 1, 2, 3$, we deduce from the latter equalities and the system (47) the equations

$$\Delta_{\mathbb{S}^3}\tilde{f} - 8\tilde{f} = \Delta_{\mathbb{S}^3}\tilde{g} - 8\tilde{g} = \Delta_{\mathbb{S}^3}\tilde{h} - 8\tilde{h} = 0.$$

Setting $\hat{f} = r^2\tilde{f}$ and likewise for \hat{g} and \hat{h} , we get that $\hat{f}, \hat{g}, \hat{h}$ are harmonic (on the whole \mathbb{R}^4) and homogeneous of degree 2. This is not hard seeing that they are thus linear combinations of the $x_1^2 - x_j^2$, $j = 2, 3, 4$, and the x_jx_k , $1 \leq j < k \leq 4$.

The θ_j are Γ -invariant; f, g and h , and consequently \hat{f}, \hat{g} and \hat{h} , must be as well. But if Γ contains a binary dihedral group as a subgroup, then there is no non-trivial linear combination of the above polynomials which is Γ -invariant. We use first the τ -invariance; if indeed $\mathcal{D}_k < \Gamma$ for some $k \geq 2$ and $u = \sum_{j=1}^3 a_j(x_1^2 - x_j^2) + \sum_{1 \leq j < \ell \leq 4} a_{j\ell}x_jx_\ell$ is Γ -invariant, then $2u = u + \tau^*u = a_2(x_1^2 - x_2^2 + x_3^2 - x_4^2) + a_3(x_1^2 - x_3^2 + x_3^2 - x_1^2) + a_4(x_1^2 - x_4^2 + x_3^2 - x_2^2) + a_{12}(x_1x_2 + x_3x_4) + a_{13}(x_1x_3 - x_3x_1) + a_{14}(x_1x_4 - x_3x_2) + a_{23}(x_2x_3 - x_4x_1) + a_{24}(x_2x_4 - x_4x_2) + a_{34}(x_3x_4 + x_1x_2)$, that is: u has shape $a(x_1^2 - x_2^2 + x_3^2 - x_4^2) + 2b(x_1x_2 + x_3x_4) + 2c(x_1x_4 - x_3x_2)$, i.e. $a\Re(z_1^2 + z_2^2) + b\Im(z_1^2 + z_2^2) + c\Im(\overline{z_1}z_2)$, $a, b, c \in \mathbb{R}$, in complex notation. We now use the ζ_k -action and write

$$\begin{aligned} ku &= \sum_{\ell=0}^k \zeta_k^\ell u = \sum_{\ell=0}^k a(\zeta_k^\ell)^* \Re(z_1^2 + z_2^2) + b(\zeta_k^\ell)^* \Im(z_1^2 + z_2^2) + c(\zeta_k^\ell)^* \Im(\overline{z_1}z_2) \\ &= \Re\left(\sum_{\ell=0}^k a(e^{2i\ell\pi/k}z_1^2 + e^{-2i\ell\pi/k}z_2^2)\right) \\ &\quad + \Im\left(\sum_{\ell=0}^k b(e^{2i\ell\pi/k}z_1^2 + e^{-2i\ell\pi/k}z_2^2) + ce^{-2i\ell\pi/k}\overline{z_1}z_2\right) \\ &= 0, \end{aligned}$$

since $e^{2i\pi/k} \neq 1$ ($k \geq 2$). In particular, the third-order variation term of ω_1^ζ vanishes; in other words, $\omega_1^\zeta = \omega_1 + \varpi_1^\zeta + O(r^{-8})$.

Since moreover J_1^ζ is determined by $h_{\zeta'}^{(3)}$ which is also 0, this third-order variation of the first complex structure vanishes as well, or $I_1^\zeta = I_1 + \iota_1^\zeta + O(r^{-8})$. □

This completes the proof of Theorem 2.1. Notice however that in view of the previous two sections, we could also have given similar statements on the second and third complex structures and Kähler forms of X_ζ . We chose to focus on the first ones since this is what is needed in our construction of Part 1, see in particular Lemma 1.6, which is just a specialisation of Theorem 2.1: take $\zeta = \xi$ verifying condition (11), and $\Phi_Y = \Phi_\zeta$.

Nonetheless, the asymptotics of the second and third complex Kähler forms are available via Proposition 2.12, from which the asymptotics of the corresponding complex structures easily follow, since the asymptotics of the metric are known.

2.5 Comments on Lemma 1.1

2.5.1 *The condition (11).* The first comment we want to make about Lemma 1.1 concerns the reason why we state it under the condition (11), which we can recall as $(|\zeta_2|^2 - |\zeta_3|^2) + 2i\langle \zeta_2, \zeta_3 \rangle = 0$ (if one takes ζ instead of ξ as parameter).

One could instead try to generalise the proof we give in § 1.3 with the help of the asymptotics given by Theorem 2.1, with ζ a generic element of $\mathfrak{h} \otimes \mathbb{R}^3 - D$. This is formally possible, but leads to include terms such as $1/r^2 z_1, 1/r^2 \bar{z}_1, 1/r^2 z_2, 1/r^2 \bar{z}_2$ in the correction terms ε_1 and ε_2 of that proof, which is obviously not compatible with the requirement that \sqsupset is a diffeomorphism of \mathbb{R}^4 .

In others words, $(|\zeta_2|^2 - |\zeta_3|^2) + 2i\langle \zeta_2, \zeta_3 \rangle$ appears as an obstruction for I_1^ζ to be approximated to higher orders by I_1 , even with some liberty on the diffeomorphism between infinities of X_ζ and \mathbb{R}^4/Γ , which reveals some link between the parametrisation of the X_ζ and the general problem of the approximation of their complex structures.

2.5.2 *Links with the parametrisation.* Conversely we interpret Lemma 1.1 as follows: when $\Gamma = \mathcal{D}_k, k \geq 2$ (this would be true also in the tetrahedral, octahedral and icosahedral cases) and $(|\zeta_2|^2 - |\zeta_3|^2) + 2i\langle \zeta_2, \zeta_3 \rangle = 0$, then the complex structure I_1^ζ can be viewed as approximating the standard complex structure I_1 with precision twice that of the general case, i.e. with an error $O(r^{-8})$ instead of $O(r^{-4})$, up to an adjustment of the ALE diffeomorphism given in Kronheimer’s construction. Now $(|\zeta_2|^2 - |\zeta_3|^2) + 2i\langle \zeta_2, \zeta_3 \rangle = \langle \zeta_2 + i\zeta_3, \zeta_2 + i\zeta_3 \rangle$, and this is precisely the coefficient a_k in the equation of X_ζ seen as a submanifold of \mathbb{C}^3 , which is

$$u^2 + v^2 w + w^{k+1} = a_0 + a_1 w + \dots + a_k w^k + b w \tag{48}$$

(a_j being given by symmetric functions of the $(k + 2)$ first diagonal values of $\zeta_2 + i\zeta_3 \in \mathfrak{h}_\mathbb{C}$ or $\mathfrak{h}_\mathbb{C}/(\text{Weyl group})$ of degree $(k + 2 - j)$, and b by their Pfaffian).

Denote by $X_{\mathcal{D}_k}$ the orbifold defined in \mathbb{C}^3 by the equation $u^2 + v^2 w + w^{k+1} = 0$, i.e. (48) with $a_0 = \dots = a_k = b = 0$. This is identified to $\mathbb{C}^2/\mathcal{D}_k$ via the map $(z_1, z_2) \mapsto (u, v, w) := (\frac{1}{2}(z_1^{2k+1} z_2 - z_2^{2k+1} z_1), (i/2)(z_1^{2k} + z_2^{2k}), z_1^2 z_2^2)$. This suggests that (u, v, w) in (48) should somehow have respective degrees $2k + 2, 2k$ and 4 in the z_1, z_2 variables, and this equation remains homogeneous if we give formal degree 2 to ζ . When $a_k = 0$, the right-hand side member of (48) is therefore formally conferred ‘pure’ degree at most $4k - 4$, instead of $4k$.

