Canad. Math. Bull. Vol. **45** (1), 2002 pp. 71–79

Images of Additive Polynomials in $\mathbb{F}_q((t))$ Have the Optimal Approximation Property

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Abstract. We show that the set of values of an additive polynomial in several variables with arguments in a formal Laurent series field over a finite field has the optimal approximation property: every element in the field has a (not necessarily unique) closest approximation in this set of values. The approximation is with respect to the canonical valuation on the field. This property is elementary in the language of valued rings.

1 Introduction

Let $\mathbb{F}_q$ denote the field with q elements, where q is a power of a prime p. The power series field $\mathbb{F}_q((t))$, also called "field of formal Laurent series over $\mathbb{F}_q$", carries a canonical valuation v_t, the t-adic valuation, with value group $\mathbb{Z}$ and $v_t(t) = 1$. In studying elementary properties of this valued field the following notion turns up, see [K].

Let (K, v) be a valued field, and S a nonempty subset of K. We say that S has the *optimal approximation property (*OA*) in* (K, v) if for every point in K there is a (not necessarily unique) closest point in S, that is, for every $x \in K$ there exists $y \in S$ such that

$$v(x - y) = \max\{v(x - z) \mid z \in S\}.$$

(We write valuations in the additive Krull style, that is, the ultrametric triangle law reads as $v(a + b) \geq \min\{va, vb\}$. Thus elements a, b are close if $v(a - b)$ is large. We denote the value group of (K, v) by vK.) The following implications hold (see Section 2):

$$S \text{ compact } \Rightarrow S \text{ has OA} \Rightarrow S \text{ is closed.}$$

This approximation property relates to the model theory of valued fields since it is elementary for (elementarily) definable S. The *image*

$$S := \{f(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in K\} \tag{1}$$

of K under a polynomial $f \in K[X_1, \ldots, X_n]$ is definable, so the question arises when this image has OA. In [K] the second author shows that the image of an algebraically maximal field under a polynomial in one variable has OA. (A valued field is said to be *algebraically maximal* if it has no proper algebraic valued field extension preserving both value group and residue field.)

Received by the editors May 26, 2000.

While working on this paper, the first author was partially supported by NSF grant DMS 98-02745. The second author was partially supported by a Canadian NSERC grant. Both authors would like to thank the MSRI for support and hospitality.

AMS subject classification: 12J10, 12L12, 03C60.

We consider here the case of additive polynomials in several variables. Let K be a field of characteristic $p > 0$. A polynomial $f(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$ is called additive if

$$f(a_1 + b_1, \ldots, a_n + b_n) = f(a_1, \ldots, a_n) + f(b_1, \ldots, b_n)$$

for all elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ in any extension field of K. If f is additive, then

$$f(X_1, \ldots, X_n) = f_1(X_1) + \cdots + f_n(X_n) \tag{2}$$

where each $f_i(X_i) := f(0, \ldots, 0, X_i, 0 \ldots, 0)$ is an additive polynomial in one variable. We refer to [L, VIII, Section 11] for the fact that the additive polynomials in one variable X over K are precisely the polynomials of the form

$$\sum_{i=0}^{m} c_i X^{p^i} \quad \text{with } c_i \in K, m \in \mathbb{N}.$$

A valued field is called *maximal* if it has no proper valued field extension preserving both value group and residue field. In [K] it is shown that if (K, v) is maximal, then under a certain additional (elementary) condition on the additive polynomial f in several variables, the image S has OA. It would be desirable to remove this additional condition. We do this here for $\bigl(\mathbb{F}_q((t)), v_t\bigr)$, using its local compactness:

Theorem 1 *If f is an additive polynomial in several variables with coefficients in $\mathbb{F}_q((t))$, then the image of $\mathbb{F}_q((t))$ under f has the optimal approximation property in $\bigl(\mathbb{F}_q((t)), v_t\bigr)$.*

It would be nice to generalize this to all maximal valued fields, in other words, to replace the use of local compactness by some other argument.

