J. Austral. Math. Soc. (*Series A*) **29** (1980), 407–416

RUDIN SYNTHESIS ON HOMOGENEOUS BANACH ALGEBRAS

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(Received 20 November 1978; revised 25 September 1979)

Communicated by E. Strzelecki

Abstract

The main results of this article are

(I) Let B be a homogeneous Banach algebra, A a closed subalgebra of B, and I the largest closed ideal of B contained in A. We assert that $\overline{\mathrm{P}(A)}^B = \overline{I+J}^B$ for some closed subalgebra J of B. Furthermore, under suitable conditions, we show that A is an R-subalgebra if and only if J is an R-subalgebra. A number of concrete closed subalgebras of a homogeneous Banach algebra therefore are R-subalgebras. For the definition of $\mathrm{P}(A)$ and that of an R-subalgebra, see the introduction in Section 1.

(II) We give sufficient and necessary conditions for a closed subalgebra of $L^p(G)$, $1 < p \leq \infty$, to be an R-subalgebra.

1980 Mathematics subject classification (*Amer. Math. Soc.*): 43 A 15, 43 A 45, 46 J 30.

1. Introduction

Throughout this article, let G be an infinite compact abelian group with character group Γ and T the circle group. Rudin (1962), Chapter 9, began to investigate the structure of closed subalgebras of $L^1(G)$ by a synthesis method, called Rudin synthesis or simply R-synthesis. As a matter of fact, Rudin's synthesis method can be applied to homogeneous Banach algebras.

DEFINITION 1.1. Let G be an infinite compact abelian group. By a homogeneous Banach algebra on G, we mean a subalgebra $B(G)$ of $L^1(G)$ which is itself a Banach algebra under suitable norm $\|\cdot\|_B$ with $\|\cdot\|_B \geqslant \|\cdot\|_1$, convolution as multiplication and possessing the following homogeneous properties:

(I) If $f \in B(G)$, $x \in G$, then $f_x \in B(G)$ and $\|f_x\|_B = \|f\|_B$ where $f_x(y) = f(y-x)$.

(II) For each f in $B(G)$, $x \to f_x$ is a continuous map of G into $(B(G), \|\cdot\|_B)$.

The research in this paper was partially supported by the National Science Council, Republic of China.

A homogeneous Banach algebra $B(G)$ is called a Segal algebra if $B(G)$ is dense in $(L^1(G), \|\cdot\|_1)$. For properties of homogeneous Banach algebras, see Šilov (1954), Reiter (1968, 1971) and Wang (1972, 1977).

A closed subalgebra A of a homogeneous Banach algebra $B(G)$ induces an equivalence relation $\sim$ on Γ where $\gamma_1 \sim \gamma_2$ if and only if $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for all f in A. Denote the equivalence classes by (Δ_α) called the Rudin classes (or simply, the R-classes) induced by A. By the Riemann–Lebesgue lemma, each Δ_α is finite except possibly for Δ_0 where $\Delta_0 = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0, \forall f \in A\}$. Let $\mathbf{P}(A)$ be the subalgebra generated by the trigonometric polynomials P_α, $\alpha \neq 0$, such that $\hat{P}_\alpha = \chi_{\Delta_\alpha}$, the characteristic function of Δ_α, and

$$A^{B(G)} = \{f \in B(G) : \hat{f}(\Delta_0) = 0, \hat{f}(\Delta_\alpha) = \text{constant}, \alpha \neq 0\}.$$

Rudin (1962), Chapter 9, proved that $\overline{\mathbf{P}(A)}^B$ and $\mathrm{A}^{B(G)}$ are the minimal and the maximal closed subalgebra of $B(G)$ inducing the same R-classes (Δ_α).

DEFINITION 1.2. A closed subalgebra A of a homogeneous Banach algebra $B(G)$ is called an R-subalgebra if $\overline{\mathbf{P}(A)}^B = A = A^{B(G)}$ or equivalently, if each $f \in B(G)$ with $\hat{f}(\Delta_0) = 0$, $\hat{f} =$ constant on Δ_α, $\alpha \neq 0$, can be approximated by trigonometric polynomials P such that $\hat{P}$ are constant on $\Delta_\alpha, \alpha \neq 0$. We say that Rudin synthesis (or simply, R-synthesis) holds for $B(G)$ if every closed subalgebra of $B(G)$ is an R-subalgebra. Otherwise we say that Rudin synthesis fails for $B(G)$.