We believe that this corresponds to the improvement by four orders in the approximation of I_1^ζ by I_1 in the sense of Lemma 1.1. It would thus be of interest to draw a rigorous picture out of these informal considerations, establishing a more direct link of the kind suggested here between the parameter ζ and the associated complex structures, without passing by the analysis of g_ζ ; we strongly suspect that this can be done using techniques similar to the ones used in [HC13, Part 5] (in § 5.2 for instance, the authors construct an approximately holomorphic diffeomorphism Φ between infinities of an affine cubic cone and one of its affine smoothings; such a construction is

very likely to be explicit enough, regarding especially the equations of the cone and its smoothing, to determine to what extent Φ resembles a biholomorphism, that is, to estimate the difference between the source and the pulled-back target complex structures). In a more algebro-geometric fashion, see also the formalism developed by Li ([Li14], in particular §7.3.3). Notice however that such constructions have no obvious reason to produce Bianchi-gauged leading-order error terms, which is nonetheless essential from the metric/Kähler form viewpoint of our analysis.

3. Proof of Theorem 1.16

We prove Theorem 1.16 in this part. In order to do so, a possible strategy would be to start from the beginning, in terms of Calabi’s celebrated *continuity method*, based on a quest for *a priori estimates*. Since the successful use by Yau of this method [Yau78], it has indeed been adapted to different non-compact settings; let us quote here the version by Joyce [Joy00, ch. 8] for ALE manifolds, especially his contribution in proving that *asymptotics are preserved* along the continuity method, provided that one starts with specific enough data.

Now, the above-mentioned result by Hein [Hei10, Proposition 4.1] (based itself on some version of the continuity method), already gives us a solution to (22) *bounded at any order* with reference to ω_y . We thus prefer here the following more economic (though essentially equivalent) approach to the continuity method, consisting in establishing *asymptotic a posteriori estimates* on such a solution. Roughly speaking, this can be done at order 0 using *linear harmonic analysis in (loose) ALF geometry* available as well in [Hei10] (Proposition 3.16); this is what one would (have to) adapt in the nonlinear analysis of the Monge–Ampère equation when using the continuity method option as mentioned above. Then, at positive orders, we use some of Joyce’s arguments, transposed from the ALE to the ALF framework.

To make things a bit more precise, with the notation and under the assumptions of 1.16, we have by Hein a smooth solution φ on \mathcal{Y} to the equation

$$(\omega_y + dd^c_y \varphi)^2 = e^f \omega_y^2, \tag{49}$$

which is bounded at every order with respect to ω_y , and such that

$$\omega_\varphi := \omega_y + dd^c_y \varphi$$

is positive, and, more precisely, ω_φ is equivalent to ω_y , that is, for some $c > 0$,

$$c\omega_y \leq \omega_\varphi \leq c^{-1}\omega_y \quad \text{on } \mathcal{Y}$$

(these last points are fairly automatic from (49), the boundedness of φ , and that of $dd^c_y \varphi$ with respect to ω_y). Our goal is thus to prove that $\varphi \in C^\infty_\beta(\mathcal{Y}, \omega_y)$, up to the addition of a constant; we do it in the following lines, along a three-step process:

- (i) we first prove that φ , correctly normalised, is in $C^\delta_0(\mathcal{Y}, \omega_y)$ for some $\delta \in (0, \beta)$;
- (ii) then, we get that $\varphi \in C^\infty_\delta(\mathcal{Y}, \omega_y)$;
- (iii) we conclude by sharpening the order of decay, that is, by proving that $\varphi \in C^\infty_\beta(\mathcal{Y}, \omega_y)$.

We henceforth organise the rest of this part accordingly, each section corresponding to one of the above steps. We keep the same notation throughout.

3.1 Order 0: $\varphi \in C^0_\delta(\mathcal{Y}, \omega_\mathcal{Y})$ for some $\delta \in (0, \beta)$

The purpose of this paragraph is to establish the following.

PROPOSITION 3.1. *Any smooth solution φ of (49) which is bounded at every order for $\omega_\mathcal{Y}$ can be written as $a + \psi$, where $a \in \mathbb{R}$ and $\psi \in C^0_\delta(\mathcal{Y}, \omega_\mathcal{Y}) \cap C^\infty(\mathcal{Y}, \omega_\mathcal{Y})$ for some (any) $\delta \in (0, \beta)$.*

As will be clear in the proof, we take advantage of the complex dimension 2 and the Ricci-flatness of ω_φ to reach the announced decay at infinity via *linear harmonic analysis*.

Proof. Equation (49) expands as

$$\omega_\mathcal{Y}^2 + 2\omega_\mathcal{Y} \wedge dd_\mathcal{Y}^c \varphi + (dd_\mathcal{Y}^c \varphi)^2 = e^f \omega_\mathcal{Y}^2,$$

and can thus be rewritten as

$$(\omega_\mathcal{Y} + \frac{1}{2} dd_\mathcal{Y}^c \varphi) \wedge dd_\mathcal{Y}^c \varphi = \frac{1}{2}(e^f - 1)\omega_\mathcal{Y}^2,$$

that is

$$\Delta_{g'_\varphi} \varphi = (1 - e^f) \frac{\omega_\mathcal{Y}^2}{(\omega'_\varphi)^2},$$

where $\omega'_\varphi = \omega_\mathcal{Y} + \frac{1}{2} dd_\mathcal{Y}^c \varphi = \frac{1}{2}(\omega_\mathcal{Y} + \omega_\varphi)$ is indeed a Kähler form, and g'_φ is the associated metric, which enjoys mutual bounds with $g_\mathcal{Y}$, and is bounded at all orders with respect to this reference metric.

Now, $(1 - e^f) \in C^1_{\beta+2}(\mathcal{Y}, \omega_\mathcal{Y})$ and $\omega_\mathcal{Y}^2 / (\omega'_\varphi)^2$ is bounded up to order 1 (at least) for $\omega_\mathcal{Y}$; therefore, if we fix $\beta_1 \in (0, \beta)$, by taking $\alpha \in (0, 1)$ small enough, we have $(1 - e^f)(\omega_\mathcal{Y}^2 / (\omega'_\varphi)^2) \in C^{0,\alpha}_{\beta_1+2}(\mathcal{Y}, g'_\varphi)$. On the other hand, by [Hei10, Proposition 3.16], (i)–(ii), one has some u bounded for g'_φ up to order $(2, \alpha)$ which satisfies

$$\Delta_{g'_\varphi} u = (1 - e^f) \frac{\omega_\mathcal{Y}^2}{(\omega'_\varphi)^2},$$

and such that $|u| \leq C\rho^{-\delta}$ for any (fixed) $\delta \in (0, \beta_1)$, where C is a constant. Here, one should mention that the ρ we use is the pull-back of the radius function R from \mathbb{R}^3 , which indeed satisfies $|d\rho|_{g'_\varphi} + \rho|\Delta_{g'_\varphi} \rho| \leq C$, as g'_φ is mutually bounded with $g_\mathcal{Y}$; moreover, the result we use is stated for ‘SOB(3)-metrics’ such as $g_\mathcal{Y}$, but a reading of its proof shows it still holds for a metric uniformly co-bounded to it up to order 2 such as g'_φ , at least when these metrics are Kähler for a common complex structure.

We are thus done if we prove that

$$h := \varphi - u,$$

which is bounded, and $\Delta_{g'_\varphi}$ -harmonic by construction, is in fact *constant*. Now, observe that:

- (i) g_φ is complete and Ricci-flat;
- (ii) for any C^2_{loc} -function w , $\Delta_{g'_\varphi} w$ can be written as

$$-m^{-1} \operatorname{div}^{g_\varphi} (m \mathcal{A} \operatorname{grad}_{g_\varphi} w),$$

with $m = (\det^{g_\varphi} g'_\varphi)^{1/2}$, and where \mathcal{A} is given in coordinates by $\mathcal{A} = (g'_\varphi)^{ij} (g_\varphi)_{jk} (\partial / \partial x^i) \otimes dx^k$. In other words, $\mathcal{A} v = (v^\sharp_{g_\varphi})^\flat_{g'_\varphi}$.