The axiom scheme which by Theorem 1 holds in $\bigl(\mathbb{F}_q((t)), v_t\bigr)$ consists of the sentences

$$\forall (c_{i,j})\, \forall x\, \exists y_1 \cdots \exists y_n\, \forall z_1 \cdots \forall z_n : v\Bigl(x - \sum_{i=1}^{n}\sum_{j=0}^{n} c_{i,j} y_i^{p^j}\Bigr) \geq v\Bigl(x - \sum_{i=1}^{n}\sum_{j=0}^{n} c_{i,j} z_i^{p^j}\Bigr)$$

in the language of valued rings. In [K] this scheme is shown to be independent of the following more familiar axioms satisfied by $\bigl(\mathbb{F}_q((t)), v_t\bigr)$: "henselian defectless (= algebraically complete) valued field with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_q$". We suspect that the elementary theory of the valued field $\bigl(\mathbb{F}_q((t)), v_t\bigr)$ is completely axiomatizable by augmenting these familiar axioms with sentences that express just properties of *additive* polynomials (like those provided by Theorem 1).

To be more explicit about this suspicion, let us briefly review a (well-known) module-theoretic interpretation of additive polynomials. Suppose the field K is infinite and of characteristic $p > 0$. Then the endomorphism ring of the additive group of K has subring $K[\varphi]$, where $\lambda \in K \subseteq K[\varphi]$ acts on K as multiplication by λ, and φ acts as the Frobenius map $x \mapsto x^p$; so $\varphi\lambda = \lambda^p\varphi$ in $K[\varphi]$. This makes K a left module over $K[\varphi]$, and the images of K under the additive polynomials in several variables over K are exactly the additive subgroups of the $K[\phi]$-module K of the form $f_1K+\cdots+f_nK$, where $f_1, \ldots, f_n \in K[\phi]$. (Since $K[\phi]$ is not commutative, these additive subgroups are not in general submodules of K.) For $(K, v) = \big(\mathbb{F}_q((t)), v_t\big)$, its elementary theory as a *valued module* over $K[\varphi]$ has yet to be determined in a satisfactory way; our theorem is exactly about this valued module. A complete description of the elementary theory of this *valued module* seems essential in reaching an understanding of the elementary theory of the *valued field* (K, v).

The second author would like to thank Hervé Perdry for discussions on Remark 1 at the end of this paper, and Trevor Green for proof-reading.

2 Compactness and OA

Let (K, v) be a non-trivially valued field, $\alpha \in vK$ and $a \in K$. The *closed ball* $B_\alpha(a)$ and the *open ball* $B^\circ_\alpha(a)$ are defined as follows:

$$B_\alpha(a) = \{b \in K \mid v(a-b) \geq \alpha\} \quad \text{and} \quad B^\circ_\alpha(a) = \{b \in K \mid v(a-b) > \alpha\}.$$

Both kinds of balls are open and closed in the topology induced by the valuation, and are easily seen to have OA in (K, v). (But they are not compact if (K, v) is not locally compact.) Note that if vK is dense, then $K \setminus B_\alpha(0)$ is closed but does not have OA in (K, v) since it contains no closest point to 0.

Lemma 2 *Suppose S is a nonempty compact subset of K. Then S has* OA *in (K, v).*

Proof Let $x \in K \setminus S$. If $z \in S$, then $z \notin B^\circ_{v(x-z)}(x)$, so the collection

$$\{K \setminus B^\circ_{v(x-z)}(x) \mid z \in S\}$$

is an open covering of S, hence contains a finite subcovering. The collection of balls $B^\circ_{v(x-z)}(x)$ is totally ordered by inclusion. It follows that the finite subcovering contains a largest set, say, $K \setminus B^\circ_{v(x-y)}(x)$, which consequently contains S. That is, $S \cap B^\circ_{v(x-y)}(x) = \varnothing$. Since $y \in S$, this means that $v(x-y) = \max\{v(x-z) \mid z \in S\}$. ■

This proof actually provides a characterization of the optimal approximation property: *A nonempty set $S \subseteq K$ has* OA *in (K, v) if and only if every covering of S by the complements of a system of open balls with common center has a finite subcovering.*