Kahane (1965) and Rider (1969) proved that R-synthesis fails for $L^1(T)$ and $L^p(T)$, $1 < p < 2$, respectively. Tseng-Wang (1975) proved that, for the $A^p(T)$-algebras, R-synthesis holds for $1 \leqslant p \leqslant 2$ and fails for $2 < p < \infty$. In this article, we study R-synthesis of homogeneous Banach algebras through their largest closed ideals. For a closed subalgebra A of a homogeneous Banach algebra $B(G)$, we first assert that $\overline{P(A)}^B = \overline{I+J}^B$ where I is the largest closed ideal of $B(G)$ contained in A and J a closed subalgebra of $B(G)$. Various characterizations of I are given. A number of concrete closed subalgebras of homogeneous Banach algebras therefore are R-subalgebras. Stimulated by the ideas of Friedberg (1970), we give sufficient and necessary conditions for a closed subalgebra A of $L^p(G)$, $1 < p < \infty$, to be an R-subalgebra. Finally, we give a simple and different proof of Edward's result: R-synthesis holds for $L^2(T)$.

2. Rudin synthesis for homogeneous Banach algebras

Let $\{I_\alpha\}$ be the family of all ideals of $B(G)$ contained in the closed subalgebra A, $\Sigma \mathrm{I}_\alpha$ the ideal of all finite sums $f_1 + \ldots + f_n, f_i \in I_{\alpha_i}$. Then the B-closure I of ΣI_α is the largest

https://doi.org/10.1017/S144678870002156X Published online by Cambridge University Press

closed ideal of $B(G)$ contained in A. Let (Δ_α) be the R-classes induced by A, $Z(I)$ the zero set of I, $P_1 = \{P_\alpha \mid \alpha \neq 0,\ \Delta_\alpha \text{ is a singleton}\}$, and $P_2 = \{P_\alpha \mid \alpha \neq 0,\ \Delta_\alpha \text{ is not a singleton}\}$. Then we have

THEOREM 2.1. (I) *Let J be a closed ideal of $B(G)$, then J is the closed linear space of the set, say K, of characters it contains. In particular, ideals of $B(G)$ are R-subalgebras.*
(II) *The zero set $Z(J)$ of a closed ideal J of $B(G)$ is the complement of K in Γ.*

The proof follows from Wang (1977), Theorem 9.1.

THEOREM 2.2. *Let A be a closed subalgebra of a homogeneous Banach algebra $B(G)$, I the largest closed ideal of $B(G)$ contained in A, and $P_1 = \{P_\alpha \mid \alpha \neq 0,\ \Delta_\alpha$ is a singleton$\}$. Then*

(I) *I is the closed linear span of P_1, $I \subset \overline{\mathrm{P}(A)}^{B}$, and $Z(I) = \Delta_0 \cup (\bigcup_\alpha \Delta_\alpha)$ where Δ_α is not a singleton.*
(II) *Any two closed subalgebras which induce the same R-classes have the same largest closed ideal.*
(III) *A is an R-subalgebra if and only if $(\overline{\mathrm{P}(A)}^{B})^\wedge|_{Z(I)} = (A^{B(G)})^\wedge|_{Z(I)}$.*

PROOF. It suffices to prove (III). If $f \in A^{B(G)}$ and $f = g$ on $Z(I)$ where $g \in \overline{\mathrm{P}(A)}^{B}$, then $f - g \in I$, so that $f - g$ is in the closed span of P_1, or $f - g \in \overline{\mathrm{P}(A)}^{B}$.

LEMMA 2.3. *Let $C(G)$ be the Segal algebra of all complex-valued continuous functions on G and $S(G)$ a Segal algebra containing $C(G)$, H a closed ideal of $S(G)$. For any μ in the dual $S(G)^*$ of $S(G)$, we have $h \perp \mu$ for all $h \in H$ if and only if $h * \mu = 0$ for all $h \in H$.*

PROOF. Since $C(G)^* = M(G)$, $S(G)^* \subset M(G)$. Recall that $M(G) * S(G) = S(G)$ (see Wang (1977)). For $g \in S(G)$, $h \in H$,

$$\begin{aligned} 0 &= \langle h * g, \mu \rangle \text{ since } H \perp \mu \text{ and } h * g \in H \\ &= \int (h * g)(-y)\, d\mu(y) = \int\int g(x) h(-y-x)\, dx\, d\mu(y) \\ &= \int g(x)\, d(h * \mu)(-x) = \langle g, h * \mu \rangle. \end{aligned}$$

Thus $h * \mu = 0$ for all $h \in H$. Conversely, suppose that $h * \mu = 0$ for all $h \in H$. Let (K_n) be an approximate identity. For $h \in H$, we have $0 = \langle K_n, h * \mu \rangle = \langle K_n * h, \mu \rangle$. Since $\| K_n * h - h \|_s \to 0$, we have $\langle h, \mu \rangle = 0$. This completes the proof.