As g_φ and g'_φ are both mutually bounded with g_Y , uniformly on \mathcal{Y} , hence mutually bounded with one another uniformly on \mathcal{Y} , giving us the existence of $\mu, \alpha \geq 1$ such that on \mathcal{Y} ,

$$\mu^{-1} \leq m \leq \mu, \quad \text{and} \quad \alpha^{-1}|v|_{g_\varphi} \leq g_\varphi(\mathcal{A}v, v) \quad \text{and} \quad |\mathcal{A}v|_{g_\varphi}^2 \leq \alpha|v|_{g_\varphi} \quad \text{for all } v \in T\mathcal{Y},$$

(see [Sal92, p. 424]), we can apply [Sal92, Theorem 7.4] to $L = \Delta_{g'_\varphi}$ and the bounded (below) L -harmonic function h . This immediately yields the conclusion that h is constant, and the proposition, as $h = \varphi - u$ with $u = O(\rho^{-\delta})$ on \mathcal{Y} . \square

Remark 3.2. As mentioned in the beginning of this part, an alternative to the above linear harmonic analysis, more in the spirit of ‘*a priori* estimates’, and adapted as well to higher dimensional or non-Ricci-flat contexts, could be followed from Hein’s arguments, and more precisely from the proof of [Hei10, Proposition 4.1] itself, as suggested in [Hei10, § 4.5] (‘*Non-parabolic manifolds*’). Indeed, the solution φ of (49) we use is itself constructed as the locally uniform limit, as $\varepsilon \rightarrow 0^+$, of solutions φ_ε to the relaxed equations

$$(\omega_Y + dd_Y^c \varphi_\varepsilon)^2 = e^{f+\varepsilon \varphi_\varepsilon} \omega_Y^2.$$

Now, a barrier argument (application of the maximum principle to functions of shape $\varphi_\varepsilon \pm C\rho^{-\delta}$, $\delta \in (0, \beta)$, on some exterior domain) can be invoked in this framework. As the construction of φ already relies on a C^0 -bound on the φ_ε uniform in ε , and as the exterior domain, the parameters C and δ above can be chosen independently of ε , the only remaining (moderate) price to pay to make this method work and get an estimate $|\varphi_\varepsilon| \leq C\rho^{-\delta}$ uniform in ε (hence surviving the $\varepsilon \rightarrow 0$, giving $|\varphi| \leq C\rho^{-\delta}$, which actually reveals that $\varphi = u$ from the beginning in the above proof), is the (qualitative) vanishing at infinity of the φ_ε . This in turn follows from a classical yet careful integration by parts/Moser iteration scheme, applied to the φ_ε together with cut-offs centred at points going to infinity as test-functions; we refer to [Hei10, § 4.5] and references therein for the details.

3.2 Higher-order weighted estimates

We now prove that our solution φ of (49), which we assume *from now on normalised so as to vanish at infinity on \mathcal{Y}* , lies in $C^\infty_\delta(\mathcal{Y}, g_Y)$, where $\delta \in (0, \beta)$ is as in Proposition 3.1.

This is a straightforward consequence of the following, which mimics [Joy00, Theorem 8.6.11].

PROPOSITION 3.3. *Let $k \geq 3$, $\alpha \in (0, 1)$, and assume that $\varphi \in C^0_\delta$, $0 < \delta < \beta$, that φ is bounded up to order $(k + 2, \alpha)$ with respect to ω_Y , and verifies $(\omega_Y + dd_Y^c \varphi)^2 = e^f \omega_Y^2$.*

Then $\varphi \in C^{k+2, \alpha}_\delta(\mathcal{Y}, g_Y)$; more precisely, there exists a constant $Q_{\alpha, \delta}^{(k)}$ depending only on δ, α , $\|\varphi\|_{C^0_\delta}$ and $\|f\|_{C^{k, \alpha}_{\beta+2}(\mathcal{Y}, g_Y)}$ such that $\|\varphi\|_{C^{k+2, \alpha}_\delta(\mathcal{Y}, g_Y)} \leq Q_{\alpha, \delta}^{(k)}$.

Proof. As in [Joy00, § 8.6.3], this statement readily follows from the inductive use of the following technical lemma.

LEMMA 3.4. *Let $K_1, K_2 > 0$, $\lambda \in [0, 1]$, and $k \geq 3$. Then there exists K_3 depending only on $\alpha, \beta, \delta, \|\varphi\|_{C^0_\delta}$ and K_1, K_2, λ, k such that the following holds.*

Under the assumptions of Proposition 3.3 and if $\|f\|_{C^{k, \alpha}_{\beta+2}} \leq K_1$, $\|(\nabla^j)^l dd_Y^c \varphi\|_{C^0_{\lambda l}} \leq K_2$, $l = 0, \dots, k$, and $\|[(\nabla^j)^k dd_Y^c \varphi]^\alpha_{\lambda k + (\lambda - 1)\alpha}\| \leq K_2$, then $\|(\nabla^j)^l \varphi\|_{C^0_{\lambda l + \delta}} \leq K_3$, $l = 0, \dots, k + 2$, and $\|[(\nabla^j)^{k+2} \varphi]^\alpha_{\delta + \lambda(k+2) + (\lambda - 1)\alpha}\| \leq K_3$.

In this statement, the Hölder moduli are those of g_y , defined by (24) applied to g_y (hence the slight shift of notation by comparison with [Joy00]). We recall the ideas of the proof of Lemma 3.4 after the current proof. Now, for any fixed $k \geq 3$, one starts with applying this lemma to φ with $\lambda_0 = 0$, thanks to Proposition 3.1; one can then apply it with $\lambda_1 = \delta/(k + \alpha)$ and so on, with $\lambda_n = \min\{[(1 + 2/(k + \alpha))^n - 1]\delta/2, 1\}$. \square

Proof of Lemma 3.4. As the proof is an (almost) immediate retranscription of Joyce’s from the ALE to the ALF framework, we will be brief and, while exposing the main lines, ensure that the change of geometry is harmless. In fact, the major change here is that the injectivity radius does not grow as fast as ρ , but instead remains bounded, essentially by half the length of the fibres of ϖ . This is not an issue. Indeed, the Riemannian exponential map still authorises the following manipulations. Given $x \in \{\rho \geq 2\rho_0\}$ (ρ_0 determined later), identify $(T_x\mathcal{Y}, J_{y,x}, g_{y,x})$ with $(\mathbb{C}^2, I_1, \mathbf{e})$. Taking $R > 0$, the map $\pi_{x,R} : B_{\mathbf{e}}(0, 1) \rightarrow B_{g_y}(x, R)$, $u \mapsto \exp_x^{g_y}(Ru)$, is not a diffeomorphism in general: large balls wrap following asymptotically the fibres. As it is a smooth covering however, we can define the operator $P_{x,R} : C^{k+2,\alpha}(B_{\mathbf{e}}(0, 1)) \rightarrow C^{k,\alpha}(B_{\mathbf{e}}(0, 1))$ by

$$P_{x,R}(v) = R^2 \frac{(\pi_{x,R}^*(dd_{\mathcal{Y}}^c))(v) \wedge \pi_{x,R}^*(\omega_{\mathcal{Y}} + \omega_{\varphi})}{\pi_{x,R}^*(\omega_{\mathcal{Y}}^2)}.$$

One then takes $R = L\rho(x)^\lambda$, with $L = L(\rho_0, \lambda, g_y)$ small enough so that $B_{g_y}(x, R) \subset \{\rho \geq \rho_0\}$; this way one has

$$\|R^{-2}\pi_{x,R}^*g_y - \mathbf{e}\|_{C^{k,\alpha}(B_{\mathbf{e}}(0,1))} \leq \frac{1}{2}$$

and

$$\|R^{-2}\pi_{x,R}^*\omega_{\mathcal{Y}} - \omega_{\mathbf{e}}\|_{C^{k,\alpha}(B_{\mathbf{e}}(0,1))} \leq \frac{1}{2} \quad \text{for all } x \in \{\rho \geq 2\rho_0\},$$

if ρ_0 is chosen large enough, thanks to the asymptotic geometry of g_y . Now the rest of Joyce’s proof applies unchanged (in particular, one is brought to using Schauder estimates between the fixed balls $B_{\mathbf{e}}(0, 2)$ and $B_{\mathbf{e}}(0, 1)$, with a $C^{3,\alpha}$ uniformly elliptic family of operators), since the identity

$$P_{x,R}((\pi_{x,R})^*\varphi) = R^2(e^{(\pi_{x,R})^*f} - 1)$$

is again just a rewriting of the pulled-back Monge–Ampère equation (49) verified by φ . This gives the desired estimates near infinity, the estimates on the fixed compact subset $\{\rho \leq 2\rho_0\}$ being immediately deduced from the uniform bounds on φ for g_y . \square

3.3 Refinement of the decay, and conclusion of the proof of Theorem 1.16

We have now a rather sharp estimate on $dd_{\mathcal{Y}}^c\varphi$, and thus, in particular, on $g'_{\varphi} - g_y$; we can therefore state the following.