From the continuity of polynomial maps and Lemma 2 we conclude the following: *If (K, v) is locally compact, then for every $f \in K[X_1, \ldots, X_n]$, $\alpha \in vK$ and $a \in K$, the image $\{f(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in B_\alpha(a)\}$ of $B_\alpha(a)$ has* OA *in (K, v).*

We also have the following easy implication for nonempty $S \subseteq K$:

$$S \text{ has OA in } (K, v) \Longrightarrow S \text{ is closed.}$$

As noted above, the converse fails if vK is dense. However, by a standard argument, this converse does hold if (K,v) is locally compact. (The validity of this converse for definable S amounts to a further set of elementary properties of the valued field $\big(\mathbb{F}_q((t)),v_t\big)$.)

3 Valuation Independence

Let (K,v) be a valued field, L a subfield of K, and $(b_i)_{i\in I}$ a system of non-zero elements in K, with $I\neq\varnothing$. We call this system *L-valuation independent* if for every choice of elements $a_i\in L$ such that $a_i\neq 0$ for only finitely many $i\in I$, we have

$$v\Big(\sum_{i\in I}a_ib_i\Big)=\min_{i\in I}v(a_ib_i).$$

If V is an L-subvector space of K, then this system is called a *valuation basis of V* if it is a basis of V and L-valuation independent.

Let $d=p^\nu$, $\nu\in\mathbb{N}$, $(K,v)=\big(\mathbb{F}_q((t)),v_t\big)$ and $L=K^d:=\{a^d\mid a\in K\}=\mathbb{F}_q((t^d))$. Note that $1,t,t^2,\dots,t^{d-1}$ is a valuation basis of K as vector space over L.

Let V be an L-subvector space of K with basis $b_1,\dots,b_m$. We now indicate how to modify this basis to a valuation basis of V. Write $b_i=\sum_{j=0}^{d-1}c_{ij}t^j$ with $c_{ij}\in L$. Take j_1 to be the unique index such that $vb_1=vc_{1j_1}t^{j_1}$. Replacing b_1 by b_1/c_{1j_1} we may assume $c_{1j_1}=1$. Next, for every $i\geq 2$, replace b_i by $b_i-c_{ij_1}b_1$, so we reduce to the case that $c_{ij_1}=0$ for $i\geq 2$. Repeat this procedure with the new elements $b_2,\dots,b_m$. By construction, the coefficients of t^{j_1} in the representations of these elements are zero. Thus, if $vb_2=vc_{2j_2}t^{j_2}$ (where c_{2j_2} denotes the new coefficient), then $j_2\neq j_1$. Hence, applying the procedure a total of m times we obtain a new basis of V which by an abuse of language we also call $b_1,\dots,b_m$, such that $vb_1,\dots,vb_m$ are distinct elements of $\{0,1,\dots,d-1\}$. In particular, this new basis is a valuation basis of V.

4 Proof of Theorem 1

Throughout this section, $(K,v)=\big(\mathbb{F}_q((t)),v_t\big)$. Let

$$S:=\{f(a_1,\dots,a_n)\mid a_1,\dots,a_n\in K\},$$

be the image of K under some additive polynomial $f\in K[X_1,\dots,X_n]$.

We choose a system of non-zero additive polynomials $h_1,\dots,h_k\in K[X]$ in one variable such that

$$h_1(K)+\cdots+h_k(K)=S \tag{3}$$

for which $\sum_{i=1}^k\deg h_i$ is minimal. This is possible by (2). The idea is to modify this system to one that makes it so to say "visible" that S has OA in (K,v).