For $H \subset S(G)$, $K \subset S(G)^*$ and $\mu \in S(G)^*$, define

$$H^\perp = \{\varphi \in S(G)^* \colon \langle h, \varphi \rangle = 0\ \forall\ h \in H\},\ K^\perp = \{f \in S(G) \colon \langle f, \varphi \rangle = 0,\ \forall\ \varphi \in K\}$$

and

$$\mathscr{I}_\mu = \{g \in S(G) : g*\mu = 0\}.$$

Then $H^\perp$ and $K^\perp$ are closed linear subspace of $S(G)$ and $S(G)^*$ respectively. Moreover, if H and K are ideals then so are $H^\perp$ and $K^\perp$. Clearly $\mathscr{I}_\mu$ is a closed ideal of $S(G)$ contained in $\{\mu\}^\perp$.

THEOREM 2.4. *Let $S(G)$ be a Segal algebra containing $C(G)$, A a closed subalgebra of $S(G)$ and I the largest closed ideal of $S(G)$ contained in A. Then $I = \bigcap_{\mu\in A^\perp}\mathscr{I}_\mu$.*

PROOF. By Lemma 2.3, $I \subset \bigcap_{\mu\in A^\perp}\mathscr{I}_\mu$. Since

$$\bigcap_{\mu\in A^\perp}\mathscr{I}_\mu \subset \bigcap_{\mu\in A^\perp}\{\mu\} = (A^\perp)^\perp = \bar{A} = A$$

and $\bigcap_{\mu\in A^\perp}\mathscr{I}_\mu$ is a closed ideal of $S(G)$, $\bigcap_{\mu\in A^\perp}\mathscr{I}_\mu \subset I$. Therefore $I = \bigcap_{\mu\in A^\perp}\mathscr{I}_\mu$.

Let A be a closed subalgebra of a homogeneous Banach algebra $B(G)$ inducing the R-classes (Δ_α), $P_2 = \{P_\alpha \mid \alpha \neq 0,\ \Delta_\alpha \text{ is not a singleton}\}$, and J be the closed span of P_2. Clearly $I \cap J = \{0\}$. Note that, for the R-classes (∇_λ) induced by J, then $\nabla_0 = \Delta_0 \cup \Delta_\alpha$ where Δ_α are singletons. We claim that $\overline{\mathrm{P}(A)}^B = \overline{I+J}^B$. In fact, $I+J$ and $\mathrm{P}(A)$ are both spanned by $P_1 \cup P_2$.

THEOREM 2.5. *If $Z(I)$ is in the coset ring of Γ, then $\mathrm{A}^{B(G)} = \mathrm{I}+\mathrm{J}^{B(G)}$ and $\overline{\mathrm{P}(A)}^B = I+J$. In this case, A is an R-subalgebra if and only if J is an R-subalgebra.*

PROOF. By a well-known result of Cohen (see Rudin (1962), 3.1.3), there exists $\mu \in M(G)$ with $\hat{\mu} = \chi_{Z(I)}$. For $f \in A^{B(G)}$, we have $f-f*\mu \in \mathrm{B}$ and $(f-f*\mu)\hat{}\,|_{Z(I)} = 0$. Hence $f-f*\mu \in I$. But $f*\mu \in J^{B(G)}$ so $f \in I+J^{B(G)}$ or $A^{B(G)} \subset I+J^{B(G)}$. Clearly $I+J^{B(G)} \subset A^{B(G)}$. We conclude that $A^{B(G)} = I+J^{B(G)}$.

For $g \in \overline{\mathrm{P}(A)}^B$, there exists $(P_m) \subset \mathrm{P}(A)$ with $\|P_m - g\|_B \to 0$. Since

$$\|P_m*\mu - g*\mu\|_B \leqslant \|\mu\|_M \|P_m - g\|_B \to 0$$

and $P_m*\mu\mathrm{P}(J)$, we have $g*\mu \in \overline{\mathrm{P}(J)} = J$. Since $(g-g*\mu)\hat{}\,|_{Z(I)} = 0$, $g-g*\mu \in I$. Thus $g = (g-g*\mu)+g*\mu \in I+J$. We conclude that $\overline{\mathrm{P}(A)}^B \subset I+J$. But $I+J \subset \overline{\mathrm{P}(A)}^B$, so $\overline{\mathrm{P}(A)}^B = I+J$.

Suppose that A is an R-subalgebra. For f in J^B, there exists $(P_n) \subset \mathrm{P}(A)$ with $\|P_n - f\|_B \to 0$ since $J^B \subset A^B = \mathrm{P}(A)^B$. But $\mu * P_n \in \mathrm{P}(J)$ and

$$\|f-\mu*P_n\|_B = \|\mu*f-\mu*P_n\|_B \to 0,$$

so $f \in \overline{\mathrm{P}(J)}^B$ or J is an R-subalgebra. Conversely, let J be an R-subalgebra, we have $A^B = I+J^B = I+J = \overline{\mathrm{P}(A)}^B$. This completes the proof.