PROPOSITION 3.5. *For all $k \geq 0$, $\alpha \in (0, 1)$, and $\nu \in (0, 1)$, the map*

$$\Delta_{g'_{\varphi}} : C_{\nu}^{k+2,\alpha}(\mathcal{Y}, g'_{\varphi}) \longrightarrow C_{\nu+2}^{k,\alpha}(\mathcal{Y}, g'_{\varphi})$$

is an isomorphism.

Proof. This is deduced from [BM11, Appendix], thanks to the estimate $g'_{\varphi} - g_y \in C_{\delta+2}^{\infty}(\mathcal{Y}, g_y)$. \square

The proof of Theorem 1.16 is now easily concluded as follows. Recall that

$$\Delta_{g'_\varphi} \varphi = (1 - e^f) \frac{\omega_{\mathcal{Y}}^2}{(\omega'_\varphi)^2};$$

this can now be rewritten, knowing that $\omega'_\varphi - \omega_{\mathcal{Y}} \in C_{\delta+2}^\infty(\mathcal{Y}, g'_\varphi)$, as $\Delta_{g'_\varphi} \varphi \in C_{\beta+2}^\infty(\mathcal{Y}, g'_\varphi)$. An immediate use of Proposition 3.5 (together with the fact that g'_φ -harmonic functions vanishing at infinity are trivial) thus provides that $\varphi \in C_\beta^\infty(\mathcal{Y}, g'_\varphi)$, which is equivalent to: $\varphi \in C_\beta^\infty(\mathcal{Y}, g_{\mathcal{Y}})$.

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Appendix A. The Taub-NUT metric on \mathbb{C}^2

A.1 A potential for the Taub-NUT metric on \mathbb{C}^2

In [LeB91] LeBrun leaves the following exercise to his reader: let m be a positive parameter, and u and v implicitly defined on \mathbb{C}^2 by the following:

$$\begin{aligned} |z_1| &= e^{m(u^2-v^2)}u, \\ |z_2| &= e^{m(v^2-u^2)}v \end{aligned} \tag{A.1}$$

(we do not make the dependence on m apparent here, since for now we see this parameter as fixed; we shall only add m as an index by places to emphasise this dependence).

PROPOSITION A.1 (LeBrun). *The metric \mathbf{f} associated to the form*

$$\omega_{\mathbf{f}} := \frac{1}{4} dd^c(u^2 + v^2 + m(u^4 + v^4))$$

for the standard complex structure I_1 on \mathbb{C}^2 is the Taub-NUT metric.

We shall give our own, direct proof here. Before this, we shall mention that LeBrun’s potential may be obtained by hyperkähler quotient considerations; we chose to give a less conceptual proof though since it exhibits several objects we use back in this article.

LEMMA A.2. *The metric \mathbf{f} is Ricci-flat; more precisely, $\omega_{\mathbf{f}}^2 = 2\Omega_{\mathbf{e}}$, where we recall that $\Omega_{\mathbf{e}}$ is the standard volume form $(idz_1 \wedge d\bar{z}_1 \wedge idz_2 \wedge d\bar{z}_2)/4$.*

Proof. We start by the computation of $\omega_{\mathbf{f}}$, which goes through that of $\partial u/\partial z_j, \partial v/\partial z_j, j = 1, 2$. One has

$$\begin{aligned} \frac{\partial u}{\partial z_1} &= \frac{1 + 2mv^2}{(2z_1)(1 + 2m(u^2 + v^2))}u, & \frac{\partial u}{\partial z_2} &= \frac{muv^2}{z_2(1 + 2m(u^2 + v^2))}, \\ \frac{\partial v}{\partial z_1} &= \frac{mu^2v}{z_1(1 + 2m(u^2 + v^2))}, & \frac{\partial v}{\partial z_2} &= \frac{1 + 2mu^2}{(2z_2)(1 + 2m(u^2 + v^2))}v. \end{aligned} \tag{A.2}$$

Indeed, differentiating the relation $|z_1| = e^{m(u^2-v^2)}u$ with $\partial/\partial z_1$ yields

$$\frac{1}{2} \frac{|z_1|}{z_1} = \left[m \left(2u^2 \frac{\partial u}{\partial z_1} - 2uv \frac{\partial v}{\partial z_1} \right) + \frac{\partial u}{\partial z_1} \right] e^{m(u^2-v^2)},$$

hence, writing $e^{m(u^2-v^2)} = |z_1|/u$, $u = 2z_1[(1 + 2mu^2)(\partial u/\partial z_1) - 2muv(\partial v/\partial z_1)]$. Similarly, applying $\partial/\partial z_1$ to the relation $|z_2| = e^{m(v^2-u^2)}v$, one gets $0 = (1 + 2mv^2)(\partial v/\partial z_1) - 2muv(\partial u/\partial z_1)$, that is $\partial v/\partial z_1 = (2muv/(1 + 2mv^2))(\partial u/\partial z_1)$. Substituting in the previous equality, one gets (A.2) for $\partial u/\partial z_1, \dots, \partial v/\partial z_2$.

Now set $\varphi = \frac{1}{4}(u^2 + v^2 + m(u^4 + v^4))$. According to (A.2),

$$2 \frac{\partial \varphi}{\partial z_1} = u(1 + 2mu^2) \frac{\partial u}{\partial z_1} + v(1 + 2mv^2) \frac{\partial v}{\partial z_1} = \frac{(1 + 2mv^2)u^2}{2z_1}$$

and $2(\partial \varphi/\partial z_2) = (1 + 2mu^2)v^2/2z_2$, i.e. $\partial \varphi/\partial \bar{z}_1 = (1 + 2mv^2)u^2/4\bar{z}_1$ and $\partial \varphi/\partial \bar{z}_2 = (1 + 2mu^2)v^2/4\bar{z}_2$ by conjugation. Apply again $\partial/\partial z_1$ and $\partial/\partial z_2$ to those equalities, as well as the relation $uv = |z_1 z_2|$ and (A.2); then, setting $R = \frac{1}{2}(u^2 + v^2)$,

$$\begin{aligned} \omega_f &= dd^c \varphi \\ &= \left(\frac{u^2(1 + 2mv^2)}{2|z_1|^2(1 + 4mR)} + m|z_2|^2 \right) idz_1 \wedge d\bar{z}_1 + m\bar{z}_2 z_1 \left(1 + \frac{1}{1 + 4mR} \right) idz_2 \wedge d\bar{z}_1 \\ &\quad + m\bar{z}_1 z_2 \left(1 + \frac{1}{1 + 4mR} \right) idz_1 \wedge d\bar{z}_2 + \left(\frac{v^2(1 + 2mu^2)}{2|z_2|^2(1 + 4mR)} + m|z_1|^2 \right) idz_2 \wedge d\bar{z}_2. \end{aligned}$$

A direct computation of ω_f^2 , using again $uv = |z_1 z_2|$, brings the conclusion. □

Remark A.3. With the above definition of R and (A.1), one gets

$$2R \leq r^2 \leq 2Re^{4mR}, \tag{A.3}$$

(with equality along $\{|z_1| = |z_2|\}$ and $\{z_1 z_2 = 0\}$, respectively); this implies that R is proper on \mathbb{C}^2 .

Recall S^1 acts on \mathbb{C}^2 by $\alpha \cdot (z_1, z_2) = (e^{i\alpha} z_1, e^{-i\alpha} z_2)$; the associated infinitesimal action is generated by the vector field $\xi = i(z_1(\partial/\partial z_1) + \bar{z}_2(\partial/\partial \bar{z}_2) - z_2(\partial/\partial z_2) - \bar{z}_1(\partial/\partial \bar{z}_1))$. By invariance of u and v under this circle action, clearly, $\xi \cdot u = \xi \cdot v = \xi \cdot \varphi = 0$, and similarly $\mathcal{L}_\xi \omega_f = 0$. This holds as well for the holomorphic symplectic $(2, 0)$ -form $\Theta := dz_1 \wedge dz_2$ (notice that $\Theta \wedge \bar{\Theta} = 4\Omega_e = 2\omega_f^2$), thus $\mathcal{L}_\xi \Theta = 0$. More precisely, $\iota_\xi \Theta = (z_1 dz_2 + z_2 dz_1) = d(iz_1 z_2)$; a complex hamiltonian $H = y_2 + iy_3$ for the S^1 -action on (\mathbb{C}^2, Θ) is thus given by $y_2 := \Im(z_1 z_2)$ and $y_3 := -\Re(z_1 z_2)$.