Denote by c_i the leading coefficient and by d_i the degree of h_i, for $1\leq i\leq k$. Then:

Lemma 3 *For every choice of $a_1, \dots, a_k \in K$, not all zero, we have*

$$\sum_{i=1}^{k} c_i a_i^{d_i} \neq 0.$$

Proof Suppose there are $a_1, \dots, a_k \in K$, not all zero, such that $\sum_{i=1}^{k} c_i a_i^{d_i} = 0$. After renumbering we may assume $a_1 \neq 0$ and $d_1 = \max\{d_i \mid 1 \leq i \leq k \text{ and } a_i \neq 0\}$. Replacing every a_i by $a_i a_1^{-d_1/d_i}$, we may even assume that $a_1 = 1$. Now we set

$$\tilde{h}_1(X) := \sum_{i=1}^{k} h_i(a_i X^{d_1/d_i}).$$

Since each polynomial $h_i(a_i X^{d_1/d_i})$ has degree d_1 and leading coefficient $c_i a_i^{d_i}$, and since $\sum_{i=1}^{k} c_i a_i^{d_i} = 0$, we obtain $\deg \tilde{h}_1 < d_1 = \deg h_1$. Therefore,

$$\deg \tilde{h}_1 + \sum_{i=2}^{k} \deg h_i < \sum_{i=1}^{k} \deg h_i. \tag{4}$$

On the other hand, for every choice of $b_i \in K$ we have

$$\begin{aligned} h_1(b_1) + \cdots + h_k(b_k) &= \sum_{i=1}^{k} h_i(a_i b_1^{d_1/d_i}) + \sum_{i=2}^{k} \big(h_i(b_i) - h_i(a_i b_1^{d_1/d_i}) \big) \\ &= \tilde{h}_1(b_1) + \sum_{i=2}^{k} h_i(b_i - a_i b_1^{d_1/d_i}). \end{aligned}$$

Thus $S \subseteq \tilde{h}_1(K) + h_2(K) + \cdots + h_k(K)$. The converse inclusion follows from the definition of $\tilde{h}_1$ and the fact that S is an additive subgroup of K which contains the images $h_i(K)$ for all i. So

$$S = \tilde{h}_1(K) + h_2(K) + \cdots + h_k(K),$$

which in view of (4) contradicts the minimality of the system $h_1, \dots, h_k$. ■

Lemma 4 *There are additive polynomials $g_1, \dots, g_m \in K[X]$ in one variable such that*

a) $S = g_1(K) + \cdots + g_m(K)$,
b) *all polynomials g_i have the same degree $d = p^\nu$, for some non-negative integer ν,*
c) *the leading coefficients $b_1, \dots, b_m$ of $g_1, \dots, g_m$ are such that $vb_1, \dots, vb_m$ are distinct elements of $\{0, 1, \dots, d-1\}$.*

Proof We set

$$d := \max_i d_i \quad \textit{and} \quad \delta_i := d/d_i.$$

Since the h_i are additive polynomials, these numbers are powers of p. Hence

$$K = K^{\delta_i} + tK^{\delta_i} + \cdots + t^{\delta_i - 1}K^{\delta_i}.$$

Therefore,

$$h_i(K) = h_i(K^{\delta_i}) + h_i(tK^{\delta_i}) + \cdots + h_i(t^{\delta_i-1}K^{\delta_i}) = h_{i,0}(K) + \cdots + h_{i,\delta_i-1}(K)$$

where

$$h_{i,j}(X) := h_i(t^j X^{\delta_i}) \in K[X].$$

Consequently,

$$S = \sum_{i=1}^{k} \sum_{j=0}^{\delta_i - 1} h_{i,j}(K)$$

with all polynomials $h_{i,j}$ having degree d.

We claim that the leading coefficients $c_{ij} = c_i t^{jd_i}$ of the polynomials $h_{i,j}$ are K^d-linearly independent. Suppose that for $a_{ij} \in K$,

$$0 = \sum_{i=1}^{k} \sum_{j=0}^{\delta_i-1} c_{ij} a_{ij}^d = \sum_{i=1}^{k} c_i \sum_{j=0}^{\delta_i-1} t^{jd_i} a_{ij}^{\delta_i d_i} = \sum_{i=1}^{k} c_i \Big(\sum_{j=0}^{\delta_i-1} t^j a_{ij}^{\delta_i} \Big)^{d_i}.$$

Lemma 3 then gives

$$\sum_{j=0}^{\delta_i-1} t^j a_{ij}^{\delta_i} = 0 \quad \text{for } 1 \le i \le k.$$

As $1, t, \dots, t^{\delta_i-1}$ are K^{δ_i}-linearly independent, it follows that $a_{ij} = 0$ for all i and j. This proves our claim.