REMARK 2.6. If we replace $Z(I)$ by $Z(I)\setminus\Delta_0$ in Theorem 2.5, the conclusion still holds.

COROLLARY 2.7. *Let $B(G)$ be a homogeneous Banach algebra with maximal ideal space $\mathscr{M}$, and A a closed subalgebra of $B(G)$. If $\hat{A}$ separates points of $\mathscr{M}$, then A is an R-subalgebra.*

PROOF. Let (Δ_α) be the R-classes induced by A. Note that $\Delta_\alpha \cap \mathscr{M} = \Delta_\alpha$ for all $\alpha \neq 0$ since $\Gamma \backslash \mathscr{M} \subset \Delta_0$. $\hat{A}$ separates points of $\mathscr{M}$ if and only if Δ_α is a singleton for all $\alpha \neq 0$ and $\Delta_0 \cap \mathscr{M}$ contains at most one element. We claim : (I) If $\Delta_0 \cap \mathscr{M} = \emptyset$, then A is the closed ideal of $B(G)$ with zero set $\Gamma \backslash \mathscr{M} = \Delta_0$, and $A = B(G)$. (II) If $\Delta_0 \cap \mathscr{M} = \{\gamma\}$, then $A = \{f \in B(G) : \hat{f}(\gamma) = 0\}$. Both cases follow from the fact that $B(G)$ admits an approximate identity (see Wang (1977), p. 95) and so A is a closed ideal of $B(G)$. Thus A is an R-subalgebra.

THEOREM 2.8. *A maximal subalgebra A of a homogeneous Banach algebra $B(G)$ with maximal ideal space $\mathscr{M}$ is either*

(I) $A = \{f \in B(G) : \hat{f}(\gamma) = 0\}$ *for some γ in $\mathscr{M}$ or*

(II) $A = \{f \in B(G) : \hat{f}(\gamma_1) = \hat{f}(\gamma_2)\}$ *for some γ_1, γ_2 in $\mathscr{M}$.*

PROOF. The proof is along the lines of Edwards (1967), p. 17.

COROLLARY 2.9. *A maximal subalgebra of a homogeneous Banach algebra $B(G)$ is an R-subalgebra.*

PROOF. The proof follows from Theorem 2.8.

THEOREM 2.10. *Let B_i $(i = 1, 2)$ be a homogeneous Banach algebra with maximal ideal space $\mathscr{M}_i$ and the family $\mathscr{F}_i$ of all closed subalgebras. If $B_1 \subset B_2$, define $\pi : \mathscr{F}_1 \to \mathscr{F}_2$, $\lambda : \mathscr{F}_2 \to \mathscr{F}_1$ by $\pi(A) = \bar{A}^{B_2}$ for $A \in \mathscr{F}_1$, $\lambda(C) = C \cap B_1$ for $C \in \mathscr{F}_2$. Then we have*

(I) *π preserves the R-classes in the sense that A and $\pi(A)$ induce the same R-classes for all A in $\mathscr{F}_1$.*

(II) *If π is injective and R-synthesis fails for B_1, then R-synthesis fails for B_2.*

(III) *If π is surjective, then $\mathscr{M}_1 = \mathscr{M}_2$ and $\overline{(A^{B_1})}^{B_2} = (A^{B_2})^{B_2}$, $\forall\ A \in \mathscr{F}_1$.*

(IV) *λ preserves the R-classes if and only if $\mathscr{M}_1 = \mathscr{M}_2$.*

(V) *If λ is injective then $\mathscr{M}_1 = \mathscr{M}_2$.*

(VI) *If λ is surjective, then $\overline{\mathrm{P}(A)}^{B_2} \cap B_1 = \overline{\mathrm{P}(A)}^{B_1}$, $\forall\, A \in \mathscr{F}_1$.*

(VII) *If R-synthesis holds for B_1, then π is injective. λ is surjective and $\lambda \circ \pi$ is the identity map of $\mathscr{F}_1$ and $A = \bar{A}^{B_2} \cap B_1$, $\forall\, A \in \mathscr{F}_1$.*

(VIII) *If R-synthesis holds for B_2 and $\mathscr{M}_1 = \mathscr{M}_2$, then $\pi \circ \lambda$ is the identity map of $\mathscr{F}_2$ and $C = \overline{C \cap B_1}^{B_2}$, $\forall\ C \in \mathscr{F}^s{}_2$.*

PROOF. It suffices to prove (II), (III), (IV) and (VII).