In the same way, $\mathcal{L}_\xi d^c \varphi = 0$; as $\mathcal{L}_\xi d^c \varphi = \iota_\xi dd^c \varphi + d(\iota_\xi d^c \varphi)$ (Cartan's formula), i.e. $\iota_\xi \omega_f = -d(d^c \varphi(\xi))$, we are led to setting $y_1 = d^c \varphi(\xi)$. All computations are done.

LEMMA A.4. One has $y_1 = \frac{1}{2}(u^2 - v^2)$, and thus R indeed equals $(y_1^2 + y_2^2 + y_3^2)^{1/2}$.

Proof. To see that, $y_1 = \frac{1}{2}(u^2 - v^2)$, write, according to the proof of Lemma A.2,

$$d^c \varphi = i(1 + 2mv^2)u^2 \left(\frac{d\bar{z}_1}{2\bar{z}_1} - \frac{dz_1}{2z_1} \right) + i(1 + 2mu^2)v^2 \left(\frac{d\bar{z}_2}{2\bar{z}_2} - \frac{dz_2}{2z_2} \right),$$

hence the result, from the identity $\xi = i(z_1(\partial/\partial z_1) + \bar{z}_2(\partial/\partial \bar{z}_2) - z_2(\partial/\partial z_2) - \bar{z}_1(\partial/\partial \bar{z}_1))$.

Noticing that $y_2^2 + y_3^2 = |z_1 z_2|^2 = u^2 v^2$ suffices to get $y_1^2 + y_2^2 + y_3^2 = \frac{1}{4}(u^2 + v^2)^2$. □

LEMMA A.5. Set $V = |\xi|_f^{-2}$. Then $|\xi|_f^2 = 2R/(1 + 4mR)$, and hence $V = 2m(1 + (1/4mR))$.

Proof. One has $I_1 \xi = -z_1(\partial/\partial z_1) - \bar{z}_1(\partial/\partial \bar{z}_1) + \bar{z}_2(\partial/\partial \bar{z}_2) + z_2(\partial/\partial z_2)$; the easy but tedious calculation of $|\xi|_f^2 = \omega_f(\xi, I_1 \xi)$ then follows, which can be made easier by noticing that $idz_1 \wedge d\bar{z}_1(\xi, I_1 \xi) = 2|z_1|^2$, $idz_1 \wedge d\bar{z}_2(\xi, I_1 \xi) = -2z_1 \bar{z}_2$, and so on. □

To get the Taub-NUT metric back under its classical form, we need finally a 1-form η , which is also a connection 1-form for the circle fibration $\varpi : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$, $(z_1, z_2) \mapsto (y_1, y_2, y_3)$. The natural candidate is given by $\eta := VI_1 dy_1$.

LEMMA A.6. On $\mathbb{C}^2 \setminus \{z_1 z_2 = 0\}$, one has

$$\eta = \frac{i}{4R} \left[u^2 \left(\frac{d\bar{z}_1}{\bar{z}_1} - \frac{dz_1}{z_1} \right) - v^2 \left(\frac{d\bar{z}_2}{\bar{z}_2} - \frac{dz_2}{z_2} \right) \right], \tag{A.4}$$

and $\eta(\xi) = 1$ outside of 0.

Proof. By definition, $\eta = Vd^c y_1 = \frac{1}{2}iV(2u(\bar{\partial}u - \partial u) - 2v(\bar{\partial}v - \partial v))$. We then apply (A.2), which we rewrite as

$$V \frac{\partial u}{\partial z_1} = \frac{1 + 2mv^2}{4z_1 R} u, \quad V \frac{\partial u}{\partial z_2} = \frac{mu^2 v}{2z_2 R}, \quad V \frac{\partial v}{\partial z_1} = \frac{mu^2 v}{2z_1 R}, \quad V \frac{\partial v}{\partial z_2} = \frac{1 + 2mu^2}{4z_2 R} v,$$

hence the component of η in the dz_1 direction is $-iu((1 + 2mv^2)/4z_1 R)u + iv(mu^2 v/2z_1 R) = -iu^2/4z_1 R$, and so on. A straightforward computation suffices to see that $\eta(\xi) = 1$. \square

We shall now recover the Taub-NUT metric under a more familiar shape.

LEMMA A.7. On $\mathbb{C}^2 \setminus \{0\}$, $\omega_{\mathbf{f}} = dy_1 \wedge \eta + Vdy_2 \wedge dy_3$, hence $\mathbf{f} = V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta^2$.

Proof. Clearly, $\{dy_2, dy_3\}$ is linearly independent at all points of $\mathbb{C}^2 \setminus \{0\}$, and those forms vanish against ξ by \mathbb{S}^1 -invariance. They vanish as well against $I_1\xi$, as $I_1 dy_2 = dy_3$. Since $dy_1(\xi) = 0$ (y_1 is \mathbb{S}^1 -invariant) and $dy_1(I_1\xi) = -V^{-1} \neq 0$ as $I_1 dy_1 = V^{-1}\eta$, and since $\eta(I_1\xi) = 0$ and $\eta(\xi) = 1$, we deduce that $\{dy_1, dy_2, dy_3, \eta\}$ is linearly independent outside $\{0\}$. Consequently, on $\mathbb{C}^2 \setminus \{0\}$, one can write

$$\omega_{\mathbf{f}} = \alpha dy_1 \wedge \eta + \beta dy_2 \wedge \eta + \gamma dy_3 \wedge \eta + \delta dy_1 \wedge dy_2 + \varepsilon dy_1 \wedge dy_3 + \zeta dy_2 \wedge dy_3$$

for some functions α, \dots, ζ . Now $\alpha dy_1 + \beta dy_2 + \gamma dy_3 = -\iota_{\xi} \omega_{\mathbf{f}} = dy_1$, thus $\alpha = 1$ and $\beta = \gamma = 0$; as $\omega_{\mathbf{f}}$ is of type $I_1 - (1, 1)$, one also has $\delta = \varepsilon = 0$.

To determine ζ , one evaluates $\zeta dy_1 \wedge \eta \wedge dy_2 \wedge dy_3 = \frac{1}{2}\omega_{\mathbf{f}}^2 = \Omega_{\mathbf{e}}$ on $(-I_1\xi, \zeta)$; this gives $V^{-1}\zeta dy_2 \wedge dy_3 = \Omega_{\mathbf{e}}(-I_1\xi, \xi, \cdot, \cdot) = \frac{1}{2}(|z_2|^2 idz_1 \wedge d\bar{z}_1 + |z_1|^2 idz_2 \wedge d\bar{z}_2 + \bar{z}_1 z_2 idz_1 \wedge d\bar{z}_2 + z_1 \bar{z}_2 idz_2 \wedge d\bar{z}_1) = dy_2 \wedge dy_3$, hence $\zeta = V$. \square

One easily checks that η is a connection 1-form away from 0 for the fibration $\varpi = (y_1, y_2, y_3)$: it is \mathbb{S}^1 -invariant, and at any point p but $0 \in \mathbb{C}^2$, as $\{\eta, dy_1, dy_2, dy_3\}$ is a basis of $T_p^*\mathbb{C}^2$, necessarily, $T_p\mathbb{C}^2 = \ker \eta + \ker T\varpi$. Finally, $d\eta$ has the expected shape.