We have now found additive polynomials $\tilde{h}_1, \dots, \tilde{h}_m$ in $K[X]$ of degree d, with K^d-linearly independent leading coefficients $\tilde{c}_1, \dots, \tilde{c}_m$ and such that $S = \tilde{h}_1(K) + \cdots + \tilde{h}_m(K)$. The previous section shows that the K^d-vector space generated by $\tilde{c}_1, \dots, \tilde{c}_m$ admits a valuation basis $b_1, \dots, b_m$, say, for which $vb_1, \dots, vb_m$ are distinct elements of $\{0, 1, \dots, d-1\}$. Write $b_i = \sum_{j=1}^m r_{ij}^d \tilde{c}_j$ with $r_{ij} \in K$. Now we set

$$g_i(X) := \sum_{j=1}^{m} \tilde{h}_j(r_{ij}X)$$

and observe that for each i the polynomial g_i is of degree d with leading coefficient b_i. It only remains to show that condition a) is satisfied. Since S is an additive subgroup of K and contains the images $\tilde{h}_j(K)$ for all j it follows that $g_1(K) + \cdots + g_m(K) \subseteq \tilde{h}_1(K) + \cdots + \tilde{h}_m(K) = S$. On the other hand, both $\tilde{c}_1, \dots, \tilde{c}_m$ and $b_1, \dots, b_m$ are bases, so the matrix (r_{ij}^d) is invertible. Thus, also the matrix (r_{ij}) is invertible. Denote its inverse by (s_{ij}), with $s_{ij} \in K$. A simple computation then shows that $\tilde{h}_i = \sum_{j=1}^m g_j(s_{ij}X)$. Hence $S = \tilde{h}_1(K) + \cdots + \tilde{h}_m(K) \subseteq g_1(K) + \cdots + g_m(K)$, which concludes the proof. ■

https://doi.org/10.4153/CMB-2002-007-5 Published online by Cambridge University Press

Lemma 5 *Suppose the additive polynomials $g_1, \ldots, g_m \in K[X]$ satisfy conditions b) and c) of Lemma 4. Then there exists $\alpha \in vK = \mathbb{Z}$ such that if B is the additive subgroup $B_\alpha(0)$ of K and C a group complement of B in K, then for all $b \in g_1(B) + \cdots + g_m(B)$ and all non-zero $c \in g_1(C) + \cdots + g_m(C)$,*

$$vc < d\alpha \leq vb.$$

Proof Let $f(X) = c_n X^n + \cdots + c_1 X + c_0 \in K[X]$ be any polynomial, $c_n \neq 0$. Take $\alpha \in vK$ such that if $a \in K$ and $va \leq \alpha$ then

$$vc_n a^n = vc_n + nva < vc_j + jva = vc_j a^j \quad \text{for } 0 \leq j < n,$$

which implies

$$vf(a) = \min_{0 \leq j \leq n} vc_j a^j = vc_n a^n = vc_n + nva. \tag{5}$$

This in turn implies that for $a \in K$ with $va \geq \alpha$ we have

$$vf(a) \geq \min_{0 \leq j \leq n} vc_j a^j \geq vc_n + n\alpha. \tag{6}$$

Now choose α such that (5) holds simultaneously for all $f = g_i$, $1 \leq i \leq m$, and all $a \in K$ with $va \leq \alpha$. Hence (6) holds simultaneously for all $f = g_i$ and all $a \in K$ with $va \geq \alpha$. As before, denote by b_i the leading coefficient of g_i. Then every $a \in K$ with $va \leq \alpha$ satisfies $vg_i(a) = vb_i + dva$. Since $vb_1, \ldots, vb_m$ are distinct elements of $\{0, 1, \ldots, d-1\}$, we find that for all choices of $a_i \in K$ with $va_i < \alpha$ for at least one i, we have $vb_i + dva_i < d\alpha$ for this i, and

$$v\big(g_1(a_1) + \cdots + g_m(a_m)\big) = \min_i vg_i(a_i) = \min_i vb_i + dva_i < d\alpha$$

by (5). But if $va_i \geq \alpha$ for all i, then by (6),

$$v\big(g_1(a_1) + \cdots + g_m(a_m)\big) \geq \min_i vg_i(a_i) \geq \min_i vb_i + d\alpha \geq d\alpha.$$