(II) Suppose that π is injective and that R-synthesis fails for B_1. Then there is A in $\mathscr{F}_1$ such that $\overline{\mathbf{P}(A)}^{B_1} \subsetneq A^{B_1}$. Thus $\pi(\overline{\mathbf{P}(A)}^{B_1}) \subsetneq \pi(A^{B_1})$. Since $\pi(\overline{\mathbf{P}(A)}^{B_1}) = \overline{\mathbf{P}(A)}^{B_2}$ and $\pi(A^{B_1}) \subset (\bar{A}^{B_2})^{B_2}$, we have $\overline{\mathbf{P}(\bar{A}^{B_2})}^{B_2} = \overline{\mathbf{P}(A)}^{B_2} \neq (\bar{A}^{B_2})^{B_2}$ so R-synthesis fails for B_2.

(III) Suppose that $\mathscr{M}_1 \neq \mathscr{M}_2$. Take γ in $\mathscr{M}_2 \setminus \mathscr{M}_1$, then $\mathbf{C}\gamma \in \mathscr{F}_2$. Obviously thus there is no A in $\mathscr{F}_1$ with $\pi(A) = \bar{A}^{B_2} = \mathbf{C}\gamma$. Hence π cannot be surjective. Now suppose that π is surjective. For $A \in \mathscr{F}_1$, take $D \in \mathscr{F}_1$ with $\pi(D) = (\bar{A}^{B_2})^{B_2}$. Since $D \subset A^{B_1}$, we have $(\bar{A}^{B_2})^{B_2} = \pi(D) \subset \pi(A^{B_1}) = \overline{(A^{B_1})}^{B_2} \subset (\bar{A}^{B_2})^{B_2}$, so $\overline{(A^{B_1})}^{B_2} = (\bar{A}^{B_2})^{B_2}$.

(IV) Suppose that $\mathscr{M}_1 \neq \mathscr{M}_2$. Take γ in $\mathscr{M}_2 \setminus \mathscr{M}_1$, then $\lambda(\mathbf{C}\gamma) = \{0\}$. Since $\mathbf{C}\gamma$ and $\{0\}$ induce different *R-classes,* λ cannot preserve the R-classes. Conversely, if $\mathscr{M}_1 = \mathscr{M}_2$ then $\mathbf{P}(B_1) = \mathbf{P}(B_2)$ and $\mathbf{P}(C \cap B_1) = \mathbf{P}(C \cap B_2) = \mathbf{P}(C)$ for all C in $\mathscr{F}_2$. Thus $\overline{\mathbf{P}(C \cap B_1)}^{B_1} = \overline{\mathbf{P}(C)}^{B_1}$. Since $\overline{\mathbf{P}(C)}^{B_2}$ induces the same R-classes as C, $\overline{\mathbf{P}(C)}^{B_1}$ induces the same R-calsses as $\pi(\overline{\mathbf{P}(C)}^{B_1}) = \overline{\mathbf{P}(C)}^{B_2}$ by (I), we see that the R-classes induced by $C \in \mathscr{F}_2$ and $\lambda(C) = C \cap B_1$ are the same.

(VII) Suppose that R-synthesis holds for B_1.

(1) Assume that π is not injective. Then there are A, A' in $\mathscr{F}_1$ with $A \neq A'$ but $\pi(A) = \pi(A')$. Since R-synthesis holds for B_1, the R-classes induced by A and A' are different. By (I), the R-classes induced by $\pi(A)$ and $\pi(A')$ are different, contradicting our hypothesis that $\pi(A) = \pi(A')$. Thus π is injective.

(2) For each A in $\mathscr{F}_1$, since $\bar{A}^{B_2} \cap B_1$ induces the same R-classes as A, we have $\lambda(\bar{A}^{B_2}) = \bar{A}^{B_2} \cap B_1 = A$. Thus λ is surjective. Combining (1) and (2) we see that $\lambda \circ \pi(A) = \bar{A}^{B_2} \cap B_1 = A$ for all A in $\mathscr{F}_1$ and that $\lambda \circ \pi$ is the identity map of $\mathscr{F}_1$.

Kahane (1965) proved that if A is a closed subalgebra of $L^1(T)$ with a constant $C > 0$ such that $|m-n| < C$ for $m, n \in \Delta_\alpha$, $\alpha \neq 0$, then A is an R-subalgebra of $L^1(T)$. Tseng-Wang (1975) extended Kahane's result to any Segal algebra containing $C(T)$. As a matter of fact, with the same proof their criterion is applicable for certain non-Segal homogeneous Banach algebras : $L^p_\Delta(T)$, $C_\Delta(T)$, $A^p_\Delta(T)$ where $\Delta \subsetneq Z$ and $C^k_\Omega(T)$, $L^{(k)}_\Omega(T)$ where Ω is a finite subset of Z. (See Wang (1977) for these algebras.)