LEMMA A.8. The differential of η is given on $\mathbb{C}^2 \setminus \{0\}$ by

$$d\eta = *_{\mathbb{R}^3} dV.$$

Proof. The 1-form η is \mathbb{S}^1 -invariant and $\eta(\xi)$ is constant; by Cartan’s formula, $0 = \mathcal{L}_{\xi}\eta = \iota_{\xi}d\eta + d(\iota_{\xi}\eta) = \iota_{\xi}d\eta$, i.e. the components of $d\eta$ in the $dy_j \wedge \eta$ -directions vanish. Moreover $d\omega_{\mathbf{f}} = 0$ thus according to Lemma A.7, $d\eta = (\partial V/\partial y_1)dy_2 \wedge dy_3 + \alpha_2 dy_3 \wedge dy_1 + \alpha_3 dy_1 \wedge dy_2$. For the computation of α_2 and α_3 , observe that

$$4\eta = \left(1 + \frac{y_1}{R}\right) d^c \log(|z_1|^2) - \left(1 - \frac{y_1}{R}\right) d^c \log(|z_2|^2),$$

as $u^2 = R + y_1$ and $v^2 = R - y_1$. Since $dd^c \log(|z_1|^2) = dd^c \log(|z_2|^2) = 0$ outside of $\{z_1 z_2 = 0\}$, we thus have $d\eta = \frac{1}{4}d(y_1/R) \wedge d^c \log(|z_1 z_2|^2) = \frac{1}{4}d(y_1/R) \wedge d^c \log(y_2^2 + y_3^2)$. Now

$$d\left(\frac{y_1}{R}\right) = \frac{1}{R^3}((y_2^2 + y_3^2)dy_1 - y_1 y_2 dy_2 - y_1 y_3 dy_3)$$

and

$$d^c \log(y_2^2 + y_3^2) = I_1 d \log(y_2^2 + y_3^2) = 2 \frac{y_2 dy_3 - y_3 dy_2}{y_2^2 + y_3^2};$$

this clearly provides $\alpha_j = -y_j/2R^3 = \partial V/\partial y_j, j = 2, 3$. The lemma is proved, outside of $\{z_1 z_2 = 0\}$, and the formula extends at once to $\mathbb{C}^2 \setminus \{0\}$ by continuity. \square

A.2 Comparison of the Euclidean and the Taub-NUT metrics

A.2.1 *Mutual control.* The metrics \mathbf{e} and \mathbf{f} are far from being globally mutually bounded; an example of this geometric gap can be read in the scale of the ball volume growth: r^4 in the Euclidean regime, but R^3 for Taub-NUT; notice that R plays the role of the distance to 0 on $(\mathbb{C}^2, \mathbf{f})$. Another example of the geometric gap is given by the length of the orbit of the S^1 -action on \mathbb{C}^2 used above: the orbit of $x \in \mathbb{C}^2 \setminus \{0\}$ has length $2\pi|x|_{\mathbf{e}}$ under \mathbf{e} , and length $2\pi V(x)^{-1/2}$ when measured by \mathbf{f} ; this latter length tends to $\pi\sqrt{2/m}$ when x goes ∞ , which gives us a geometric interpretation of the parameter m . We can nonetheless still compare \mathbf{e} and \mathbf{f} as follows.

PROPOSITION A.9. *There exists some constant $C > 0$ such that on \mathbb{C}^2 minus its unit ball,*

$$C^{-1}r^{-2}\mathbf{e} \leq \mathbf{f} \leq Cr^2\mathbf{e}.$$

Proof. As $\mathbf{f} = V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta^2$, with $\eta = I_1 V dy_1$ and $dy_3 = I_1 dy_2$, we evaluate $|dy_1|_{\mathbf{e}}$ and $|dy_2|_{\mathbf{e}}$ first; since $dy_2 = (i/2)(z_1 dz_2 + z_2 dz_1 - \bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1)$, we readily get $|dy_2|_{\mathbf{e}} = cr$. Now, we rearrange (A.1) to write

$$dy_1 = \frac{1}{2(1 + 4mR)}(e^{-4my_1}(\bar{z}_1 dz_1 + z_1 d\bar{z}_1) - e^{4my_1}(\bar{z}_2 dz_2 + z_2 d\bar{z}_2)). \tag{A.5}$$

This provides $|dy_1|_{\mathbf{e}}^2 = (c/(1 + 4mR)^2)(|z_1|^2 e^{-8my_1} + |z_2|^2 e^{8my_1})$. But $|z_1|^2 e^{-4my_1} = u^2$ and $|z_2|^2 e^{4my_1} = v^2$, so $|dy_1|_{\mathbf{e}}^2 = (c/(1 + 4mR)^2)(e^{-4my_1} u^2 + e^{4my_1} v^2) = (c'/(1 + 4mR)^2)(R \cosh(4my_1) - y_1 \sinh(4my_1))$. Now $R \cosh(4my_1) - y_1 \sinh(4my_1) \leq R \cosh(4my_1) + y_1 \sinh(4my_1)$ is obvious, and rearranging (A.1) gives also

$$2(R \cosh(4my_1) + y_1 \sinh(4my_1)) = r^2, \tag{A.6}$$

so finally $|dy_1|_{\mathbf{e}}^2 \leq c(r^2/R^2)$. Those estimates give us the bound $\mathbf{f} \leq Cr^2\mathbf{e}$.

The reverse bound $\mathbf{e} \leq Cr^2\mathbf{f}$ follows at once, as \mathbf{e} and \mathbf{f} are hermitian, have the same volume form, and as we are in complex dimension 2. \square

A.2.2 *Expressing Euclidean objects in Taub-NUT vocabulary.* We give here some further material useful in the comparison between \mathbf{e} and \mathbf{f} on \mathbb{C}^2 . In Lemma A.10 we introduce a vector field ζ helping to complete the dual frame of $(V^{-1/2}\eta, V^{1/2}dy_1, V^{1/2}dy_2, V^{1/2}dy_3)$ for \mathbf{f} . Then in Lemma A.11, we express the canonical frames of 1-forms and vector fields of \mathbf{e} , i.e. the dx_j and the $\partial/\partial x_j$, in terms of those of \mathbf{f} . The essential point in those expressions lies in their computational consequences; indeed, they allow to compute objects like $\nabla^{\mathbf{f}} dx_j$, and estimate quantities like

$|\nabla^f dx_j|_f$, which is required when manipulating Euclidean objects in the Taub-NUT setting; see e.g. the proof of Proposition 1.7.

In § A.1, we used the vector field ξ on \mathbb{C}^2 , which verified $\eta(\xi) = 1$, $dy_j(\xi) = 0$, $j = 1, 2, 3$, and $dy_1(I_1\xi) = -1/V$, $\eta(I_1\xi) = dy_2(I_1\xi) = dy_3(I_1\xi) = 0$. We shall complete our dual frame with the help of another vector field.

LEMMA A.10. Define on $\mathbb{C}^2 \setminus \{0\}$ the vector field

$$\zeta = \frac{1}{2iR} \left(e^{4my_1} \left(z_2 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial z_1} \right) + e^{-4my_1} \left(z_1 \frac{\partial}{\partial \bar{z}_2} - \bar{z}_1 \frac{\partial}{\partial z_2} \right) \right).$$

Then $dy_2(\zeta) = 1$ whereas $\eta(\zeta) = dy_1(\zeta) = dy_3(\zeta) = 0$, and $dy_3(I_1\zeta) = 1$ whereas $\eta(I_1\zeta) = dy_1(I_1\zeta) = dy_2(I_1\zeta) = 0$. Moreover, $[\xi, \zeta] = 0$.

Proof. We only need to check the first list of equalities, as $dy_3 = I_1 dy_2$ and $\eta = I_1 V dy_1$. Since $dy_2 = (1/2i)(z_1 dz_2 + z_2 dz_1 - \bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1)$, we get

$$dy_2(\zeta) = \frac{1}{2R} (e^{4my_1} |z_2|^2 + e^{-4my_1} |z_1|^2);$$

now $e^{4my_1} |z_2|^2 = v^2$, $e^{-4my_1} |z_1|^2 = u^2$, and $R = \frac{1}{2}(u^2 + v^2)$, hence $dy_2(\zeta) = 1$. Using that $dy_3 = -\frac{1}{2}(z_1 dz_2 + z_2 dz_1 + \bar{z}_1 d\bar{z}_2 + \bar{z}_2 d\bar{z}_1)$ readily gives $dy_3(\zeta) = 0$. Now for the equality $dy_1(\zeta) = 0$, we use (A.5) to write $dy_1(\zeta) = (1/4iR(1 + 4mR))(z_1 z_2 - \bar{z}_1 \bar{z}_2 - z_1 z_2 + \bar{z}_1 \bar{z}_2) = 0$; likewise, equality $\eta(\zeta) = 0$ follows from formula (A.4).