Let B and C be as in the lemma. Then $B \cap C = \{0\}$, so every non-zero $c \in C$ satisfies $vc < \alpha$. Now the lemma follows from the inequalities above. ■

Lemma 6 *Let (F, v) be a valued field with value group $vF = \mathbb{Z}$. Suppose $\mathcal{B}$ and $\mathcal{C}$ are non-trivial additive subgroups of F such that $\mathcal{B}$ has* OA *in (F, v) and $vc < vb$ for all $b \in \mathcal{B}$ and all non-zero $c \in \mathcal{C}$. Then $\mathcal{B} + \mathcal{C}$ has* OA *in (F, v).*

Proof Take any $x \in F$. Since $\mathcal{B} \neq \{0\}$, the set $\{vc \mid 0 \neq c \in \mathcal{C}\}$ is bounded from above in $vF = \mathbb{Z}$, so $v\mathcal{C}$ has a maximum γ.

Suppose first that $v(x - z) \leq \gamma$ for all $z \in \mathcal{C}$. Then $\{v(x - z) \mid z \in \mathcal{C}\}$ has a maximum in $\mathbb{Z}$. Also, if $b \in \mathcal{B}$ and $c \in \mathcal{C}$, then $v(x - c) \leq \gamma < vb$, and therefore,

$$v\big(x - (b + c)\big) = \min\{v(x - c), vb\} = v(x - c) \leq \max\{v(x - z) \mid z \in \mathcal{C}\},$$

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showing that $\{v(x-z) \mid z \in \mathcal{B}+\mathcal{C}\}$ has a maximum, namely $\max\{v(x-z) \mid z \in \mathcal{C}\}$.

Now assume that $v(x-c_0) > \gamma$, with $c_0 \in \mathcal{C}$. Our assumption on $\mathcal{B}$ implies that the set $\{v(x-c_0-z) \mid z \in \mathcal{B}\}$ has a maximum, say, $v(x-c_0-b_0)$ with $b_0 \in \mathcal{B}$. Note that $v(x-c_0-b_0) > \gamma$. Take any $b \in \mathcal{B}$ and $c \in \mathcal{C}$. Then

$$v\bigl(x-(b+c)\bigr) \geq \min\{v(x-c_0-b_0), v(c_0-c), v(b_0-b)\}.$$

If $c_0 \neq c$, then $v(c_0-c) < v(b_0-b)$ and $v(c_0-c) \leq \gamma < v(x-c_0-b_0)$, showing that $v\bigl(x-(b+c)\bigr) = v(c_0-c) < v(x-c_0-b_0)$. If $c = c_0$, then $v\bigl(x-(b+c)\bigr) \leq v(x-c_0-b_0)$ holds by our choice of b_0. Thus $\{v(x-z) \mid z \in \mathcal{B}+\mathcal{C}\}$ has a maximum, namely $v(x-c_0-b_0)$. ■

Proof of Theorem 1 Let f be an additive polynomial in several variables with coefficients in $K = \mathbb{F}_q((t))$. Write the image of K under f as $S = g_1(K)+\cdots+g_m(K)$ with additive polynomials $g_1,\dots,g_m \in K[X]$ in one variable which satisfy conditions b) and c) of Lemma 4. We choose α, B and C as in Lemma 5. Since B and C are additive subgroups of K, the additivity of the g_i implies that $\mathcal{B} := g_1(B)+\cdots+g_m(B)$ and $\mathcal{C} := g_1(C)+\cdots+g_m(C)$ are again additive subgroups of K. Moreover, as B is compact, so is $\mathcal{B}$, and thus $\mathcal{B}$ has OA. Lemma 5 implies that the hypothesis of Lemma 6 is satisfied, so $\mathcal{B}+\mathcal{C}$ has OA. But by the additivity of the g_i,

$$\mathcal{B}+\mathcal{C} = g_1(B+C)+\cdots+g_m(B+C) = g_1(K)+\cdots+g_m(K) = S.$$