THEOREM 2.11. *Let $B(T)$ be a homogeneous Banach algebra with maximal ideal space $\mathscr{M}$ and d a constant such that $\|\gamma_n\|_B \leqslant d \; \forall \, n \in \mathscr{M}$, where $\gamma_n(x) = e^{inx}$. Suppose that A is a closed subalgebra of $B(T)$ with the R-classes (Δ_α), $k > 0$ a constant such that $|m-n| < k$ whenever $m, n \in \Delta_\alpha$, $\alpha \neq 0$. Then A is an R-subalgebra of $B(T)$.*

3. Rudin synthesis for $L^p(G)$, $1 < p < \infty$

Friedberg (1970) proved that for any closed subalgebra A of $L^1(G)$, $\overline{(\mathbf{P}(A)}^{L^1}) = \bar{B}^{L^2} \cap L^\infty$ and $(A^{L^1}) = \bar{B}^{w^*}$ for some subspace B of $L^\infty(G)$. It turns out that A is an R-subalgebra of $L^1(G)$ if and only if $\bar{B}^{L^2} \cap L^\infty = \bar{B}^{w^*}$. We apply Friedberg's theory to $L^p(G)$-algebras, $1 < p < \infty$.

https://doi.org/10.1017/S144678870002156X Published online by Cambridge University Press

Let A be a closed subalgebra of $L^p(G)$, $1 < p < \infty$, which induces the R-classes (Δ_α). For $\alpha \neq 0$, $\Delta_\alpha = \{\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}\}$, define

$$F_\alpha = \left\{\varphi \in L^q(G) : \varphi = \sum_{i=1}^{n} a_i \gamma_{\alpha i}, \ a_i \in \mathbf{C}, \ \sum_{i=1}^{n} a_i = 0\right\}$$

where $p^{-1} + q^{-1} = 1$, and $F_0 = \Delta_0$. Let F be the closed linear subspace of $L^q(G)$ generated by $\bigcup_\alpha F_\alpha$. We call F the Friedberg space of A. Note that supp $\hat{\varphi} \subset Z(I)$ for all φ in F and $F_\alpha = \{0\}$, $\alpha \neq 0$, whenever Δ_α is a singleton.

LEMMA 3.1. *Let A be a closed subalgebra of $L^p(G)$, $1 < p < \infty$, with R-classes (Δ_α). If F is the Friedberg space of A, then $F \subset A^\perp$.*

PROOF. Fix $f \in A$. For

$$\varphi = \sum_{j=1}^{n} a_j \gamma_{\alpha_j} \in F_\alpha,$$

we have

$$\begin{aligned}
\langle f, \varphi \rangle &= \int_G f(x)\varphi(-x)\,dx \\
&= \int_G f(x) \sum_{j=1}^{n} a_j \gamma_{\alpha_j}(-x)\,dx \\
&= \sum_{j=1}^{n} a_j \int_G f(x) \langle -x, \gamma_{\alpha_j} \rangle\,dx \\
&= \sum_{j=1}^{n} a_j \hat{f}(\gamma_{\alpha_j}) \\
&= \sum_{j=1}^{n} a_j m_\alpha \quad \text{where } m_\alpha = \hat{f}(\Delta_\alpha) \\
&= m_\alpha \cdot \sum_{j=1}^{n} a_j \\
&= 0.
\end{aligned}$$

It is clear that $\langle f, \gamma \rangle = \hat{f}(\gamma) = 0$ for all $\gamma \in F_0$. Therefore $F \subset A^\perp$.

THEOREM 3.2. *Let F be the Friedberg space of A. Then $(A^{L^p})^\perp = F$, $1 < p < \infty$.*

PROOF. By Lemma 3.1, $F \subset (A^{L^p})^\perp$. L^p is reflexive for $1 < p < \infty$, so $(F^\perp)^\perp = F$. Let $g \notin A^{L^p}$. Suppose that $\hat{g}(\gamma_1) \neq \hat{g}(\gamma_2)$ for some $\gamma_1, \gamma_2 \in \Delta_\alpha$, $\alpha \neq 0$. Then $\gamma_1 - \gamma_2 \in F_\alpha$ but $\langle \gamma_1 - \gamma_2, g \rangle \neq 0$. Suppose that $\hat{g}(\gamma) \neq 0$ for some $\gamma \in \Delta_0$, then $\gamma \in F_0$ but $\langle g, \gamma \rangle \neq 0$. We conclude that $g \notin F^\perp$. Thus $F^\perp \subset A^{L^p}$ or $(A^{L^p})^\perp \subset F$. This completes the proof.