Finally, the \mathbb{S}^1 -invariance of ζ provides $[\xi, \zeta] = 0$. □

LEMMA A.11. One has the following formulas for 1-forms:

$$\begin{aligned} dx_1 &= V x_1 dy_1 - x_2 \eta + \frac{e^{4my_1}}{2R} (x_4 dy_2 - x_3 dy_3), \\ dx_2 &= V x_2 dy_1 + x_1 \eta + \frac{e^{4my_1}}{2R} (x_3 dy_2 + x_4 dy_3), \\ dx_3 &= -V x_3 dy_1 + x_4 \eta + \frac{e^{-4my_1}}{2R} (x_2 dy_2 - x_1 dy_3), \\ dx_4 &= -V x_4 dy_1 - x_3 \eta + \frac{e^{-4my_1}}{2R} (x_1 dy_2 + x_2 dy_3); \end{aligned}$$

and for vector fields:

$$\begin{aligned} \frac{\partial}{\partial x_1} &= -\frac{e^{-4my_1}}{2R} (x_2 \xi + x_1 I_1 \xi) + (x_4 \zeta - x_3 I_1 \zeta), \\ \frac{\partial}{\partial x_2} &= \frac{e^{-4my_1}}{2R} (x_1 \xi - x_2 I_1 \xi) + (x_3 \zeta + x_4 I_1 \zeta), \\ \frac{\partial}{\partial x_3} &= \frac{e^{4my_1}}{2R} (x_4 \xi + x_3 I_1 \xi) + (x_2 \zeta - x_1 I_1 \zeta), \\ \frac{\partial}{\partial x_4} &= \frac{e^{4my_1}}{2R} (-x_3 \xi + x_4 I_1 \xi) + (x_1 \zeta + x_2 I_1 \zeta). \end{aligned}$$

Proof. We shall only see how those formulas arise for dx_1 and $\partial/\partial x_1$; the other identities are then easily deduced with the relations $dx_2 = I_1 dx_1$, $dx_3 = \tau^* dx_1$, $dx_4 = I_1 dx_3$, etc., on the Euclidean side, and $\tau^* y_j = -y_j$, $\tau^* \eta = -\eta$, $\tau^* \xi = -\xi$, $\tau^* \zeta = -d\zeta$, etc., on the Taub-NUT side.

Write $dx_1 = \alpha dy_1 + \beta \eta + \gamma dy_2 + \delta dy_3$. By duality between $(\xi, -VI_1\xi, \zeta, I_1\zeta)$ and (η, dy_1, dy_2, dy_3) , $dx_1(\xi) = \beta$, $dx_1(I_1\xi) = -\alpha/V$, $dx_1(\zeta) = \gamma$ and $dx_1(I_1\zeta) = \delta$. On the other hand, $dx_1(\xi) = \frac{1}{2}i(z_1 - \bar{z}_1) = -x_2$, $dx_1(I_1\xi) = -\frac{1}{2}(z_1 + \bar{z}_1) = -x_1$, $dx_1(\zeta) = (1/2iR)(e^{4my_1} \frac{1}{2}(z_2 - \bar{z}_2) + 0) = (e^{4my_1}/2R)x_4$ and similarly $dx_1(I_1\zeta) = (i/2R)e^{4my_1}(i/2)(z_2 + \bar{z}_2) = -(e^{4my_1}/2R)x_3$, hence the result.

Similarly, if $\partial/\partial x_1 = \alpha\xi + \beta I_1\xi + \gamma\zeta + \delta I_1\zeta$, then $\alpha = \eta(\partial/\partial x_1) = -e^{-4my_1}x_2/2R$, $\beta = -Vdy_1(\partial/\partial x_1) = -e^{-4my_1}x_1/2R$, $\gamma = dy_2(\partial/\partial x_1) = x_4$ and $\delta = dy_3(\partial/\partial x_1) = -x_3$. \square

A.2.3 Derivatives. Consider the \mathbf{f} -orthonormal frame $(e_j)_{j=0,\dots,3}$ of vector fields given by

$$(e_0, e_1, e_2, e_3) = (V^{1/2}\xi, -V^{1/2}I_1\xi, V^{-1/2}\zeta, V^{-1/2}I_1\zeta)$$

away from 0. In Part 1, we have to estimate the $\nabla_{e_i}^{\mathbf{f}}e_j$. This we do in the following lemma.

LEMMA A.12. One has $[e_0, e_i] = (y_i/4R^3V^{3/2})e_0$ for $i = 1, 2, 3$, and

$$[e_i, e_j] = \frac{1}{4R^3V^{3/2}}(y_i e_j - y_j e_i + 2y_k e_0)$$

for any triple $(i, j, k) \in \mathcal{I} = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. As a consequence,

$$\nabla^{\mathbf{f}}e_0 = \frac{1}{4R^3V^{3/2}} \sum_{(i,j,k) \in \mathcal{I}} e_i \otimes (y_k e_j^* - y_j e_k^* - y_i e_0^*)$$

with $(e_0^*, e_1^*, e_2^*, e_3^*) = (V^{-1/2}\eta, V^{1/2}dy_1, V^{1/2}dy_2, V^{1/2}dy_3)$.

Remark A.13. Defining J_j by $\mathbf{f}(J_j, \cdot) = \omega_j^e$, $j = 2, 3$, we get two complex structures verifying with $J_1 := I_1$ the quaternionic relations, just as we did for J_2^Y and J_3^Y at the end of § 1.4.4. By Lemma A.11, we see moreover that ω_2 is exactly $dy_2 \wedge \eta + Vdy_1 \wedge dy_3$, and likewise for ω_3 , so that, for instance, $e_0 = J_1e_1 = J_2e_2 = J_3e_3$.

Proof of Lemma A.12. Once the statement on the Lie brackets is proved, the formula for $\nabla^{\mathbf{f}}e_0$ follows from Koszul formula for the Levi-Civita connection $\nabla^{\mathbf{f}}$ and the orthonormality of the frame (e_i) . Moreover, because of the symmetric roles of e_1, e_2, e_3 , we shall only see how to compute $[e_0, e_1]$ and $[e_1, e_2]$.

- $[e_0, e_1]$: this bracket is rather easy to compute. Recall that $e_0 = V^{1/2}\xi$, $e_1 = -V^{1/2}I_1\xi$, and ξ is holomorphic for I_1 , so that $[\xi, I_1\xi] = 0$. Moreover, as V is \mathbb{S}^1 -invariant, $\xi \cdot V = 0$, and $(I_1\xi) \cdot V = -V^{-1}(\partial V/\partial y_1)$. Thus,

$$\begin{aligned} [e_0, e_1] &= \mathcal{L}_{e_0}(-V^{1/2}I_1\xi) = (-e_0 \cdot V^{1/2})I_1\xi - V^{1/2}\mathcal{L}_{e_0}(I_1\xi) = 0 + V^{1/2}\mathcal{L}_{I_1\xi}e_0 \\ &= V^{1/2}((I_1\xi) \cdot V^{1/2})\xi + V\mathcal{L}_{I_1\xi}e_0 = \frac{1}{2}((I_1\xi) \cdot V)\xi + 0 = -\frac{1}{2V} \frac{\partial V}{\partial y_1} \xi, \end{aligned}$$

hence the result, as $\partial V/\partial y_1 = (\partial R/\partial y_1)(dV/dR) = -(y_1/R)(1/2R^2)$.