This concludes our proof. ■

Remark 1 Theorem 1 goes through when $\bigl(\mathbb{F}_q((t)), v_t\bigr)$ is replaced by any henselian valued subfield $(L, v_t|_L)$ of $\bigl(\mathbb{F}_q((t)), v_t\bigr)$ such that $\mathbb{F}_q(t) \subset L$ and $[L : L^p] = p$.

To see this, note that by Greenberg's approximation theorem for discrete henselian valuation rings [G], such a subfield $(L, v_t|_L)$ is existentially closed in $\bigl(\mathbb{F}_q((t)), v_t\bigr)$ (the conditions $\mathbb{F}_q(t) \subset L$ and $[L : L^p] = p$ imply that $\mathbb{F}_q((t))|L$ is separable). Let f be an additive polynomial with coefficients in L, and let $x \in L$. By Theorem 1 there are $a_1,\dots,a_m \in K$ such that $v_t\bigl(x-f(a_1,\dots,a_m)\bigr)$ is maximal. Since $v_tL = v_t\mathbb{F}_q((t))$ we can choose $c \in L$ such that $v_tc = v_t\bigl(x-f(a_1,\dots,a_m)\bigr)$. So the existential sentence $\exists x_1\cdots\exists x_m v_tc = v_t\bigl(x-f(x_1,\dots,x_m)\bigr)$ holds in $\bigl(\mathbb{F}_q((t)), v_t\bigr)$. But then it also holds in $(L, v_t|_L)$, that is, $a_1,\dots,a_m$ can be chosen to lie in L.

An example of such a valued subfield is the henselization of $\mathbb{F}_q(t)$ inside $\mathbb{F}_q((t))$, where both fields carry the t-adic valuation.

Remark 2 Theorem 1 does not hold for $\mathbb{F}_q(t)$ with its t-adic valuation: the additive polynomial $X - X^p$ does not have t in its image on this field, but t can be approximated arbitrarily closely by elements in this image, since $t = x - x^p$ for $x = \sum_{n=0}^{\infty} t^{p^n} \in \mathbb{F}_q((t))$.

Remark 3 The proofs of Lemmas 3 and 4 do not use the local compactness of $(K, v) = \bigl(\mathbb{F}_q((t)), v_t\bigr)$. Actually, we just need that $1, t, \dots, t^{p-1}$ is a basis for $K|K^p$ and that vt is not divisible by p in vK. Hence, in every extension (L, v) of $\bigl(\mathbb{F}_q((t)), v_t\bigr)$

https://doi.org/10.4153/CMB-2002-007-5 Published online by Cambridge University Press

with these properties one can derive polynomials $g_1, \dots, g_m$ over L as in Lemma 4 from any given additive polynomial f over L. The proof of Lemma 5 then shows that there exists $\alpha_0 \in vL$ such that for each $\alpha \leq \alpha_0$ in vL, any complement C of $B_\alpha(0)$ in L, and all $a_1, \dots, a_m \in C$ we have

$$v \sum_{i=1}^{m} g_i(a_i) = \min_i vg_i(a_i)$$

(so the sum $g_1(C) + \cdots + g_m(C)$ is "valuation direct"). If (L, v) is maximal, one can then prove along the lines of [K] that $g_1(C) + \cdots + g_m(C)$ has OA in (L, v).

References

[G] M. J. Greenberg, *Rational points in henselian discrete valuation rings.* Inst. Hautes Études Sci. Publ. Math. **31**(1966), 59–64.

[K] F.-V. Kuhlmann, *Elementary properties of power series fields over finite fields.* J. Symbolic Logic (2) **66**(2001), 771–791.

[L] S. Lang, *Algebra*. Addison-Wesley, New York, 1965.

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