THEOREM 3.3.
(I) *For* $1 < p < \infty$, $p^{-1}+q^{-1} = 1$, $\bar{F}^{L^1} \cap L^q \subset (\overline{\mathbf{P}(A)}^{L^p})^{\perp}$.
(II) *For* $1 < p \leqslant 2$, $p^{-1}+q^{-1} = 1$, $\bar{F}^{L^2} \cap L^q = (\overline{\mathbf{P}(A)}^{L^p})^{\perp}$.

PROOF. (I) Let $\varphi \in \bar{F}^{L^1} \cap L^q$. Take $(\varphi_n) \subset F$ with $\|\varphi_n - \varphi\|_1 \to 0$. For $p \in \mathbf{P}(A)$, we have

$$|\langle \varphi, p\rangle| \leqslant |\langle \varphi - \varphi_n, p\rangle| + |\langle \varphi_n, p\rangle|$$
$$\leqslant \|\varphi - \varphi_n\| \cdot \|p\|_\infty \to 0.$$

Hence $\langle \varphi, p\rangle = 0$. Therefore $\langle \varphi, f\rangle = 0$ for f in $\overline{\mathbf{P}(A)}^{L^p}$. Therefore $\bar{F}^{L^1} \cap L^q \subset (\overline{\mathbf{P}(A)}^{L^p})^{\perp}$.

(II) Clearly $\bar{F}^{L^2} \cap L^q \subset \bar{F}^{L^1} \cap L^q \subset (\overline{\mathbf{P}(A)}^{L^p})^{\perp}$. We need to prove that $(\overline{\mathbf{P}(A)}^{L^p}) \subset \bar{F}^{L^2}$. Let $\varphi \in (\overline{\mathbf{P}(A)}^{L^p})^{\perp} \subset L^2(G)$. By the Plancherel Theorem, $\varphi = \Sigma_{\gamma\in\Gamma} \hat{\varphi}(\gamma)\gamma$ in L^2-norm. If $\alpha \neq 0$, $\Delta_\alpha = \{\gamma_1, \ldots, \gamma_m\}$, then $0 = \langle \varphi, p_\alpha\rangle = \Sigma_{j=1}^m \hat{\varphi}(\gamma_j)$. We conclude that $\varphi_\alpha = \Sigma_{j=1}^m \hat{\varphi}(\gamma_j)\gamma_j \in F_\alpha$. For $\gamma \in \Delta_0 = F_0$, let $\varphi_\gamma = \hat{\varphi}(\gamma)\gamma$, we have $\varphi_\gamma \in F$. Hence $\phi \in \bar{F}^{L^2}$. Together with the fact that $(\overline{\mathbf{P}(A)}^{L^p})^{\perp} \subset L^q$, we conclude that $(\overline{\mathbf{P}(A)}^{L^p})^{\perp} \subset \bar{F}^{L^2} \cap L^q$. This completes the proof.

COROLLARY 3.4. *Let F be the Friedberg space of A. Then*
(I) *For* $p > 2$, *if A is an R-subalgebra of* $L^p(G)$, *then* $F^{L^1} \cap L^q = F$.
(II) *For* $p > 2$, *if* $(\overline{\mathbf{P}(A)}^{L^p})^{\perp} \subset F$, *then A is an R-subalgebra.*
(III) *For* $1 < p \leqslant 2$, *A is an R-subalgebra of* $L^p(G)$ *if and only if* $F = \bar{F}^{L^2} \cap L^q$.

PROOF. By Theorem 3.2, and 3.3.

By Corollary 3.4, (III), we have that R-synthesis holds for $L^2(G)$. Note that for $G = T$, this fact was first proved by Edwards (1967), pp. 15–16, with different methods.

THEOREM 3.5. *Let A be a closed subalgebra of* $L^p(G)$, *F the Friedberg space of A, and I the largest closed ideal of* $L^p(G)$ *contained in A; then the following are equivalent:*
(I) $Z(I) = \Delta_0$.
(II) $J = \{0\}$.
(III) *J is an ideal of* $L^p(G)$.
(IV) $A = I$.
(V) *F is an ideal of* $L^q(G)$, *where* $p^{-1}+q^{-1} = 1$.

PROOF. Clearly (I), (II) and (III) are equivalent.

(III) $\Rightarrow$ (IV). Let $P_\alpha \in J$. If $P_\alpha \neq 0$, let $\gamma \in \Delta_\alpha$, we have $\gamma = \gamma * P_\alpha \in J$. Thus $\Delta_\alpha = \{\gamma\}$, a contradiction. This implies $J = \{0\}$. By Theorem 2.5, $\overline{\mathbf{P}(A)}^{L^p} = I$. Spectral synthesis holds for $L^p(G)$ and $\hat{f}(Z(I)) = \hat{f}(\Delta_0) = 0$ for all f in A, so $A = I$.