- $[e_1, e_2]$: as $e_1 = -V^{1/2}I_1\xi$ and $e_2 = V^{-1/2}\zeta$, by Leibniz rule,

$$\begin{aligned} [e_1, e_2] &= (e_2 \cdot V^{1/2})I_1\xi + V^{1/2}\mathcal{L}_{e_2}(I_1\xi) \\ &= -V^{-1}(\zeta \cdot V^{1/2})e_1 - V((I_1\xi) \cdot V^{-1/2})e_2 - \mathcal{L}_{I_1\xi}\zeta. \end{aligned} \tag{A.7}$$

We already know $(I_1\xi) \cdot V^{-1/2} = (1/2V^{5/2})(\partial V/\partial y_1) = -y_1/4R^3V^{1/2}$; similarly, $\zeta \cdot V^{1/2} = \frac{1}{2}V^{-1/2}(\partial V/\partial y_2) = -y_2/4R^3V^{1/2}$. We are thus left with $\mathcal{L}_{I_1\xi}\zeta$. Now $e^{tI_1\xi}(z_1, z_2) = (e^{-t}z_1, e^tz_2)$, hence $\omega_2 = dx_1 \wedge dx_3 + dx_4 \wedge dx_2$ is invariant under this flow: $\mathcal{L}_{I_1\xi}\omega_2 = 0$. Besides, $\omega_2 = dy_2 \wedge \eta + dy_3 \wedge dy_1$, so that $\omega_2(\zeta, \cdot) = \eta$, hence

$$(d\eta)(I_1\xi, \cdot) = \mathcal{L}_{I_1\xi}\eta = \mathcal{L}_{I_1\xi}(\omega_2(\zeta, \cdot)) = \omega_2(\mathcal{L}_{I_1\xi}\zeta, \cdot),$$

the first equality coming from Cartan's formula and the identity $\eta(I_1\xi) = 0$. Now by Lemma A.8, $(d\eta)(I_1\xi, \cdot) = V^{-1}((\partial V/\partial y_2)dy_3 - (\partial V/\partial y_3)dy_2)$, and thus

$$\mathcal{L}_{I_1\xi}\zeta = -\frac{1}{2R^3V}(y_3\xi + y_2I_1\xi) = -\frac{1}{2R^3V^{3/2}}(y_3e_0 - y_2e_1).$$

The conclusion follows from plugging this back into (A.7). □

A.2.4 *Proof of Lemma 1.8.* We conclude this appendix with a proof of Lemma 1.8 of Part 1; we actually prove it under the following shape:

For all $\ell \geq 0$, and $j = 1, 2, 3, 4$,

$$(a) |(\nabla^{\mathbf{f}})^\ell x_j|_{\mathbf{f}} = O(r) \quad \text{and} \quad (b) |(\nabla^{\mathbf{f}})^\ell (e^{4\varepsilon_j my_1} x_j)|_{\mathbf{f}} = O(r), \tag{A.8}$$

where $\varepsilon_1 = \varepsilon_2 = -1$ and $\varepsilon_3 = \varepsilon_4 = +1$.

With help of the Leibniz rule, and Lemmas A.11 and A.12, the verification of Lemma 1.8 then boils down to an easy verification: if for instance $\alpha = \sum_{j,k=1}^4 \alpha_{jk} dx_j \otimes dx_k$ with $\alpha = O(r^{-2a})$ and $|\nabla^e \alpha| = O(r^{-2a-1})$, $a \geq 1$, then

$$\nabla^{\mathbf{f}} \alpha = \sum_{j,k,p=1}^4 \frac{\partial \alpha_{jk}}{\partial x_p} dx_p \otimes dx_j \otimes dx_k + \sum_{j,k=1}^4 \alpha_{jk} [(\nabla^{\mathbf{f}} dx_j) \otimes dx_k + dx_j \otimes (\nabla^{\mathbf{f}} dx_k)]$$

is immediately seen to be $O(r^{-2a+2})$, hence $O(R^{1-a})$, for \mathbf{f} , thanks to these lemmas and estimates (A.8)(a); estimates (A.8)(b) are actually essentially useful in proving estimates (A.8)(a).

Let us establish these estimates. First, the case $\ell = 0$ is obvious for (a), and (b) follows from (A.6), providing $e^{\pm 2my_1} = O(r/R^{1/2})$, together with the identities

$$e^{-4my_1}(x_1^2 + x_2^2) + e^{4my_1}(x_3^2 + x_4^2) = u^2 + v^2 = 2R,$$

(direct consequence of (A.1) and $y_1 = \frac{1}{2}(u^2 - v^2)$), providing $e^{-2my_1} x_j = O(R^{1/2})$, $j = 1, 2$ and $e^{2my_1} x_j = O(R^{1/2})$, $j = 3, 4$.

We now come to the $\ell = 1$ case. Here we build on (A.8), $\ell = 0$, (a) and (b), and Lemma A.11: for instance, for $j = 1$, we get that dx_1 has $O(r)$ coefficient for $e_0^* = V^{-1/2}\eta$ and $e_1^* = V^{1/2}dy_1$, and $O(r/R)$, hence $O(r)$, coefficients for $e_2^* = V^{1/2}dy_2$ and $e_3^* = V^{1/2}dy_3$. Moreover, as

$$d(e^{-4my_1} x_1) = e^{-4my_1}(dx_1 - 4mx_1 dy_1) = (V - 4m)x_1 e^{-4my_1} dy_1 - x_2 e^{-4my_1} \eta + \frac{1}{2R}(x_4 dy_2 - x_3 dy_3),$$

one sees that $d(e^{-4my_1} x_1)$ has $O(r/R)$, hence $O(r)$, coefficients for e_0^* and e_1^* , and $O(r)$ coefficients for e_2^* and e_3^* . The verification for $j = 2, 3, 4$ goes the same way, the only occurrences of $e^{\pm 4my_1} x_p$ to be dealt with being exactly the $e^{4\varepsilon_p my_1} x_p$, already estimated in the $\ell = 0$ case.

In the $\ell = 2$ case, we build on (A.8), $\ell = 0, 1$, (a) and (b), and Lemma A.11 again, and on Lemma A.12. Indeed, by Lemma A.11, one can write

$$V^{1/2} dx_j = \sum_{k=0}^3 Q_{j,k}(x_\bullet, e^{4\varepsilon_\bullet my_1} x_\bullet, R) e_k^*,$$

with $Q_{j,k}(\xi_1, \dots, \xi_4, \xi'_1, \dots, \xi'_4, \rho) = A_{j,k}(\xi_1, \dots, \xi_4, \xi'_1, \dots, \xi'_4)B_{j,k}(\rho)$, where the $A_{j,k}$ are affine, and the $B_{j,k}$ are rational fractions with $\deg B_{j,k} \leq 0$. This way,

$$V^{1/2}\nabla^f dx_j = \nabla^f(V^{1/2}dx_{1j}) - \frac{1}{2}V^{-1/2}dV \otimes dx_j = \sum_{k=0}^3(dQ_{j,k} \otimes e_k^* + Q_{j,k}\nabla^f e_k^*) - \frac{1}{2}\frac{dV}{V} \otimes V^{1/2}dx_j.$$

Using Lemmas A.11 and A.12, this can be rewritten as

$$V^{1/2}\nabla^f dx_j = \sum_{k_1, k_2=0}^3 Q_{j, k_1, k_2}(x_\bullet, e^{4\varepsilon_\bullet my_1}, y_\bullet, R)e_{k_1}^* \otimes e_{k_2}^*, \tag{A.9}$$

with $Q_{j, k_1, k_2} = A_{j, k_1, k_2}(\xi_1, \dots, \xi_4, \xi'_1, \dots, \xi'_4, \eta_1, \eta_2, \eta_3)B_{j, k_1, k_2}(\rho)$, where the A_{j, k_1, k_2} are affine in ξ and ξ' , with coefficients polynomials of degree ≤ 1 in η , and the B_{j, k_1, k_2} rational fractions of degree ≤ -1 (we use the $V^{1/2}$ factor in (A.9) to have a simpler description here). This gives precisely $\nabla^f dx_j = O(r)$, and an analogous analysis gives $\nabla^f(e^{4\varepsilon_j my_1} dx_j) = O(r)$ as well.

For the general case, one uses the same technique inductively, leading to

$$(V^{1/2}\nabla^f)^\ell dx_j = \sum_{k_1, \dots, k_{\ell+1}=0}^3 Q_{j, k_1, \dots, k_{\ell+1}}(x_\bullet, e^{4\varepsilon_\bullet my_1}, y_\bullet, R)e_{k_1}^* \otimes \dots \otimes e_{k_{\ell+1}}^*,$$

with $Q_{j, k_1, \dots, k_{\ell+1}} = A_{j, k_1, \dots, k_{\ell+1}}(\xi_1, \dots, \xi_4, \xi'_1, \dots, \xi'_4, \eta_1, \eta_2, \eta_3)B_{j, k_1, \dots, k_{\ell+1}}(\rho)$ where the $A_{j, k_1, \dots, k_{\ell+1}}$ are affine in ξ and ξ' , with coefficients of degree $\leq \ell$ in η , and where $\deg B_{j, k_1, \dots, k_{\ell+1}} \leq -\ell$. Analogous statements hold for the $(V^{1/2}\nabla^f)^\ell(e^{4\varepsilon_j my_1} dx_j)$, for all $\ell \geq 1$. \square

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