(IV) $\Rightarrow$ (V). Suppose that $A = I$. Then Δ_α is a singleton for all $\alpha \neq 0$. In this case, $F_\alpha = \{0\}$ for all $\alpha \neq 0$. Therefore F is the closed linear subspace generated by

$\{\gamma : \gamma \in \Delta_0 = Z(I)\}$. Routine arguments reveal that F is a closed ideal of $L^q(G)$.

(V) $\Rightarrow$ (I) Assume that $Z(I)\backslash\Delta_0 \neq \emptyset$. Take $\Delta_\alpha = \{\gamma_1, \ldots, \gamma_m\} \subset Z(I), m \geqslant 2$. We have $\gamma_1 - \gamma_2 \in F$. Moreover $\gamma_1 = \gamma_1 * (\gamma_1 - \gamma_2) \in F$ since F is an ideal of L^q. By Lemma 3.1, $\langle \gamma_1, P_\alpha \rangle = 0$ since $P_\alpha \in A$. But $\langle \gamma_1, P_\alpha \rangle = \hat{P}_\alpha(\gamma_1) = 1$, a contradiction.

Finally we give a sufficient condition of closed subalgebras of $L^p(G)$, $p > 2$, to be R-subalgebras.

THEOREM 3.6. *Let A be a closed subalgebra of $L^p(G)$, $p > 2$, $p^{-1} + q^{-1} = 1$. If $(\overline{\mathrm{P}(A)}^{L^p})^{\perp} \cap L^\infty(G)$ is L^q-dense in $(\overline{\mathrm{P}(A)}^{L^p})^{\perp}$, then A is an R-subalgebra of $L^p(G)$.*

PROOF. Suppose that $(\overline{\mathrm{P}(A)}^{L^p})^{\perp} \cap L^\infty$ is L^q-dense in $(\overline{\mathrm{P}(A)}^{L^p})^{\perp}$. We claim that $\overline{\mathrm{P}(A)}^{L^1} \cap L^p = \overline{\mathrm{P}(A)}^{L^p}$. Clearly $\overline{\mathrm{P}(A)}^{L^p} \subset \overline{\mathrm{P}(A)}^{L^1} \cap L^p$. Let $\varphi \in (\overline{\mathrm{P}(A)}^{L^p})^{\perp} \cap L^\infty$. For $f \in \overline{\mathrm{P}(A)}^{L^1} \cap L^p$, take $(f_n) \subset \mathrm{P}(A)$ with $\| f_n - f \|_1 \to 0$, we have

$$|\langle f, \varphi \rangle| \leqslant |\langle f_n - f, \varphi \rangle| + |\langle f_n, \varphi \rangle| \leqslant \| f_n - f \|_1 \| \varphi \|_\infty \to 0$$

so $\langle f, \varphi \rangle = 0$. Thus $(\overline{\mathrm{P}(A)}^{L^p})^{\perp} \cap L^\infty \subset (\overline{\mathrm{P}(A)}^{L^1} \cap L^p)^{\perp}$. By the hypothesis, $(\overline{\mathrm{P}(A)}^{L^p})^{\perp} \cap L^\infty$ is L^q-dense in $(\overline{\mathrm{P}(A)}^{L^p})^{\perp}$. Since $(\overline{\mathrm{P}(A)}^{L^1} \cap L^p)^{\perp}$ is closed in $L^q(G)$, $(\overline{\mathrm{P}(A)}^{L^p})^{\perp} \subset (\overline{\mathrm{P}(A)}^{L^1} \cap L^p)^{\perp}$ or $\overline{\mathrm{P}(A)}^{L^p} \supset \overline{\mathrm{P}(A)}^{L^1} \cap L^p$. Thus $\overline{\mathrm{P}(A)}^{L^1} \cap L^p = \overline{\mathrm{P}(A)}^{L^p}$.

Now

$$\begin{aligned} A^{L^p} &= \{ f \in L^p(G) : \hat{f}(\Delta_0) = 0,\ \hat{f}(\Delta_\alpha) = \text{constant},\ \alpha \neq 0 \} \\ &= \{ f \in L^2(G) : \hat{f}(\Delta_0) = 0,\ \hat{f}(\Delta_\alpha) = \text{constant},\ \alpha \neq 0 \} \cap L^p(G) \\ &= \overline{\mathrm{P}(A)}^{L^2} \cap L^p \text{ (since } R\text{-synthesis holds for } L^2(G)) \\ &\subset \overline{\mathrm{P}(A)}^{L^1} \cap L^p = \overline{\mathrm{P}(A)}^{L^p}. \end{aligned}$$

Therefore $A^{L^p} = \overline{\mathrm{P}(A)}^{L^p}$. Thus A is an R-subalgebra of $L^p(G)$.

Acknowledgement

We would like to thank the referee for his valuable suggestions in the first draft of this paper.

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